Dense sphere packings from new codes

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July 15, 2002

Abstract

The idea behind the **coset code** construction (see [6, 7]) is to reduce the construction of sphere packings to error-correcting codes in a unified way. We give here a short self-contained description of this method. In recent papers [1, 2, 3, 4] we constructed a large number of new binary, ternary and quaternary linear error-correcting codes. In a number of dimensions our new codes yield improvements. Recently Vardy [8, 9] has found a construction, which yields record densities in dimensions 20,27,28,29 and 30. We give a short description of his method using the language of coset codes. Moreover we are able to apply this method in dimension 18 as well, producing a sphere packing with a record center density of $(3/4)^9$.

Key Words

Sphere packings, lattices, codes, center density, hexagonal lattice, dual codes, Mordell's inequality, Leech lattice.

1 Sphere packings and coset codes

Let $E = I\!R^N$ be the N-dimensional Euclidean space, $\Gamma \subset E$ a discrete subset. Denote by $\|x\|$ the Euclidean distance of x from the origin, by $\mu(\Gamma)$ the minimum norm (= square of the distance) between different elements of Γ. The value $\rho(\Gamma) = \sqrt{\mu(\Gamma)}/2$ is called the **packing radius** of Γ. The meaning of ρ is that open balls of radius ρ centered at the lattice points do not intersect, and ρ is the maximum such radius. We will be mainly interested in the parameter

$$\delta = \delta(\Gamma) = \frac{\rho^N}{vol(\Gamma)},$$

the **center density** of Γ. As the discrete sets Γ constructed in this paper will be unions of cosets of lattices the determination of the volume will be no problem (if Γ is the union of M different cosets of a lattice of volume ν, then Γ has volume ν/M). Observe that δ is unchanged if a constant positive nonzero multiplicative factor is applied: $\delta(c \cdot \Gamma) = \delta(\Gamma)$. We can therefore assume $\rho = 1$. Then δ is the reciprocal of the volume of Γ. Our objective is to construct sphere packings with a high center density.

1.1 Coset codes

Let $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset \mathcal{A}_l$ be a chain of m-dimensional lattices, where the factor group $\mathcal{A}_{i-1}/\mathcal{A}_i$ is isomorphic to the abelian group A_i of order $a_i, i = 1, 2 \ldots, l$. Let further C_i be an a_i-ary code with M_i elements and minimum distance d_i. We choose representatives $\alpha_{ij}, j = 1, 1, \ldots, a_i$ for the cosets of $\mathcal{A}_i$ in $\mathcal{A}_{i-1}$. Choose $\alpha_{i1} = 0$. Put $A_i = \{\alpha_{ij}, j = 1, 2, \ldots, a_i\}$. Choose A_i as the alphabet over which the code C_i is defined. It is convenient and no loss of generality to assume that the all-0 word belongs to C_i. The $N = nm$-dimensional packing

$$\Gamma = \Gamma(\mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset \mathcal{A}_l; C_1, C_2, \ldots, C_l)$$

is defined as the union of $M_1 M_2 \dots M_l$ cosets of the sublattice $(\mathcal{A}_l)^n$. The cosets are parametrized by l-tupels of codewords $(v_1, v_2, \dots, v_l)$, where $v_i \in C_i$. Let $v_i = (v_{i1}, \dots, v_{in})$, where $v_{ij} \in A_i$. Then the coset $N(v_1, v_2, \dots, v_l)$ is defined as

$$N(v_1, v_2, \dots, v_l) = (\sum_{i=1}^{l} v_{ij})_{j=1}^{n} + (\mathcal{A}_l)^n.$$

Observe that $N(0, 0, \dots, 0) = (\mathcal{A}_l)^n$. It is clear that these cosets are distinct so that

$$vol(\Gamma) = \frac{vol(\mathcal{A}_l)^n}{M_1 \dots M_l}.$$

How about the minimal norm? Let $x, y \in \Gamma, x \neq y$. If x and y belong to the same coset, then their difference is in $(\mathcal{A}_l)^n$. It follows $\|x - y\| \geq \sqrt{\mu(\mathcal{A}_l)}$. So assume they are in different cosets. Let $x \in N(v_1, v_2, \dots, v_l), y \in N(v_1', v_2', \dots, v_l')$ and i minimal such that $v_i \neq v_i'$. As C_i has minimum distance d_i it follows that $x - y$ has in d_i of its n components an entry in $\mathcal{A}_{i-1} \backslash \mathcal{A}_i$. It follows $\|x - y\| \geq \sqrt{d_i \cdot \mu(\mathcal{A}_{i-1})}$.

