Higher Class Field Theory

Oberseminar im Sommersemester 2011

Content

The aim of higher class field theory is to describe the abelian fundamental group of an arithmetic scheme using a suitable, arithmetically defined class group $C(X)$. In the case $\dim(X) = 1$ this is achieved as follows:

Let K be a number field and S a finite set of places containing all infinite places. We denote the complement of S in the spectrum of the ring of integers of K by X. The class group $C(X)$ is the restricted idele class group

$$C(X) = \operatorname{coker}\big(K^\times \longrightarrow \bigoplus_{x\in X^0} \mathbb{Z} \oplus \bigoplus_{v\in S} K_v^\times\big).$$

The main theorem of global class field theory describes in this case a reciprocity homomorphism $\rho\colon C(X) \longrightarrow \pi_1^{ab}(X)$, inducing a bijection between the open subgroups of $C(X)$ and those of $\pi_1^{ab}(X)$.

In our seminar we follow the approach of G. Wiesend to generalise this construction of the class group and the reciprocity homomorphism to higher dimensional schemes. In contrast to the earlier approaches of Parshin, Kato and Saito, the construction of the class group $C(X)$ is of more elementary nature and takes as input only data associated to points and curves on X.

Instead of the partly inaccurate original works [Wie06], [Wie07], and [Wie08] we take the well readable article [KS09] as basis for our seminar. A survey can be found in [Sza09a].

Time and Space

Thursday 3:30 p. m. – 5:00 p. m. ; INF 288 / MathI HS 1

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Talks

Talk 1: Introduction and distribution of talks; elementary fibrations.
(14.04.11)

After a short introduction we distribute the talks. Immediately after, we will treat elementary and special fibrations according to M. Artin, including the proof of [AGV72, XI, Prop. 3.3] and of [KS09, Lemma 1.1, 1.2].

Talk 2: Bloch's approximation lemma
(21.04.11)

For the proof of Bloch's approximation lemma [KS09, 1.5] we follow [Ker10, §2]. Probably, we can only cite the Bertini theorem [Poo04]. We cite as well the Chebotarev theorem [KS09, Prop. 1.3, 1.4] (A complete proof can be found in [Pin97, Theorem B.9]).

Talk 3: Tamely ramified coverings
(28.04.11)

Definition and first properties of tamely ramified coverings and of the tame fundamental group [KS09, §2 up to Remark 2.5], including the proof of the key lemma [KS10, Lemma 2.4]. We must cite an assertion on the purity of the branch locus [Gro05, X.3.4].

Talk 4: The finiteness theorem of Katz-Lang
(05.05.11)

We need [KL81, Theorem 1,2]. To save time, we should concentrate on the version [Sza09b, Theorem 5.8.14] for the proof.

Talk 5: Finiteness theorems for tame fundamental groups; good curves
(12.05.11)

Building upon the preceding talk, prove [KS09, Thm. 2.7 - 2.9]. For this, we need to cite [dJ96].

Talk 6: Covering data
(19.05.11)

Introduction of the term covering data; trivialisation and effectivity [KS09, §3, §4]. From [Gro03, XIII] we cite the specialisation isomorphism and the assertion that the tame fundamental group of an algebraic curve is finitely generated (a readable proof can also be found in [OV00]).

Talk 7: Abelian covering data
(26.05.11)

Proof of [KS09, Theorem 5.1]: Abelian covering data are effective. The proof uses an assertion from [Gro03, XIII], stating that a certain sheaf is locally constant constructible.

Talk 8: Wiesend's class group
(09.06.11)

After some topological preparations, we can introduce the class group [KS09, §6, §7].

Talk 9: The main theorems: arithmetic case
(16.06.11)

Proof of [KS09, Theorem 8.1].

Talk 10: The main theorems: geometric case; applications
(30.06.11)

Proof of [KS09, Theorem 8.2, 8.3]. In the remaining time we can go into [KS09, §9].

Literatur

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