NONCOMMUTATIVE L-FUNCTIONS FOR p-ADIC REPRESENTATIONS OVER TOTALLY REAL FIELDS

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Abstract. We prove a unicity result for the L-functions appearing in the noncommutative Iwasawa main conjecture over totally real fields. We then consider continuous representations ρ of the absolute Galois group of a totally real field F on adic rings in the sense of Fukaya and Kato. Using our unicity result, we show that there exists a unique sensible definition of a noncommutative L-function for any such ρ that factors through the Galois group of a possibly infinite totally real extension.

1. Introduction

Let F_∞/F be an (in the sense of [Kak13]) admissible p-adic Lie extension of a totally real field F that is unramified over the open dense subscheme U of the spectrum X of the algebraic integers of F and write $G = \mathrm{Gal}(F_\infty/F)$ for its Galois group. We further assume that p is invertible on U. The noncommutative main conjecture of Iwasawa theory for F_∞/F predicts the existence of a noncommutative p-adic L-function $\mathcal{L}_{F_\infty/F}(U,\mathbb{Z}_p(1))$ living in the first algebraic K-group $\mathrm{K}_1(\mathbb{Z}_p[[G]]_S)$ of the localisation at Venjakob's canonical Ore set S of the profinite group ring

$$\mathbb{Z}_p[[G]] = \varprojlim_{\substack{H \lhd G \\ \text{open}}} \mathbb{Z}_p[[G/H]].$$

This L-function is supposed to satisfy the following two properties:

(1) It is a characteristic element for the total complex $\mathrm{R}\,\Gamma_c(U, f_! f^* \mathbb{Z}_p(1))$ of étale cohomology with proper support with values in the sheaf $f_! f^* \mathbb{Z}_p(1)$.
(2) It interpolates the values of the complex L-functions $L_{X-U}(\rho, s)$ for all Artin representations ρ factoring through G.

We refer to Theorem 12.1 for a more precise formulation.

Under the assumption that

(a) $p \neq 2$,
(b) the Iwasawa μ-invariant of any totally real field is zero,

the noncommutative main conjecture is now a theorem, first proved by Ritter and Weiss [RW11]. Almost simultaneously, Kakde [Kak13] published an alternative proof, building upon unpublished work of Kato and the seminal article [Bur15] by Burns. The book [CSSV13] is a comprehensive introduction to Kakde's work. The vanishing of the μ-invariant is still an open conjecture. We refer to [Mih16] for a recent attempt to settle it.

It turns out that properties (1) and (2) are not sufficient to guarantee the uniqueness of $\mathcal{L}_{F_\infty/F}(U,\mathbb{Z}_p(1))$. It is only well-determined up to an element of the subgroup

$$\mathrm{SK}_1(\mathbb{Z}_p[[G]]) \subset \mathrm{K}_1(\mathbb{Z}_p[[G]]_S).$$

The first objective of this article is to eradicate this indeterminacy. Under the assumptions (a) and (b) we show in Theorem 12.2 that if one lets F_∞ vary over

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all admissible extensions of F and requires a natural compatibility property for the elements $\mathcal{L}_{F_\infty/F}(U,\mathbb{Z}_p(1))$, there is indeed a unique choice of such a family.

In the course of their formulation of a very general version of the equivariant Tamagawa number conjecture, Fukaya and Kato introduced in [FK06] a certain class of coefficient rings which we call adic rings for short. This class includes among others all finite rings, the Iwasawa algebras of p-adic Lie groups and power series rings in a finite number of indeterminates. Our second objective concerns continuous representations ρ of the absolute Galois group Gal_F over some adic $\mathbb{Z}_p$-algebra Λ. Assume that ρ factors through the Galois group of some (possibly infinite) totally real extension of F with finite ramification locus. As a consequence of Theorem 12.2, we show in Theorem 12.4 and Corollary 12.8 that there exists a unique sensible assignment of a noncommutative L-function

$$\mathcal{L}_{F_\infty/F}(U,\rho(1))\in \mathrm{K}_1(\Lambda[[G]]_S)$$

to any such ρ. For this purpose, we also introduce and study noncommutative Euler factors. In the sequel [Wit] to the present article, we will use our result to prove the existence of the ζ-isomorphism for ρ as predicted by Fukaya's and Kato's central conjecture [FK06, Conj. 2.3.2].

In fact, Theorem 12.4 applies more generally to perfect complexes $\mathcal{F}^\bullet$ of Λ-adic sheaves on U which are "smooth at ∞". Moreover, in order to include noncommutative versions of the type of main conjectures treated in [GP15], we also consider the total derived pushforward $\mathrm{R}\,k_*\mathcal{F}^\bullet$ for $k\colon U\to W$ the open immersion into another dense open subscheme W of X. The extension F_∞/F may be ramified over $W-U$, but places over p remain excluded from W.

We give a short overview on the content of the article. In Section 2 we recall some of the K-theoretic machinery behind the formulation of the main conjecture. Section 4 contains an investigation of the base change properties of certain splittings of the boundary map

$$\partial\colon \mathrm{K}_1(\mathbb{Z}_p[[G]]_S)\to \mathrm{K}_0(\mathbb{Z}_p[[G]],\mathbb{Z}_p[[G]]_S),$$

extending results from [Bur09] and [Wit13b]. With the help of these splittings we are able produce characteristic elements with good functorial properties, which we call noncommutative algebraic L-functions. A central input to the proof of Theorem 12.2 is Section 5, in which we show that $\mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$ vanishes for sufficiently large extensions F_∞/F. The results of this section apply not only to admissible extensions and might be useful in other contexts as well. Section 6 contains the definition of a Waldhausen category modeling the derived category of perfect complexes of Λ-adic sheaves and the explanation of the property of being smooth at ∞. In Section 7 we recall the notion of admissible extensions and the definition of the complexes $f_!f^*\mathcal{F}^\bullet$ induced by the procovering $f\colon U_{F_\infty}\to U$. We then show in Section 8 that our hypothesis (b) on the vanishing of the μ-invariant implies that for any perfect complex of Λ-adic sheaves $\mathcal{F}^\bullet$ smooth at ∞ and any admissible extension F_∞/F, the complexes $\mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ have S-torsion cohomology. We also prove a unconditional local variant thereof. This local variant permits us to introduce the notion of noncommutative Euler factors by producing canonical characteristic elements for the complexes $\mathrm{R}\Gamma(x,i^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ for any closed point $i\colon x\to W$ of W. Comparing Euler factors with the noncommutative algebraic L-functions of these complexes, we obtain certain elements in $\mathrm{K}_1(\Lambda[[G]])$, which we call local modification factors. The investigation of the Euler factors and local modification factors is carried out in Section 9 and Section 10, first in general, then in the special case of the cyclotomic extension. This is followed by a short reminder on L-functions of Artin representations in Section 11.

Section 12 contains the main results of this article. We use Kakde's noncommutative L-functions and the noncommutative algebraic L-function of the complex $\mathrm{R}\,\Gamma_c(U, f_! f^* \mathbb{Z}_p(1))$ to define global modification factors. Changing the open dense subscheme U is reflected by adding or removing local modification factors. This compatibility allows us to pass to field extensions with arbitrary large ramification. We can then use the results of Section 5 to prove the uniqueness of the family of modification factors for all pairs (U, F_∞) with F_∞/F admissible and unramified over U. The corresponding noncommutative L-functions are the product of the global modification factors and the noncommutative algebraic L-functions. We then extend the definition of global modification factors and noncommutative L-functions to perfect complexes $\mathcal{F}^\bullet$ of Λ-adic sheaves smooth at ∞ by requiring a compatibility under twists with certain bimodules. In Theorem 12.6 we show that the global modification factors are also compatible under changes of the base field F and Corollary 12.8 subsumes the transformation properties of the noncommutative L-functions for the complexes $\mathcal{F}^\bullet$.

The findings of this article were inspired by the author's analogous results [Wit16] on geometric main conjectures in the $\ell = p$ case. Parts of it were developed during his stay at the University of Paderborn in the academic year 2014. He thanks the mathematical faculty and especially Torsten Wedhorn for the hospitality. This work is part of a larger project of the research group *Symmetry, Geometry, and Arithmetics* funded by the DFG.

2. K-theory of adic rings

All rings in this article will be associative with identity; a module over a ring will always refer to a left unitary module. For the convenience of the reader, we repeat the essentials of the K-theoretic framework introduced in [Wit14].