1.2 The case $m = 1$

We have $\mathcal{A}_0 = \mathbb{Z}, \mathcal{A}_i = q_1 \dots q_i \mathbb{Z}$ $(i = 1, 2, \dots, l), \mu(\mathcal{A}_i) = (q_1 \dots q_i)^2$, thus

$$\mu(\Gamma) \geq Min\{d_1, d_2 q_1^2, \dots, d_l (q_1 \dots q_{l-1})^2, (q_1 q_2 \dots q_l)^2\}.$$

If we use linear codes $[n, k_i, d_i]_{q_i}$ we obtain

$$\delta(\Gamma) \geq \frac{1}{2^n} \prod_{i=1}^{l} q_i^{k_i - n} \cdot \mu(\Gamma)^{n/2}.$$

1.3 The case $m = 2$

Let $\mathcal{A}_0 = \langle (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle = \langle a_0, b_0 \rangle$ be the hexagonal lattice (as generated by root systems of types A_2 and G_2). The lattice $\mathcal{A}_0$ has volume $\frac{\sqrt{3}}{2}$ and minimum norm 1. The image $\mathcal{A}_{1,0}$ of $\mathcal{A}_0$ under the linear mapping with matrix $M = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ (with respect to basis a_0, b_0) has index 3 in $\mathcal{A}_0$, is generated by $a_0 + b_0$ and $-a_0 + 2b_0$ and has minimum distance $\|a_0 + b_0\| = \sqrt{3}$. As $a_0 + b_0$ and $-a_0 + 2b_0$ have the same length and include an angle of $\pi/3$

we see that $\mathcal{A}_{1,0}$ is similar to $\mathcal{A}_0$. Applying the same matrix repeatedly we get $\mathcal{A}_{j,0} = \mathcal{A}_0 M^j$, for instance $\mathcal{A}_{2,0} = 3\langle b_0, -a_0 + b_0\rangle = 3\mathcal{A}_0$. Aside of this operation we also consider sublattices of index 4 obtained by multiplication with the constant 2. This leads to the following Definition: $\mathcal{A}_{j,k} = 2^k \mathcal{A}_0 M^j$. It is clear that $vol(\mathcal{A}_{j,k}) = 4^k 3^j \frac{\sqrt{3}}{2}$ and $\mu(\mathcal{A}_{j,k}) = 4^k 3^j$. We apply the coset code construction with $\mathcal{A}_i = \mathcal{A}_{j(i),k(i)}$, where $j(i)+k(i) = i$ and either $\mathcal{A}_{i+1} = \mathcal{A}_{j(i)+1,k(i)}$ or $\mathcal{A}_{i+1} = \mathcal{A}_{j(i),k(i)+1}$, of index 3 or 4 in $\mathcal{A}_i$. We have

$$vol(\Gamma) = \frac{(2^{2k(l)-1} 3^{j(l)+1/2})^n}{\mid C_1 \mid \ldots \mid C_l \mid}$$

and

$$\mu(\Gamma) \geq Min\{\mu(\mathcal{A}_l); d_{i+1}\mu(\mathcal{A}_i), i = 0, 1, \ldots, l-1\}.$$

2 A variant of the coset code-construction

We use the following chain of 1-dimensional lattices: $\mathcal{A}_0 = \mathbb{Z} \supset \mathcal{A}_1 = 2\mathbb{Z} \supset \mathcal{A}_2 = 4\mathbb{Z} \supset \mathcal{A}_3 = 8\mathbb{Z}$ and the following codes: $C_1 = [n, 1, n]$ (the repetition code),$C_3 = C_1^\perp = [n, n-1, 2]$ and binary codes C_2, C_2' of length n, minimum distances $\geq d$ and $\geq d'$, respectively. Observe that C_2, C_2' are not required to be linear codes. As alphabets for our codes we use $A_1 = \{0, 1\}, A_2 = \{0, -2\}, A_3 = \{0, 4\}$. With this notation we define $\Gamma = \Gamma^*(\mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3; C_1, (C_2, C_2'), C_3)$ as the union of the following cosets of $(8\mathbb{Z})^n$ in $\mathbb{Z}^n$:

