# Lengthening and the Gilbert-Varshamov bound

Yves Edel and Jürgen Bierbrauer

## Abstract

We use lengthening and an enhanced version of the Gilbert-Varshamov lower bound for linear codes to construct a large number of record-breaking codes. Our main theorem may be seen as a closure operation on data bases.

### Index Terms

Linear codes, lengthening, Gilbert Varshamov-bound.

## 1 Introduction

[1] Let $q$ be a prime-power, which will be fixed throughout the discussion. Denote by $\mathbb{F}_q$ the field of $q$ elements and by $V(n,i)$ the number of vectors of weight at most $i$ in $\mathbb{F}_q^n$. It is clear that

$$V(n,i)=\sum_{j=0}^{i}\binom{n}{j}(q-1)^j. \tag{1}$$

Let $\mathcal{C}$ be a $q$-ary code with parameters $[n,k-1,d]$. As $\mathcal{C}$ has $q^{k-1}$ elements it follows that if $q^{k-1}V(n,d-1)<q^n$, then there is a vector $v\in\mathbb{F}_q^n$, which has distance $\geq d$ from every code-word $\in\mathcal{C}$. This leads to the Gilbert-Varshamov bound:

[1]authors'addresses: Yves Edel, Mathematisches Institut der Universität, Im Neuenheimer Feld 288, 69120 Heidelberg (Germany); Jürgen Bierbrauer, Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931 (USA)

**Theorem 1 (Gilbert-Varshamov bound)** *If $V(n, d-1) < q^{n-k+1}$, then a $q$-ary linear code with parameters $[n, k, d]$ exists.*

Using orthogonal arrays the following can be proved.

**Theorem 2** *If $V(n-1, d-2) < q^{n-k}$, then a $q$-ary linear code with parameters $[n, k, d]$ exists. Moreover every code $[n-1, k-1, d]$ can be embedded in a code $[n, k, d]$.*

This is to be found in the book by Mac Williams and Sloane ([3], page 34). For the sake of completeness we shall give a proof in the final section. It is easy to see that this is always stronger than the Gilbert-Varshamov bound. Combining Theorem 2 with the method of lengthening yields new codes:

**Theorem 3** *Assume $V(n-1, d-2) < q^{n-k}$. If there exist codes $[n-i, k-i, d+\delta]$ and $[e, i, \delta]$, then a code $[n+e, k, d+\delta]$ can be constructed.*

A proof of Theorem 3 will be given in the following section. It should be noted that Theorem 3 uses only the code parameters. No information on subcodes is needed. We like to think of it as of a closure operation on data bases. In order to illustrate its use we give a binary example: a code $\mathcal{D}$ with parameters $[126, 36, 34]$ is known to exist. It can be derived from a $[128, 36, 36]$ constructed in [4]. As $V(126, 26) < 2^{90}$ it follows from Theorem 2 that $\mathcal{D}$ can be embedded in a code $\mathcal{C}$ with parameters $[127, 37, 28]$. Applying construction X to the pair $\mathcal{C} \supset \mathcal{D}$ with $[6, 1, 6]$ as auxiliary code yields the new code $[133, 37, 34]$.
In Table 1 we list some more applications of Theorem 3. In all cases $i = 1$, so that the auxiliary code is the repetition code $[e, i, \delta] = [\delta, 1, \delta]$. The following parameters are given:

- $q \in \{2, 3, 4\}$,
- the parameters $[n-1, k-1, d+\delta]$ of the known code $\mathcal{D}$,
- $\delta$,
- the parameters $[n, k, d+\delta]$ of the resulting code $\mathcal{E}$.

It is easy to write a program which operates on any given data base and produces the closure of the data base under Theorem 3. All in all Theorem 3 leads to hundreds of improvements in the present version of the data base.