The formulation of the noncommutative Iwasawa main conjecture involves certain K-groups. We are mainly interested in the first K-group of a certain class of rings introduced by Fukaya and Kato in [FK06]. It consists of those rings Λ such that for each $n \geq 1$ the n-th power of the Jacobson radical $\mathrm{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_{n \geq 1} \Lambda / \mathrm{Jac}(\Lambda)^n.$$

We will call these rings *adic rings* as in [Wit14]. Hopefully, this will not lead to confusion, as this term is now widely used in a slightly different sense by the school of Scholze. By definition, an adic ring Λ carries a natural profinite topology. We will write $\mathfrak{I}_\Lambda$ for the set of open two-sided ideals of Λ, partially ordered by inclusion.

Classically, the first K-group of Λ may be described as the quotient of the group

$$\mathrm{Gl}_\infty(\Lambda) \coloneqq \varinjlim_n \mathrm{Gl}_n(\Lambda)$$

by its commutator subgroup, but for the formulation of the main conjecture, it is more convenient to follow the constructions of higher K-theory. Among the many roads to higher K-theory, Waldhausen's S-construction turns out to be particularly well-suited for our purposes.

To construct the K-groups of Λ, one can simply apply the S-construction to the category of finitely generated, projective modules over Λ, but the true beauty of Waldhausen's construction is that we can choose among a multitude of different Waldhausen categories that all give rise to the same K-groups. Our choice below, taken from [Wit14], was designed in order to make all necessary constructions as natural as possible.

We recall that for any ring R, a complex $M^\bullet$ of R-modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of R^{op}-modules, the total complex $(N \otimes_R M)^\bullet$ is acyclic. In particular, any bounded above complex of flat R-modules is DG-flat. The notion of DG-flatness can be used to define derived tensor products without this boundedness condition. As usual, the complex $M^\bullet$ is called *strictly perfect* if M^n is finitely generated and projective for all n and $M^n = 0$ for almost all n. A complex of R-modules is a *perfect* complex if it is quasi-isomorphic to a strictly perfect complex.

Definition 2.1. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ satisfying the following conditions:

(1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat perfect complex of Λ/I-modules,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$$

induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I : P_I^\bullet \to Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective and the system $(\operatorname{coker} f_I)$ is in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

Definition 2.2. Let Λ' be another adic ring and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules. We define $\Psi_{M^\bullet}$ to be the following Waldhausen exact functor

$$\Psi_{M^\bullet} : \mathbf{PDG}^{\mathrm{cont}}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'), \qquad P^\bullet \to (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet)_{I \in \mathfrak{I}_{\Lambda'}}.$$

By [Wit14, Prop. 3.7] the K-groups $\mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ of the Waldhausen category coincide with the Quillen K-groups $\mathrm{K}_n(\Lambda)$ of the adic ring Λ and the homomorphism

$$\Psi_{M^\bullet} : \mathrm{K}_n(\Lambda) \to \mathrm{K}_n(\Lambda')$$

induced by the Waldhausen exact functor $\Psi_{M^\bullet}$ coincides with the homomorphism induced by the derived tensor product with $M^\bullet$. The essential point in this observation is that $\mathfrak{I}_\Lambda$ is a countable set and that all the transition maps φ_{IJ} are surjective such that passing to the projective limit

$$\varprojlim_{I \in \mathfrak{I}_\Lambda} P_I^\bullet$$

is a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ to the Waldhausen category of perfect complexes of Λ-modules. We write

$$\mathrm{H}^i((P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) \coloneqq \mathrm{H}^i(\varprojlim_{I \in \mathfrak{I}_\Lambda} P_I^\bullet)$$

for its cohomology groups and note that

$$\mathrm{H}^i((P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) = \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{H}^i(P_I^\bullet)$$

[Wit08, Prop. 5.2.3].

We will also need to consider localisations of certain adic rings: Note that for any adic $\mathbb{Z}_p$-algebra Λ and any profinite group G such that G has an open pro-p-subgroup which is topologically finitely generated, the profinite group algebra $\Lambda[[G]]$ is again an adic ring. Assume further that $G = H \rtimes \Gamma$ is the semidirect product of a closed normal subgroup H which is itself topologically finitely generated

and a subgroup Γ which is isomorphic to $\mathbb{Z}_p$. If Λ is commutative and noetherian and H is a p-adic Lie group then $\Lambda[[G]]$ is noetherian and the set
(2.1)

$$S \coloneqq \{f \in \Lambda[[G]] \mid \Lambda[[G]] \xrightarrow{\cdot f} \Lambda[[G]] \text{ is perfect as complex of } \Lambda[[H]]\text{-modules}\}$$

is a left denominator set in $\Lambda[[G]]$ in the sense of [GW04, Ch. 10] such that the localisation $\Lambda[[G]]_S$ exists [CFK+05, Thm. 2.4], [Wit13b, Cor. 3.4]. For general Λ and H, this is no longer true, as the following example shows.

Example 2.3. Assume that either $\Lambda = \mathbb{F}_p$ is the finite field with p elements and H is the free pro-p group on two generators with trivial action of Γ or $\Lambda = \mathbb{F}_p\langle\langle x, y\rangle\rangle$ is the power series ring in two non-commuting indeterminates x, y and H is trivial. In both cases, $\Lambda[[G]] = \mathbb{F}_p\langle\langle x, y\rangle\rangle[[t]]$ is the power series ring over $\mathbb{F}_p\langle\langle x, y\rangle\rangle$ with t commuting with x and y and the set S is the set of those power series $f(x, y, t)$ with $f(0, 0, t) \neq 0$. Set $s \coloneqq x - t \in S$. If S was a left denominator set, we could find

$$a \coloneqq \sum_{i=0}^{\infty} a_i t^i \in \mathbb{F}_p\langle\langle x, y\rangle\rangle[[t]], \quad b \coloneqq \sum_{i=0}^{\infty} b_i t^i \in S$$

such that $as = by$, i. e.

$$a_0 x = b_0 y, \qquad a_i x - a_{i-1} = b_i y \quad \text{for } i > 0.$$

The only solution for this equation is $a = b = 0$, which contradicts the assumption $b \in S$.

Nevertheless, using Waldhausen K-theory, we can still give a sensible definition of $\mathrm{K}_1(\Lambda[[G]]_S)$ even if $\Lambda[[G]]_S$ does not exist.

Definition 2.4. We write $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ of objects $(P^\bullet_J)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ such that

$$\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} P^\bullet_J$$

is a perfect complex of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

We may then identify for all $n \geq 0$

$$\mathrm{K}_n(\Lambda[[G]], \Lambda[[G]]_S) \coloneqq \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])),$$
$$\mathrm{K}_{n+1}(\Lambda[[G]]_S) \coloneqq \mathrm{K}_{n+1}(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

[Wit14, § 4].

Thanks to a result of Muro and Tonks [MT08], the groups K_0 and K_1 of any Waldhausen category can be described as the cokernel and kernel of a homomorphism between two groups that are given by explicit generators and relations in terms of the structure of the underlying Waldhausen category. In particular, $\mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S)$ is the abelian group generated by the symbols $[P^\bullet]$ with $P^\bullet$ an object in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ modulo the relations

$$[P^\bullet] = [Q^\bullet] \qquad \text{if } P^\bullet \text{ and } Q^\bullet \text{ are weakly equivalent,}$$
$$[P^\bullet_2] = [P^\bullet_1] + [P^\bullet_3] \qquad \text{if } 0 \to P^\bullet_1 \to P^\bullet_2 \to P^\bullet_3 \to 0 \text{ is an exact sequence.}$$

If $f\colon P^\bullet \to P^\bullet$ is an endomorphism which is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ or in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, we can assign to it a class $[f]$ in $\mathrm{K}_1(\Lambda[[G]])$ or $\mathrm{K}_1(\Lambda[[G]]_S)$, respectively. By the classical definition of the first K-group as factor group of the general linear group it is clear that these classes generate $\mathrm{K}_1(\Lambda[[G]])$ and if $\Lambda[[G]]$ is noetherian, also $\mathrm{K}_1(\Lambda[[G]]_S)$. The relations that are satisfied by these generators can be read off from the description of Muro and Tonks, see also [Wit14, Def. A.4].