$N(\mathbf{0}, v_2, v_3)$,	where	$v_2 \in \mathbf{1} + C_2, v_3 \in C_3$	(vectors of **even type**)
$N(\mathbf{1}, v_2, v_3)$,	where	$v_2 \in C_2', v_3 \notin C_3$	(vectors of **odd type**)

Here $\mathbf{0}$ and $\mathbf{1}$ stand for the vectors of length n with all entries 0 and 1, respectively. It is clear that the addition of cosets if as follows:

$$\begin{aligned} N(0, v_2, v_3) + N(0, w_2, w_3) &= N(0, v_2 + w_2, v_3 + w_3 + v_2 \cap w_2) \\ N(0, v_2, v_3) + N(1, w_2, w_3) &= N(1, v_2 + w_2, v_3 + w_3 + v_2 \cap w_2) \\ N(1, v_2, v_3) + N(1, w_2, w_3) &= N(0, v_2 + w_2 + 1, v_3 + w_3 + v_2 \cup w_2 + 1) \end{aligned}$$

Let us determine the minimum Euclidean distance between different elements of Γ. Assume at first x, y are both of even type, $x \in N(0, v_2, v_3), y \in$

$N(0, w_2, w_3)$. If $v_2 \neq w_2$, then $\|x - y\| \geq 2\sqrt{d}$. If $v_2 = w_2, v_3 \neq w_3$, then $\|x - y\| \geq 2\sqrt{2} = \sqrt{32}$. If finally $v_2 = w_2, v_3 = w_3$, then $\|x - y\| \geq 8$. The same arguments apply if x and y are both of odd type. We just have to replace d by d'. Let finally $x \in N(1, v_2, v_3)$ be of odd type and $y \in N(0, w_2, w_3)$ of even type. All entries of $x - y$ are odd integers. We wish to impose conditions on C_2, C_2' ensuring that for at least one coordinate the entry of $x - y$ is $\pm 3 (\text{mod } 8)$. If this is the case, then $\|x - y\| \geq \sqrt{n - 1 + 9} = \sqrt{n + 8}$. Assume to the contrary all entries of $x - y$ are $\pm 1 (\text{mod } 8)$. Fix a coordinate. Consider the 16 possibilities of how it may be distributed on the vectors v_2, v_3, w_2, w_3. Eight of these are excluded as they lead to a difference $\pm 3 (\text{mod } 8)$. Write $v_2 = 1 + u_2$, where $u_2 \in C_2$. The eight remaining cases are the following:

u_2	v_3	w_2	w_3	$N(1, v_2, v_3) - N(0, w_2, w_3)$
1	0	0	0	1-0=1
1	0	1	1	1-2=-1
1	1	0	1	5-4=1
1	1	1	0	5-(-2)=-1
0	0	0	0	-1-0=-1
0	0	1	0	-1-(-2)=1
0	1	0	1	3-4=-1
0	1	1	1	3-2=1

Here the entry in the last column is to be taken as an integer mod 8, whereas the entries in the first four columns are 1 or 0. As an example consider the second row of this table: as $u_2 = 1$ (equivalently $v_2 = 0$) and $v_3 = 0$, the entry in $N(1, v_2, v_3)$ is 1+0+0=1. As $w_2 = w_3 = 1$, the entry in $N(0, w_2, w_3)$ is -2+4=2. This explains the last entry $1 - 2 = -1$ (mod 8).

This table shows $v_3 + w_3 = w_2 \cap u_2$ (here $v + w$ is the symmetric difference of v) Observe that $v_3 + w_3$ has odd weight. We will get the desired contradiction if $w_2 \cap u_2$ is even, equivalently if C_2 and C_2' are orthogonal codes.