Table 1:

| $q$ | $\mathcal{D}$ | $\delta$ | $\mathcal{E}$ |
|---|---|---|---|
| 2 | [123,29,39] | 8 | [132,30,39] |
| 2 | [126,29,42] | 10 | [137,30,42] |
| 2 | [135,29,45] | 10 | [146,30,45] |
| 2 | [197,65,41] | 3 | [201,66,41] |
| 2 | [206,96,31] | 3 | [210,97,31] |
| 3 | [40,24,9] | 2 | [43,25,9] |
| 3 | [43,24,10] | 2 | [46,25,10] |
| 3 | [52,13,22] | 3 | [56,14,22] |
| 3 | [59,32,13] | 2 | [62,33,13] |
| 3 | [64,17,24] | 2 | [67,18,24] |
| 3 | [65,16,25] | 2 | [68,17,25] |
| 3 | [81,16,41] | 10 | [92,17,41] |
| 3 | [83,16,42] | 10 | [94,17,42] |
| 4 | [44,22,14] | 3 | [48,23,14] |
| 4 | [40,14,15] | 1 | [42,15,15] |
| 4 | [42,14,17] | 2 | [45,15,17] |
| 4 | [59,27,17] | 2 | [62,28,17] |
| 4 | [63,27,21] | 4 | [68,28,21] |
| 4 | [65,27,23] | 5 | [71,28,23] |

# 2 Proofs

Let $\mathcal{A}$ be a linear subspace of dimension $n-k$ of $\mathbb{F}_q^{n-1}$, which is an orthogonal array of strength $t$, and let $A$ be a generator matrix of $\mathcal{A}$. We wish to add an additional column to $A$ such that the resulting subspace of $\mathbb{F}_q^n$ still is an orthogonal array of strength $t$. The columns which do not do the job are precisely those vectors in $\mathbb{F}_q^{n-k}$, which can be written as linear combinations of at most $t-1$ columns of $A$. The number of such linear combinations is at most $\sum_{i=0}^{t-1}\binom{n-1}{i}(q-1)^i$. This number happens to equal $V(n-1,t-1)$. Thus, if $V(n-1,t-1) < q^{n-k}$, then our orthogonal array can be extended in the required manner. By Delsarte theory a linear subspace of $\mathbb{F}_q^n$ is an orthogonal array of strength $t$ if and only if its dual has minimum distance $\geq t+1$. Considering duals we see that we have proved the following: if there is a code $\mathcal{C}$ with parameters $[n-1,k-1,d]$ and if $V(n-1,d-2) < q^{n-k}$, then $\mathcal{C}$ can be extended to a code $[n,k,d]$. Just as in the case of the Gilbert-Varshamov bound it is easy to see by induction that the condition of the existence of an $[n-1,k-1,d]$ is not needed. Theorem 2 is proved.
In order to show that Theorem 2 is always better than Theorem 1 it suffices to show the inequality

$$qV(n-1,d-2) < V(n,d-1). \tag{2}$$

In fact, consider the $V(n-1,d-2)$ vectors of length $n-1$ and weight $\leq d-2$. Adding a coordinate and extending each of these vectors in all $q$ possible ways yields $qV(n-1,d-2)$ different (but obviously not all) vectors of length $n$ and weight $\leq d-1$. This proves our last claim concerning Theorem 2.
Consider Theorem 3: we use a basic fact on lengthening known as construction X ([3], see also [1]):

**Lemma 1 (construction X)** *Let $\mathcal{C}$ be a $q$-ary code with parameters $[n,k,d]$ and $\mathcal{D}$ a subcode of $\mathcal{C}$ of codimension $\kappa$ and minimum distance $\geq d+\delta$ for some $\delta > 0$. If there is a code with parameters $[e,\kappa,\delta]$ then there is a code $[n+e,k,d+\delta]$ which projects onto $\mathcal{C}$.*

The assumptions of Theorem 3 show that the code $[n-i,k-i,d+\delta]$ can be embedded in a code $[n,k,d]$. Application of construction X to this pair of codes leads to the conclusion of Theorem 3.

## References

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