Let Λ and Λ' be two adic $\mathbb{Z}_p$-algebras and $G = H \rtimes \Gamma$, $G = H' \rtimes \Gamma'$ be profinite groups, such that H and H' contain open, topologically finitely generated pro-p subgroups and $\Gamma \cong \mathbb{Z}_p \cong \Gamma'$. Suppose that $K^\bullet$ is a complex of $\Lambda'[[G']]$-$\Lambda[[G]]$-bimodules, strictly perfect as complex of $\Lambda'[[G']]$-modules and assume that there exists a complex $L^\bullet$ of $\Lambda'[[H']]$-$\Lambda[[H]]$-bimodules, strictly perfect as complex of $\Lambda'[[H']]$-modules, and a quasi-isomorphism of complexes of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet.$$

Here,

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \coloneqq \varprojlim_{I \in \mathfrak{I}_{\Lambda'[[G']]}} \varprojlim_{J \in \mathfrak{I}\Lambda[[G]]} L/IL^\bullet \otimes_{\Lambda[[H]]} \Lambda[[G]]/J$$

denotes the *completed tensor product*.

In the above situation, the Waldhausen exact functor

$$\Psi_{K^\bullet} \colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']]) \tag{2.2}$$

takes objects of the category $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ to objects of the category $\mathbf{PDG}^{\mathrm{cont},w_{H'}}(\Lambda'[[G']])$ and weak equivalences of $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ to weak equivalences of $w_{H'}\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']])$ [Wit14, Prop. 4.6]. Hence, it also induces homomorphisms between the corresponding K-groups. In particular, this applies to the following examples:

Example 2.5. [Wit14, Prop. 4.7]

1. Assume $G = G'$, $H = H'$. For any complex $P^\bullet$ of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, let $K^\bullet$ be the complex $P[[G]]^{\delta^\bullet} \coloneqq \Lambda'[[G]] \otimes_{\Lambda'} P^\bullet$ of $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodules with the right G-operation given by the diagonal action on both factors. This applies in particular for any complex $P^\bullet$ of Λ'-Λ-bimodules, strictly perfect as complex of Λ'-modules and equipped with the trivial G-operation.
2. Assume $\Lambda = \Lambda'$. Let $\alpha \colon G \to G'$ be a continuous homomorphism such that α maps H to H' and induces a bijection of G/H and G'/H'. Let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$.
3. Assume that G' is an open subgroup of G and set $H' \coloneqq H \cap G'$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the complex concentrated in degree 0 given by the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]]$.

Example 2.6. The assumptions in Example 2.5.(2) are in fact stronger than necessary. We may combine it with the following result. Assume that G is an open subgroup of G' such that $H \coloneqq H' \cap G = H'$ and $\Gamma = (\Gamma')^{p^n}$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$. Fix a topological generator $\gamma' \in \Gamma'$ and let $L^\bullet$ be the $\Lambda[[H]]$-$\Lambda[[H]]$-subbimodule of $\Lambda[[G']]$ generated as left $\Lambda[[H]]$-module by $1, \gamma', (\gamma')^2, \dots, (\gamma')^{p^n-1}$. Then $L^\bullet$ is a strictly perfect complex of $\Lambda[[H]]$-modules concentrated in degree 0 and the canonical map

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet, \qquad \ell \otimes \lambda \mapsto \ell\lambda$$

is an isomorphism of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules such that [Wit14, Prop. 4.6] applies. In combination with Example 2.5.(2) this implies that any continuous group homomorphism $\alpha \colon G \to G'$ such that $\alpha(G) \notin H'$ induces Waldhausen exact functors between all three variants of the above Waldhausen categories.

Example 2.7. As a special case of Example 2.5.(1), assume that $\Lambda = \mathbb{Z}_p$ and that ρ is some continuous representation of G on a finitely generated and projective Λ'-module. Let P_ρ be the Λ'-$\mathbb{Z}_p[[G]]$-bimodule which agrees with ρ as Λ'-module, but on which $g \in G$ acts from the right by the left operation of g^{-1} on ρ. We thus obtain Waldhausen exact functors

$$\Phi_\rho \coloneqq \Psi_{\Lambda'[[\Gamma]]} \circ \Psi_{P_\rho[[G]]^\delta} \tag{2.3}$$

from all three variants of the Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_p[[G]])$ to the corresponding variant of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[\Gamma]])$. If Λ' is a commutative adic $\mathbb{Z}_p$-algebra, then the image of

$$\left[\mathbb{Z}_p[[G]] \xrightarrow{\cdot g} \mathbb{Z}_p[[G]]\right] \in \mathrm{K}_1(\mathbb{Z}_p[[G]]), \quad g \in G,$$

under the composition of Φ_ρ with

$$\det\colon \mathrm{K}_1(\Lambda'[[\Gamma]]) \xrightarrow{\cong} \Lambda'[[\Gamma]]^\times$$

is $\bar{g}\det(\rho(g))^{-1}$, where $\bar{g}$ denotes the image of g under the projection $G \to \Gamma$. Note that this differs from [CFK+05, (22)] by a sign. So, our evaluation at ρ corresponds to the evaluation at the representation dual to ρ in terms of the cited article.

3. A property of S-torsion complexes

In this section, we prove Proposition 3.1, which is an abstract generalisation of [Wit13a, Prop. 2.1]. We will apply this proposition later in Section 8.

With the notation of the previous section, fix a topological generator $\gamma \in \Gamma$ and set $t \coloneqq \gamma - 1$. Assume for the moment that Λ is a finite $\mathbb{Z}_p$-algebra and that H is a finite group. Then

$$S' \coloneqq \{t^n \mid n \in \mathbb{Z}, n \geq 0\} \subset S$$

is a left and right denominator set and the localisation $\Lambda[[G]]_{S'}$ agrees with $\Lambda[[G]]_S$. Indeed, for any $s \in S$, we may consider $\Lambda[[G]]/\Lambda[[G]]s$ as a finite $\mathbb{Z}_p[[t]]$-module and conclude that for any $a \in \Lambda[[G]]$ there exists an integer $n \geq 0$ and a $b \in \Lambda[[G]]$ such that

$$t^n a = bs$$

and the same argument also works with right and left exchanged. In particular, we have

$$\Lambda[[G]]_S = \varinjlim_{n\geq 0} \Lambda[[G]]t^{-n}$$

as $\Lambda[[G]]$-modules.

Assume that $p^{i+1} = 0$ in Λ. Then

$$\binom{p^{n+i}}{k} = 0$$

in Λ whenever $p^n \nmid k$. Hence,

$$\gamma^{p^{n+i}} - 1 = (t+1)^{p^{n+i}} - 1 = t^{p^n} \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n} t^{p^n(k-1)},$$

$$\begin{aligned} t^{p^{n+i}} &= (\gamma - 1)^{p^{n+i}} - (1-1)^{p^{n+i}} = \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n} (\gamma^{kp^n} - 1)(-1)^{p^n(p^i-k)} \\ &= (\gamma^{p^n} - 1) \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n} (-1)^{p^n(p^i-k)} \sum_{\ell=0}^{k-1} \gamma^{\ell p^n} \end{aligned}$$

and therefore,

$$\Lambda[[G]]_S = \varinjlim_{n\geq 0} \Lambda[[G]](\gamma^{p^n} - 1)^{-1}.$$

Since H was assumed to be finite, the same is true for the automorphism group of H. We conclude that γ^{p^n} is a central element of G and $\Gamma^{p^n} \subset G$ a central subgroup for all $n \geq n_0$ and n_0 large enough. Set

$$N_n \coloneqq \sum_{k=0}^{p-1} \gamma^{p^n k}.$$

The homomorphism

$$\Lambda[[G]](\gamma^{p^n}-1)^{-1} \to \Lambda[[G/\Gamma^{p^n}]], \qquad \lambda(\gamma^{p^n}-1)^{-1} \mapsto \lambda + \Lambda[[G]](\gamma^{p^n}-1)$$

induces an isomorphism $\Lambda[[G]](\gamma^{p^n}-1)^{-1}/\Lambda[[G]] \cong \Lambda[[G/\Gamma^{p^n}]]$ such that the diagram

$$\begin{array}{ccc} \Lambda[[G]](\gamma^{p^n}-1)^{-1}/\Lambda[[G]] & \xrightarrow{\subset} & \Lambda[[G]](\gamma^{p^{n+1}}-1)^{-1}/\Lambda[[G]] \\ \downarrow \cong & & \downarrow \cong \\ \Lambda[[G/\Gamma^{p^n}]] & \xrightarrow{\cdot N_n} & \Lambda[[G/\Gamma^{p^{n+1}}]] \end{array}$$

commutes. Hence, we obtain an isomorphism of (left and right) $\Lambda[[G]]$-modules

$$\Lambda[[G]]_S/\Lambda[[G]] \cong \varinjlim_n \Lambda[[G/\Gamma^{p^n}]].$$

We note that this isomorphism may depend on the choice of the generator γ.