Theorem 1 *Let C_2, C_2' be binary codes of length n and minimum distances d, d', respectively, which are orthogonal to each other. Then the n-dimensional sphere packing*

$$\Gamma = \Gamma^*(\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z}; [n, 1, n], (C_2, C_2'), [n, n - 1, 2])$$

has minimum Euclidean distance $min\{2\sqrt{d}, 2\sqrt{d'}, \sqrt{32}, \sqrt{n + 8}\}$ *and volume* $vol(\Gamma) = 2^{2n+1}/\{|C_2| + |C_2'|\}$. *If $C_2 = C_2'$ is a self-orthogonal linear code containing the all-1 vector, then Γ is a lattice.*

Proof: The statements concerning the minimum Euclidean distance and volume are by now obvious. Γ is a lattice if and only if the cosets it consists of form a subgroup of $(\mathbb{Z})^n/(8\mathbb{Z})^n$. The last claim follows from the addition rules given earlier. ■

This method yields the densest known packings in dimensions $18, 20, 24, 27, 28, 29, 30$. In each case C_2, C_2' are an orthogonal pair of linear codes with the same parameters. These parameters are

$$[18, 9, 6], [20, 9, 7], [24, 12, 8], [27, 13, 8], [28, 14, 8], [29, 14, 8], [30, 15, 8].$$

Only in dimension 24 can we choose $C_2 = C_2'$. This is the extended binary Golay code and we obtain a construction of the famous Leech lattice. All the other packings are non-lattice packings. The orthogonal pair with parameters $[20, 9, 7]$ may be derived from the extended Golay code G : choose C_2 to be the subcode vanishing in the first three coordinates, projected to the last 20 coordinates, and C_2' the subcode vanishing at coordinates 1,2 and 4, also projected to the last 20 coordinates. The orthogonal pair in dimension 18 can be chosen as extended quadratic residue codes.

3 Recursive constructions

The following are relatively straightforward recursive constructions.

Lemma 1 *If there are packings of center densities δ_N, δ_j in dimensions N and j, then there is an $(N+j)-$dimensional packing of center density $\delta_1\delta_2$.*

Proof: Let Γ_1, Γ_2 be the packings whose existence is assumed above. We can choose the minimum distance of both packings to be $= 2$. The $(N+j)-$dimensional packing $\Gamma_1 \oplus \Gamma_2$ still has minimum Euclidean distance 2, hence $\delta(\Gamma_1 \oplus \Gamma_2) = vol(\Gamma_1 \oplus \Gamma_2)^{-1} = \delta_1\delta_2$. ■

The following Theorem may be proved along the lines of [5], page 167:

Theorem 2 (Mordell's inequality) *Let $\Gamma \subset \mathbb{R}^n$ be an n-dimensional lattice of center density δ, not less dense than its dual Γ^*. Let $0 \neq x \in \Gamma^*$ be a vector of minimum norm. Then $\langle x\rangle^\perp \cap \Gamma$ is an $(n-1)$-dimensional lattice of center density $\geq \frac{1}{2}\delta^{(n-2)/n}$.*

4 Some packings in high dimensions

We note that in a number of dimensions use of new codes constructed by us in [1, 2, 3, 4] as ingredients in the coset-codes construction yields packings, which are denser that what can be derived from known packings via Lemma 1 or Theorem 2. The new codes used in these constructions can be derived from the following codes: $[144, 51, 32]_2, [140, 50, 32]_2, [155, 132, 8]_2, [162, 138, 8]_2,$ $[86, 77, 5]_3, [85, 74, 6]_3, [86, 54, 14]_3$. Naturally it has to be expected that more sophisticated constructions will yield improvements in all these cases. Still it is noteworthy that the coset-code construction in its simplest form is capable of producing dense packings in low dimensions as well as in rather high dimensions. We conclude with a couple of examples.

In dimension 110 case $m = 2$ of the coset-code construction applied to ternary codes $[55, 1, 54]_3, [55, 25, 18]_3, [55, 44, 6]_3$, and $[55, 54, 2]_3$ yields density $3^{41.5}$. In dimension 170 we can use ternary codes $[85, 16, 42]_3, [85, 53, 14]_3,$ $[85, 76, 5]_3$ and $[85, 84, 2]_3$ and obtain density $7^{85}/3^{68.5}$. In dimension 140 we can apply case $m = 1$ of the coset-code method. Binary codes $[140, 1, 128]_2,$ $[140, 50, 32]_2, [140, 117, 8]_2$ and $[140, 139, 2]_2$ yield a packing of density 2^{97}.

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