For any strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules, we thus obtain an exact sequence

$$0 \to P^\bullet \to \Lambda[[G]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules such that the cohomology of $P^\bullet$ is S-torsion, then we conclude that there exists an isomorphism

$$P^\bullet[1] \cong \varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules. In particular, the right-hand complex is perfect as complex of $\Lambda[[G]]$-modules and of $\Lambda[[H]]$-modules. This signifies that its cohomology modules

$$\mathrm{H}^k(\varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet) = \varinjlim_n \mathrm{H}^k(\Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \mathrm{H}^{k+1}(P^\bullet)$$

are finite as abelian groups.

We now drop the assumption that Λ and H are finite. Let $I \subset J$ be two open ideals of Λ and $U \subset V$ be the intersections of two open normal subgroups of G with H. Then the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Lambda/I[[G/U]] & \to & \Lambda/I[[G/U]]_S & \to & \varinjlim_n \Lambda/I[[G/U\Gamma^{p^n}]] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Lambda/J[[G/V]] & \to & \Lambda/J[[G/V]]_S & \to & \varinjlim_n \Lambda/J[[G/V\Gamma^{p^n}]] & \to & 0 \end{array}$$

commutes and the downward pointing arrows are surjections. Tensoring with $P^\bullet$ and passing to the inverse limit we obtain the exact sequence

$$0 \to P^\bullet \to \varprojlim_{I,U} \Lambda/I[[G/U]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varprojlim_{I,U} \varinjlim_n \Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules, then complex in the middle is acyclic and we obtain again an isomorphism

$$P^\bullet[1] \cong \varprojlim_{I,U} \varinjlim_{n} \Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules and hence, isomorphisms of $\Lambda[[G]]$-modules

$$\mathrm{H}^{k+1}(P^\bullet) \cong \varprojlim_{I,U} \varinjlim_{n} \mathrm{H}^k(\Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet).$$

Here, we use that the modules in the projective system on the righthand side are finite and thus $\varprojlim$-acyclic.

Finally, assume that $(Q_J^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ is a complex in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Then we can find a strictly perfect complex of $\Lambda[[G]]$-modules $P^\bullet$ and a weak equivalence

$$f\colon (\Lambda[[G]]/J \otimes_{\Lambda[[G]]} P^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}} \to (Q_J^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$$

in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda)$ [Wit08, Cor. 5.2.6]. Moreover, this complex $P^\bullet$ will also be perfect as a complex of $\Lambda[[H]]$-modules. For $I \in \mathfrak{I}_\Lambda$, U the intersection of an open normal subgroup of G with H and a positive integer n such that Γ^{p^n} is central in G/U we set

$$J_{I,U,n} \coloneqq \ker \Lambda[[G]] \to \Lambda/I[[G/U\Gamma^{p^n}]],$$

such that the $J_{I,U,n}$ form a final subsystem in $\mathfrak{I}_{\Lambda[[G]]}$. We conclude:

Proposition 3.1. *For $(Q_J^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ there exists an isomorphism*

$$\mathrm{R}\varprojlim_{J\in\mathfrak{I}_{\Lambda[[G]]}} Q_J^\bullet[1] \cong \mathrm{R}\varprojlim_{I,U} \varinjlim_{n} Q^\bullet_{J_{I,U,n}}$$

in the derived category of $\Lambda[[G]]$-modules and isomorphisms of $\Lambda[[G]]$-modules

$$\varprojlim_{J\in\mathfrak{I}_{\Lambda[[G]]}} \mathrm{H}^{k+1}(Q_J^\bullet) \cong \varprojlim_{I,U} \varinjlim_{n} \mathrm{H}^k(Q^\bullet_{J_{I,U,n}}).$$

Remark 3.2. For any $(Q_J^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ we obtain a distinguished triangle

$$\mathrm{R}\varprojlim_{J\in\mathfrak{I}_{\Lambda[[G]]}} Q_J^\bullet \to \mathrm{R}\varprojlim_{I,U} \Lambda/I[[G/U]]_S \otimes^{\mathbb{L}}_{\Lambda/I[[G/U]]} \mathrm{R}\varprojlim_{n} Q^\bullet_{J_{I,U,n}} \to \mathrm{R}\varprojlim_{I,U} \varinjlim_{n} Q^\bullet_{J_{I,U,n}}$$

in the derived category of complexes of $\Lambda[[G]]$-modules.

4. Noncommutative algebraic L-functions

Let $G = H \rtimes \Gamma$ as before. Recall the split exact sequence

$$0 \to \mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S) \xrightarrow{\partial} \mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S) \to 0.$$

[Wit13b, Cor. 3.4], which is central for the formulation of the noncommutative main conjecture: The map $\mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S)$ is the obvious one; the boundary map

$$\partial\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S)$$

on the class $[f]$ of an endomorphism f which is a weak equivalence in the Waldhausen category $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is given by

$$\partial[f] = -[\mathrm{Cone}(f)^\bullet]$$

where $\operatorname{Cone}(f)^\bullet$ denotes the cone of f [Wit14, Thm. A.5]. (Note that other authors use $-\partial$ instead.) For a fixed choice of a generator $\gamma \in \Gamma$, a splitting s_γ of ∂ is given by

$$s_\gamma([P]) \coloneqq [\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P \xrightarrow{x\hat{\otimes}y \mapsto x\hat{\otimes}y - x\gamma^{-1}\hat{\otimes}\gamma y} \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P]^{-1} \tag{4.1}$$

where the precise definition of $\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P$ as an object of the Waldhausen category $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is

$$\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P = (\varprojlim_{J\in\mathfrak{J}_{\Lambda[[G]]}} \Lambda[[G]]/I \otimes_{\Lambda[[H]]} P_J)_{I\in\mathfrak{J}_{\Lambda[[G]]}}$$

[Wit13b, Def. 2.12]. A short inspection of the definition shows that s_γ only depends on the image of γ in G/H. Following [Bur09], we may call $s_\gamma(-A)$ the *noncommutative algebraic L-function* of $A \in \mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S)$.

Proposition 4.1. *Consider an element* $A \in \mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S)$.

(1) *Let* Λ' *be another adic* $\mathbb{Z}_p$*-algebra. For any complex* $P^\bullet$ *of* Λ'-$\Lambda[[G]]$*-bimodules which is strictly perfect as complex of* Λ'*-modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(s_\gamma(A)) = s_\gamma(\Psi_{P[[G]]^{\delta\bullet}}(A))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$.

(2) *Let* $G' = H' \rtimes \Gamma'$ *such that* H' *has an open, topologically finitely generated pro-p-subgroup and* $\Gamma' \cong \mathbb{Z}_p$*. Assume that* $\alpha\colon G \to G'$ *is a continuous homomorphism such that* $\alpha(G) \notin H'$*. Set* $r \coloneqq [G' : \alpha(G)H']$*. Let* $\gamma' \in \Gamma'$ *be a generator such that* $\alpha(\gamma) = (\gamma')^r$ *in* G'/H'*. Then*

$$\Psi_{\Lambda[[G']]}(s_\gamma(A)) = s_{\gamma'}(\Psi_{\Lambda[[G']]}(A))$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$.

(3) *Assume that* G' *is an open subgroup of* G *and set* $H' \coloneqq H \cap G'$, $r \coloneqq [G : G'H]$*. Consider* $\Lambda[[G]]$ *as a* $\Lambda[[G']]$-$\Lambda[[G]]$*-bimodule. Then* γ^r *generates* $G'/H' \subset G/H$ *and*

$$\Psi_{\Lambda[[G]]}(s_\gamma(A)) = s_{\gamma^r}(\Psi_{\Lambda[[G]]}(A))$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$.

Proof. For (1), we first note that by applying the Waldhausen additivity theorem [Wal85, Prop. 1.3.2] to the short exact sequences resulting from stupid truncation, we have

$$\Psi_{P[[G]]^{\delta\bullet}} = \sum_{i\in\mathbb{Z}}(-1)^i\Psi_{P^i[[G]]^\delta}$$

as homomorphisms between the K-groups. Hence we may assume that $P = P^\bullet$ is concentrated in degree 0. We now apply [Wit13b, Prop 2.14.1] to the $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodule $M \coloneqq P[[G]]^\delta$ and its $\Lambda'[[H]]$-$\Lambda[[H]]$-sub-bimodule

$$N \coloneqq \Lambda'[[H]] \otimes_{\Lambda'} P$$

(with the diagonal right action of H) and $t_1 \coloneqq t_2 \coloneqq \gamma - 1$, $\gamma_1 \coloneqq \gamma_2 \coloneqq \gamma$.

For (2), we first assume that α induces an isomorphism $G/H \cong G/H'$ and that $\gamma' = \alpha(\gamma)$. We then apply [Wit13b, Prop 2.14.1] to $M \coloneqq \Lambda[[G']]$, $N \coloneqq \Lambda[[H']]$, and $t_1 \coloneqq \gamma - 1$, $t_2 \coloneqq \alpha(\gamma) - 1$, $\gamma_1 \coloneqq \gamma$, $\gamma_2 \coloneqq \alpha(\gamma)$.

Next, we assume that $G \subset G'$, $H = H'$, and $\gamma = (\gamma')^r$. This case is not covered by [Wit13b, Prop 2.14] and therefore, we will give more details. Consider the isomorphism of $\Lambda[[G']]$-$\Lambda[[G]]$-bimodules

$$\kappa\colon \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Lambda[[G]]^r \to \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Lambda[[G']],$$

$$\mu\hat{\otimes}\begin{pmatrix}\lambda_0\\ \vdots\\ \lambda_{r-1}\end{pmatrix} \mapsto \mu\hat{\otimes}\lambda_0 + \sum_{i=1}^{r-1}\mu(\gamma')^{-i}\hat{\otimes}(\gamma')^i(\lambda_i - \lambda_{i-1}).$$

Then the map $\mu\hat{\otimes}\lambda \mapsto \mu\hat{\otimes}\lambda - \mu(\gamma')^{-1}\hat{\otimes}\gamma'\lambda$ on the righthand side corresponds to left multiplication with the matrix

$$M \coloneqq \begin{pmatrix} \mathrm{id} & 0 & \cdots & 0 & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot)\\ 0 & \mathrm{id} & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & \vdots\\ \vdots & \ddots & \ddots & \mathrm{id} & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot)\\ 0 & \cdots & \cdots & 0 & \mathrm{id}-(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot)\end{pmatrix}$$

on the lefthand side. Let $P^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Then κ induces an isomorphism

$$\kappa\colon \Psi_{\Lambda[[G']]}(\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}(P^\bullet)^r) \to \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Psi_{\Lambda[[G']]}(P^\bullet)$$

in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$ while M corresponds to a weak equivalence such that $[M]^{-1} = s_{\gamma'}([\Psi_{\Lambda[[G']]}(P^\bullet)])$ in $\mathrm{K}_1(\Lambda[[G']]_S)$. The relations listed in [Wit14, Def. A.4] imply that the class in $\mathrm{K}_1(\Lambda[[G']]_S)$ of a triangular matrix such as M is the product of the classes of its diagonal entries. Hence, $[M]^{-1} = \Psi_{\Lambda[[G']]}(s_\gamma[P^\bullet])$, as desired.

In the general case, we note that the image of α is contained in the subgroup G'' of G' topologically generated by $(\gamma')^r$ and H' and recall that s_γ only depends on the image of γ in G/H. We are then reduced to the two cases already treated above.

For (3), we first treat the case $r = 1$. By multiplying the initial choice of γ by a suitable $h \in H$, we may assume $\gamma \in G'$. We then apply [Wit13b, Prop 2.14.1] to $M \coloneqq \Lambda[[G]]$, $N \coloneqq \Lambda[[H]]$, and t_i, γ_i as above. If $r > 1$ we can thus reduce to the case that G' is topologically generated by H and γ^r and apply [Wit13b, Prop 2.14.2].

In [Wit13b], we use a slightly different Waldhausen category for the construction of the K-theory of $\Lambda[[G]]$, but the proof of [Wit13b, Prop 2.14] goes through without changes. □

Assume that $G = \Gamma$ and that M is a $\Lambda[[\Gamma]]$-module which is finitely generated and free as a Λ-module. Then the complex

$$P^\bullet\colon\qquad \underbrace{\Lambda[[\Gamma]]\hat{\otimes}_\Lambda M}_{\text{degree }-1} \xrightarrow{\mathrm{id}-(\cdot\gamma^{-1}\hat{\otimes}\gamma\cdot)} \underbrace{\Lambda[[\Gamma]]\hat{\otimes}_\Lambda M}_{\text{degree }0}$$

is an object of $\mathbf{PDG}^{\mathrm{cont},w_1}(\Lambda[[\Gamma]])$ whose cohomology is M in degree 0 and zero otherwise. Moreover, one checks easily that

$$s_\gamma([P^\bullet]) = [\mathrm{id} - (\cdot\gamma^{-1}\hat{\otimes}\gamma\cdot) \circlearrowleft \Lambda[[\Gamma]]\hat{\otimes}_\Lambda M]$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. If Λ is commutative, then the image of $s_\gamma([P^\bullet])$ under

$$\det\colon \mathrm{K}_1(\Lambda[[\Gamma]]_S) \to \Lambda[[\Gamma]]_S^\times$$

is precisely the reverse characteristic polynomial

$$\det(\mathrm{id} - t\gamma \circlearrowleft \Lambda[t]\otimes_\Lambda M)$$

evaluated at $t = \gamma^{-1} \in \Gamma$. In fact, one may extend this to noncommutative Λ as well, using the results of Section 13.

If $M = \Lambda[[\Gamma]]/\Lambda[[\Gamma]]f$ with

$$f = (\gamma - 1)^n + \sum_{i=0}^{n-1} \lambda_i(\gamma - 1)^i \in \Lambda[[\Gamma]]$$

a monic polynomial of degree n in $\gamma - 1$ then we conclude

$$s_\gamma([P^\bullet]) = [\ \gamma^{-n} f \circlearrowleft \Lambda[[\Gamma]]\ ].$$

5. On the first special K-group of a profinite group algebra

Let p be a fixed prime number. For any profinite group G, we write $\mathfrak{N}(G)$ for its lattice of open normal subgroups and $G_r \subset G$ for the profinite and hence, closed subset of p-regular elements, i. e. the union of all q-Sylow-subgroups for all primes $q \neq p$. The group G acts continuously on G_r by conjugation. For any profinite G-set S we write $\mathbb{Z}_p[[S]]$ for the compact G-module which is freely generated by S as compact $\mathbb{Z}_p$-module.

In this section, we want to analyse the completed first special K-group

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{SK}_1(\mathbb{Z}_p[G/U])$$

of the profinite group algebra

$$\mathbb{Z}_p[[G]] \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathbb{Z}_p[G/U].$$

Note that $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ is a subgroup of the completed first K-group

$$\widehat{\mathrm{K}}_1(\mathbb{Z}_p[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{K}_1(\mathbb{Z}_p[G/U]).$$

If G has an open pro-p-subgroup which is topologically finitely generated, then $\mathbb{Z}_p[[G]]$ is an adic ring and hence,

$$\widehat{\mathrm{K}}_1(\mathbb{Z}_p[[G]]) = \mathrm{K}_1(\mathbb{Z}_p[[G]])$$

by [FK06, Prop. 1.5.3]. In the case that G is a pro-p p-adic Lie group a thorough analysis of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ has been carried out in [SV13]. Note in particular that there are examples of torsionfree p-adic Lie groups with nontrivial first special K-group. Some of the results of *loc. cit* can certainly be extended to the case that G admits elements of order prime to p. We will not pursue this further. Instead, we limit ourselves to the following results relevant to our application.

Recall from [Oli88] that there is a canonical surjective homomorphism

$$\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]]).$$

where

$$\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{H}_2(G/U, \mathbb{Z}_p[(G/U)_r])$$

denotes the second continuous homology group of $\mathbb{Z}_p[[G_r]]$. We write $X(G_r) \coloneqq \mathrm{Map}(G_r, \mathbb{Q}_p/\mathbb{Z}_p)$ for the Pontryagin dual of $\mathbb{Z}_p[[G_r]]$, such that the Pontryagin dual of $\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]])$ is $\mathrm{H}^2(G, X(G_r))$.

Lemma 5.1. *Let $G = H \rtimes \Gamma$ be a semidirect product of a finite subgroup $H \subset G$ and $\Gamma \cong \mathbb{Z}_p$. Then $\mathrm{H}^2(G, X(G_r))$ and $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ are finite.*

Proof. Note that $X(G_r) = X(H_r)$ is of finite corank over $\mathbb{Z}_p$. The Hochschild-Serre spectral sequence induces an exact sequence

$$0 \to \mathrm{H}^1(\Gamma, H^1(H, X(H_r))) \to \mathrm{H}^2(G, X(H_r)) \to \mathrm{H}^0(\Gamma, \mathrm{H}^2(H, X(H_r))) \to 0$$

where both $\mathrm{H}^1(H, X(H_r))$ and $\mathrm{H}^2(H, X(H_r))$ are finite p-groups. The lemma is an immediate consequence. ∎

We are interested in the following number theoretic situation. Assume that K is an algebraic number field and K_∞ is a $\mathbb{Z}_p$-extension of K. In particular K_∞/K is unramified over the places (including the archimedian places) of K that do not lie over p. Let L_∞ be a finite extension of K_∞ which is Galois over K. Set $G \coloneqq \operatorname{Gal}(L_\infty/K)$, $H \coloneqq \operatorname{Gal}(L_\infty/K_\infty)$, $\Gamma \coloneqq \operatorname{Gal}(K_\infty/K)$. We fix a splitting $\Gamma \to G$ such that we may write G as the semidirect product of H and Γ and let L be the fixed field of a p-Sylow subgroup of G containing Γ. Write $L^{(p)}$ for the maximal Galois p-extension of L inside a fixed algebraic closure $\overline{K}$ of K. Note that $L^{(p)} = L_\infty^{(p)}$ is the subfield of $\overline{K}$ fixed by the closed subgroup $\operatorname{Gal}_{L^{(p)}}$ generated by all q-Sylow subgroups of the absolute Galois group Gal_L with $q \neq p$. Hence, $\operatorname{Gal}_{L^{(p)}} \subset \operatorname{Gal}_{L_\infty}$ is a characteristic subgroup and therefore, $L^{(p)}/K$ is a Galois extension. The following is an adaption of the argument in the proof of [FK06, Prop. 2.3.7].

Proposition 5.2. *Set* $\mathcal{G} \coloneqq \operatorname{Gal}(L^{(p)}/K)$. *Then* $\mathrm{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = 0$.

Proof. Note that the projection $\mathcal{G} \to G$ induces a canonical isomorphism $X(\mathcal{G}_r) = X(H_r)$ and that $X(H_r)$ is of finite corank over $\mathbb{Z}_p$. We have

$$\mathrm{H}^i(\operatorname{Gal}(L^{(p)}/L), X(H_r)) = \mathrm{H}^i(\operatorname{Gal}_L, X(H_r))$$

for all i according to [NSW00, Cor. 10.4.8] applied to the class of p-groups and the set of all places of L. Moreover, $\mathrm{H}^2(\operatorname{Gal}_L, X(H_r)) = 0$ as a consequence of the fact that $\mathrm{H}^2(\operatorname{Gal}_F, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for any number field F [FK06, Prop. 2.3.7, Claim].

Since $[L : K]$ is prime to p, the restriction map

$$\mathrm{H}^2(\mathcal{G}, X(H_r)) \to \mathrm{H}^2(\operatorname{Gal}(L^{(p)}/L), X(H_r))$$

is split injective. In particular, $\mathrm{H}^2(\mathcal{G}, X(H_r)) = 0$ as claimed. □

Note that if L_∞/K is unramified over a real place of K and $p \neq 2$ then $L^{(p)}/K$ is unramified over this real place as well. For the sake of completeness we also deal with the case $p = 2$ and consider for a set of real places Σ of K such that L_∞/K is unramified over Σ the maximal subfield $L_{\Sigma^c}^{(2)}$ of $L^{(2)}$ which is unramified over Σ. Note that $L_{\Sigma^c}^{(2)}/K$ is still Galois over K.

Proposition 5.3. *Set* $\mathcal{G} \coloneqq \operatorname{Gal}(L_{\Sigma^c}^{(2)}/K)$. *Then* $\mathrm{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\mathrm{SK}}_1(\mathbb{Z}_2[[\mathcal{G}]]) = 0$.

Proof. Let L' be the subfield fixed by the intersection of the centre of G with Γ and let $Y \coloneqq \operatorname{Map}(\operatorname{Gal}(L'/K), X(H_r))$ be the induced module. We obtain a canonical surjection $Y \to X(H_r)$ with kernel Z. For any discrete $\mathcal{G}$-module A we have

$$\mathrm{H}^3(\mathcal{G}, A) = \bigoplus_{v \in \Sigma_{\mathbb{R}}^c} \mathrm{H}^3(\operatorname{Gal}_{K_v}, A)$$

where v runs through set $\Sigma_{\mathbb{R}}^c$ of real places of K not in Σ and $\operatorname{Gal}_{K_v} = \mathbb{Z}/2\mathbb{Z}$ denotes the Galois group of the corresponding local field $K_v = \mathbb{R}$ [NSW00, Prop. 10.6.5]. By the proof of the $(p = 2)$-case in [FK06, Prop. 2.3.7, Claim] we have $\mathrm{H}^2(\operatorname{Gal}_{K_v}, X(H_r)) = 0$ such that

$$\mathrm{H}^3(\mathcal{G}, Z) \to \mathrm{H}^3(\mathcal{G}, Y)$$

is injective and hence,

$$\mathrm{H}^2(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L'), X(H_r)) = \mathrm{H}^2(\mathcal{G}, Y) \to \mathrm{H}^2(\mathcal{G}, X(H_r))$$

is a surjection. Moreover, $\operatorname{Gal}_{L'}$ acts trivially on $X(H_r)$ such that it suffices to show that

$$\mathrm{H}^2(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L'), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

By the proof of [NSW00, Thm. 10.6.1] we obtain an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \mathrm{H}^1(\mathrm{Gal}(L^{(2)}/L')) \to \bigoplus_{v\in\Sigma^c_{\mathbb{R}}(L')} \mathrm{H}^1(\mathrm{Gal}_{L'_v}) \to \mathrm{H}^2(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \mathrm{H}^2(\mathrm{Gal}(L^{(2)}/L'))$$

where we have omitted the coefficients $\mathbb{Q}_2/\mathbb{Z}_2$ and $\Sigma^c_{\mathbb{R}}(L')$ denotes the real places of L' lying over $\Sigma^c_{\mathbb{R}}$. But

$$\mathrm{H}^2(\mathrm{Gal}(L^{(2)}/L')) = \mathrm{H}^2(\mathrm{Gal}_{L'}) = 0$$

by [NSW00, Cor. 10.4.8] and [Sch79, Satz 1.(ii)]. Moreover, L' is dense in the product of its real local fields such that for each real place v of L', we find an element a in L' which is negative with respect to v and positive with respect to all other real places. The element of $\mathrm{H}^1(\mathrm{Gal}(L^{(2)}/L'))$ corresponding to a square root of a maps to the nontrivial element of $\mathrm{H}^1(\mathrm{Gal}_{K_v}) = \mathbb{Z}/2\mathbb{Z}$ and to the trivial element for all other real places. This shows that

$$\mathrm{H}^1(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \bigoplus_{v\in\Sigma^c_{\mathbb{R}}(L')} \mathrm{H}^1(\mathrm{Gal}_{L'_v})$$

must be surjective. □

Corollary 5.4. *Let K_∞/K be a $\mathbb{Z}_p$-extension of a number field K and L_∞/K_∞ be a finite extension such that L_∞/K is Galois with Galois group G. Assume further that L_∞/K is unramified over a (possibly empty) set of real places Σ of K. Then there exists a finite extension L'_∞/L_∞ such that*

(i) *$[L'_\infty : L_\infty]$ is a power of p,*
(ii) *L'_∞/K is Galois with Galois group G',*
(iii) *L'_∞/K is unramified over Σ,*
(iv) *The canonical homomorphism $\mathrm{SK}_1(\mathbb{Z}_p[[G']]) \to \mathrm{SK}_1(\mathbb{Z}_p[[G]])$ is the zero map.*

In particular, L'_∞ may be chosen to be totally real if L_∞ is totally real.

Proof. With L as above, set $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(p)}/K)$ if $p \neq 2$ and $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(2)}_{\Sigma^c}/K)$ if $p = 2$ and set $\mathcal{H} \coloneqq \ker \mathcal{G} \to \mathrm{Gal}(K_\infty/K)$. According to Lemma 5.1, $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ is finite and so, the image of

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = \varprojlim_{U\in\mathfrak{N}(\mathcal{G})} \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}/U\cap\mathcal{H}]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$$

will be equal to the image of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}/U_0\cap\mathcal{H}]])$ for some $U_0 \in \mathfrak{N}(\mathcal{G})$. We let L'_∞ be the fixed field of $U_0 \cap \mathcal{H}$. Then L'_∞ clearly satisfies (i), (ii), and (iii). Since $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = 0$ by Prop. 5.2 and Prop. 5.3, it also satisfies (iv). □

Remark 5.5. The extension L'_∞/K will be unramified outside a finite set of primes, but we cannot prescribe the ramfication locus. However, assume L_∞/K is unramified outside the set S of places of K and that the Leopoldt conjecture holds for every finite extension F of K inside the maximal p-extension $L^{(p)}_S$ which is unramified outside S, i. e. that

$$\mathrm{H}^2(\mathrm{Gal}(L^{(p)}_S/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

Then the same method of proof shows that we can additionally chose L'_∞ to lie in $L^{(p)}_S$.

6. Perfect complexes of adic sheaves

We will use étale cohomology instead of Galois cohomology to formulate the main conjecture. The main advantage is that we have a little bit more flexibility in choosing our coefficient systems. Instead of being restricted to locally constant sheaves corresponding to Galois modules, we can work with constructible sheaves. An alternative would be the use of cohomology for Galois modules with local conditions.

As Waldhausen models for the derived categories of complexes of constructible sheaves, we will use the Waldhausen categories introduced in [Wit08, § 5.4–5.5] for separated schemes of finite type over a finite field. The same constructions still work with some minor changes if we consider subschemes of the spectrum of a number ring.

Fix an odd prime p and let F be a number field with ring of integers $\mathcal{O}_F$ and assume that U is an open or closed subscheme of $X \coloneqq \operatorname{Spec} \mathcal{O}_F$. Recall that for a finite ring R, a complex $\mathcal{F}^\bullet$ of étale sheaves of left R-modules on U is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. It is *perfect* if it is quasi-isomorphic to a strictly perfect complex. We call it *DG-flat* if for each geometric point of U, the complex of stalks is *DG*-flat.

Let Λ be an adic $\mathbb{Z}_p$-algebra.

Definition 6.1. The *category* $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ *of perfect complexes of adic sheaves on* U is the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ are inverse systems $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ such that:

(1) for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is *DG*-flat perfect complex of étale sheaves of Λ/I-modules on U,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism
$$\varphi_{IJ} : \mathcal{F}_I^\bullet \to \mathcal{F}_J^\bullet$$
of the system induces an isomorphism
$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are defined as in Definition 2.1.

Definition 6.2. If U is an open dense subscheme of $\operatorname{Spec} \mathcal{O}_F$, we will call a complex $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ to be *smooth at* ∞ if for each $I \in \mathfrak{I}_\Lambda$, the pullback of $\mathcal{F}_I^\bullet$ to $\operatorname{Spec} F$ is quasi-isomorphic to a strictly perfect complex with trivial action of any complex conjugation $\sigma \in \mathrm{Gal}_F$. The full subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ of complexes smooth at ∞ will be denoted by
$$\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$$

Since we assume $p \neq 2$, it is immediate that if in an exact sequence
$$0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$$
in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, the complexes $\mathcal{F}^\bullet$ and $\mathcal{H}^\bullet$ are smooth at ∞, then so is $\mathcal{G}^\bullet$. It then follows from [Wit08, Prop. 3.1.1] that $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$ is a Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

We will write $\Lambda_U = (\Lambda/I)_{I\in\mathfrak{I}_\Lambda}$ for the system of complexes concetrated in degree 0 in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ given by the constant sheaves Λ/I on U. Further, if p is invertible on U, we will write μ_{p^n} for the $\mathbb{Z}/p^n\mathbb{Z}$-sheaf of p^n-th roots of unity on U,and
$$(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}(1) = (\varprojlim_n \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$$
for the Tate twist of a complex in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

We will consider Godement resolutions of the complexes in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. To be explicit, we will fix an algebraic closure $\overline{F}$ of F and for each place x of F an embedding $\overline{F} \subset \overline{F}_x$ into a fixed algebraic closure of the local field F_x in x. In particular, we also obtain an embedding of the residue field $k(x)$ of x into the algebraically closed residue field $\overline{k(x)}$ of $\overline{F}_x$ for each closed point x of U. We write $\hat{x}$ for the corresponding geometric point $\hat{x}\colon \operatorname{Spec} \overline{k(x)} \to U$ over x and set

$$\hat{U} \coloneqq \{\hat{x} \mid x \in U\}.$$

For each étale sheaf $\mathcal{F}$ on U we set

$$(\mathrm{G}_U\,\mathcal{F})^n \coloneqq \underbrace{\prod_{\hat{u}\in\hat{U}} \hat{u}_*\hat{u}^* \cdots \prod_{\hat{u}\in\hat{U}} \hat{u}_*\hat{u}^*}_{n+1}\,\mathcal{F}$$

and turn $(\mathrm{G}_U\,\mathcal{F})^\bullet$ into a complex by taking as differentials

$$\partial^n\colon (\mathrm{G}_U\,\mathcal{F})^n \to (\mathrm{G}_U\,\mathcal{F})^{n+1}$$

the alternating sums of the maps induced by the natural transformation $\mathcal{F} \to \prod_{\hat{u}\in\hat{U}} \hat{u}_*\hat{u}^*\mathcal{F}$. The Godement resolution of a complex of étale sheaves is given by the total complex of the corresponding double complex as in [Wit08, Def. 4.2.1]. The Godement resolution of a complex $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ is given by applying the Godement resolution to each of the complexes $\mathcal{F}_I^\bullet$ individually.

We may define the total derived section functor

$$\mathrm{R}\,\Gamma(U,\cdot)\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

by the formula

$$\mathrm{R}\,\Gamma(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) = (\Gamma(U,\mathrm{G}_U\,\mathcal{F}_I^\bullet))_{I\in\mathfrak{I}_\Lambda}.$$

This agrees with the usual construction if we consider $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ as an object of the "derived" category of adic sheaves, e. g. as defined in [KW01] for $\Lambda = \mathbb{Z}_p$. In addition however, we see that $\mathrm{R}\,\Gamma(U,\cdot)$ is a Waldhausen exact functor and hence, induces homomorphisms

$$\mathrm{R}\,\Gamma(U,\cdot)\colon \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)) \to \mathrm{K}_n(\Lambda)$$

for all n [Wit08, Prop. 4.6.6, Def. 5.4.13]. Here, we use the finiteness and the vanishing in large degrees of the étale cohomology groups $\mathrm{H}^n(U,\mathcal{F})$ for constructible sheaves of abelian groups in order to assure that $\mathrm{R}\,\Gamma(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda})$ is indeed an object of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. In particular, for each $I \in \mathfrak{I}_\Lambda$, $\mathrm{R}\,\Gamma(U,\mathcal{F}_I^\bullet)$ is a perfect complex of Λ/I-modules. Note that we do not need to assume that p is invertible on U (see the remark after [Mil06, Thm. II.3.1]).

If $j\colon U \to V$ is an open immersion, we set

$$j_!(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \coloneqq (j_!\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda},$$
$$\mathrm{R}\,j_*(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \coloneqq (j_*\,\mathrm{G}_U\,\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}.$$

for any $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. While the extension by zero $j_!$ always gives us a Waldhausen exact functor

$$j_!\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V,\Lambda),$$

the total direct image

$$\mathrm{R}\,j_*\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V,\Lambda)$$

is only a well-defined Waldhausen exact functor if p is invertible on $V - U$. If $V - U$ contains places above p, then $\mathrm{R}\,j_*(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ is still a system of DG-flat complexes compatible in the sense of Definition 6.1.(2), but for $I \in \mathfrak{I}_\Lambda$ the cohomology of the complex of stalks of the complexes $\mathrm{R}\,j_*\mathcal{F}_I^\bullet$ in the geometric points over places above

p is in general not finite, such that $\mathrm{R}\, j_* \mathcal{F}_I^\bullet$ fails to be a perfect complex. In any case, we may consider $\mathrm{R}\, j_*$ as a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ to the Waldhausen category of complexes over the abelian category of inverse systems of étale sheaves of Λ-modules, indexed by $\mathfrak{I}_\Lambda$.

The pullback f^* along a morphism of schemes f and the pushforward f_* along a finite morphism of schemes are also defined as Waldhausen exact functors by degreewise application. No Godement resolution is needed, since these functors are exact on all étale sheaves.

As a shorthand, we set

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U, (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) \coloneqq \mathrm{R}(X, j_!(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda})$$

for $j: U \to X$ the open immersion into $X = \operatorname{Spec} \mathcal{O}_F$. Under our assumption that $p \neq 2$, this agrees with the definition of cohomology with proper support in [Mil06, §II.2]. If F is a totally real number field and $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}(-1)$ is smooth at ∞, then it also agrees with the definition in [FK06, §1.6.3], but this is not the case in general.

For any closed point x of X and any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(x, \Lambda)$, we set

$$\mathrm{R}\,\Gamma(\hat{x}, \mathcal{F}^\bullet) \coloneqq \Gamma(\operatorname{Spec} \overline{k(x)}, \hat{x}^* \, \mathrm{G}_x \, \mathcal{F}^\bullet)$$

and let $\mathfrak{F}_x \in \operatorname{Gal}(\overline{k(x)}/k(x))$ denote the geometric Frobenius of $k(x)$. We obtain an exact sequence

$$0 \to \mathrm{R}\,\Gamma(x, \mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x}, \mathcal{F}^\bullet) \xrightarrow{\mathrm{id} - \mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x}, \mathcal{F}^\bullet) \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ [Wit08, Prop. 6.1.2]. Note that if $\hat{x}'$ is the geometric point corresponding to another choice of an embedding $\overline{F} \subset \overline{F}_x$ and if $\mathfrak{F}'_x$ denotes the associated geometric Frobenius, then there is a canonical isomorphism

$$\sigma: \mathrm{R}\,\Gamma(\hat{x}, \mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x}', \mathcal{F}^\bullet)$$

such that

$$\sigma \circ (\mathrm{id} - \mathfrak{F}_x) = (\mathrm{id} - \mathfrak{F}'_x) \circ \sigma. \tag{6.1}$$

At some point, we will also make use of the categories $\mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec} F_x, \Lambda)$ for the local fields F_x together with the associated total derived section functors. In this case, one can directly appeal to the constructions in [Wit08, Ch. 5]. We write F_x^{nr} for the maximal unramified of F_x in $\overline{F}_x$ and note that we have a canonical identification $\operatorname{Gal}(F_x^{\mathrm{nr}}/F_x) = \operatorname{Gal}(\overline{k(x)}/k(x))$.

Lemma 6.3. *Let $j: U \to V$ denote the open immersion of two open dense subschemes of X and assume that $i: x \to V$ is a closed point in the complement of U not lying over p. Write $\eta_x: \operatorname{Spec} F_x \to U$ for the map to the generic point of U. Then there exists a canonical chain of weak equivalences*

$$\mathrm{R}\,\Gamma(\hat{x}, i^* \, \mathrm{R}\, j_* \mathcal{F}^\bullet) \xrightarrow{\cong} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^* \, \mathrm{G}_U \, \mathcal{F}^\bullet) \xleftarrow{\cong} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^* \mathcal{F}^\bullet) \tag{6.2}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ compatible with the operation of the Frobenius on each complex and hence, a canonical chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^* \, \mathrm{R}\, j_* \mathcal{F}^\bullet) \xrightarrow{\cong} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^* \, \mathrm{G}_U \, \mathcal{F}^\bullet) \xleftarrow{\cong} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^* \mathcal{F}^\bullet) \tag{6.3}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

Proof. From [Mil80, Thm. III.1.15] we conclude that for each $I \in \mathfrak{I}_\Lambda$, the complex $\eta_x^* \, \mathrm{G}_U \, \mathcal{F}_I^\bullet$ is a complex of flabby sheaves on $\operatorname{Spec} F_x$ and that

$$\mathrm{R}\,\Gamma(\hat{x}, i^* \, \mathrm{R}\, j_* \mathcal{F}_I^\bullet) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^* \, \mathrm{G}_U \, \mathcal{F}_I^\bullet)$$

is an isomorphism. Write G_{F_x} for the Godement resolution on $\operatorname{Spec} F_x$ with respect to $\operatorname{Spec} \overline{F}_x \to \operatorname{Spec} F_x$. Then

$$\eta_x^* \mathrm{G}_U \mathcal{F}_I^\bullet \to \mathrm{G}_{F_x} \eta_x^* \mathrm{G}_U \mathcal{F}_I^\bullet \leftarrow \mathrm{G}_{F_x} \eta_x^* \mathcal{F}_I^\bullet$$

are quasi-isomorphisms of complexes of flabby sheaves on $\operatorname{Spec} F_x$. Hence, they remain quasi-isomorphisms if we apply the section functor $\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -)$ in each degree. Since the Frobenius acts compatibly on F_x^{nr} and $\overline{k(x)}$, the induced operation on the complexes is also compatible. The canonical exact sequence

$$0 \to \Gamma(\operatorname{Spec} F_x, -) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \to 0$$

on flabby sheaves on $\operatorname{Spec} F_x$ implies that the morphisms in the chain (6.3) are also quasi-isomorphisms. □

Remark 6.4. Note that for x lying over p, the proof of the lemma remains still valid, except that the complexes in the chain (6.2) do not lie in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

If Λ' is another adic $\mathbb{Z}_p$-algebra and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules, we may extend $\Psi_{M^\bullet}$ to a Waldhausen exact functor

$$\Psi_{M^\bullet}: \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda'),$$
$$(\mathcal{P}_J^\bullet)_{J \in \mathfrak{I}_\Lambda} \mapsto (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_\Lambda M^\bullet \otimes_\Lambda \mathcal{P}_J{}^\bullet)_{I \in \mathfrak{I}_{\Lambda'}}$$

such that

$$\Psi_{M^\bullet} \mathrm{R}\Gamma(U, \mathcal{P}^\bullet) \to \mathrm{R}\Gamma(U, \Psi_{M^\bullet}(\mathcal{P}^\bullet))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda')$ [Wit08, Prop. 5.5.7].

7. Admissible extensions

As before, we fix an odd prime p and a number field F. Assume that F_∞/F is a possibly infinite Galois extension unramified over an open or closed subscheme $U = U_F$ of $X = \operatorname{Spec} \mathcal{O}_F$. Let $G = \operatorname{Gal}(F_\infty/F)$ be its Galois group. We also assume that G has a topologically finitely generated, open pro-p-subgroup, such that for any adic $\mathbb{Z}_p$-algebra Λ, the profinite group ring $\Lambda[[G]]$ is again an adic ring. For any intermediate number field K of F_∞/F, we will write U_K for the base change with $X_K := \operatorname{Spec} \mathcal{O}_K$ and $f_K: U_K \to U$ for the corresponding Galois covering of U, such that we obtain a system of Galois coverings $(f_K: U_K \to U)_{F \subset K \subset F_\infty}$, which we denote by

$$f: U_{F_\infty} \to U.$$

As in [Wit14, Def. 6.1] we make the following construction.

Definition 7.1. Let Λ be any adic $\mathbb{Z}_p$-algebra. For $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ we set

$$f_! f^* \mathcal{F}^\bullet := (\varprojlim_{I \in \mathfrak{I}_\Lambda} \varprojlim_{F \subset K \subset F_\infty} \Lambda[[G]]/J \otimes_{\Lambda[[G]]} f_{K!} f_K^* \mathcal{F}_I{}^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$$

As in [Wit14, Prop. 6.2] one verifies that we thus obtain a Waldhausen exact functor

$$f_! f^*: \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]]).$$

In particular, if U is open and dense in X and if $k: U \to W$ denotes the open immersion into another open dense subscheme W of X, we obtain for each complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ a complex

$$\mathrm{R}\Gamma_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

Remark 7.2. Set $V = U \cup (X - W)$ and let $j: U \to V$ denote the corresponding open immersion. Write $j': V \to X$ and $k': W \to X$ for the open immersions into X. For any étale sheaf $\mathcal{G}$ on U, the canonical morphism

$$k'_! k_* \mathrm{G}^\bullet_U \mathcal{G} \cong j'_* j_! \mathrm{G}^\bullet_U \mathcal{G} \xrightarrow{j'}_* \mathrm{G}^\bullet_V j_! \mathcal{G}$$

is seen to be a quasi-isomorphism by checking on the stalks. Hence, for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$, there is a weak equivalence

$$\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \cong \mathrm{R}\,\Gamma(V, j_! f_! f^* \mathcal{F}^\bullet).$$

We recall that the righthand complex is always in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. Hence, the same is true for the lefthand complex without any condition on U and W, even if $\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet$ fails to be a perfect complex. In particular, we may use the two complexes interchangeably in our results.

We recall how the functor $f_! f^*$ transforms under the change of the extension F_∞/F and under changes of the coefficient ring Λ.

Proposition 7.3. *Let $f: U_{F_\infty} \to U$ be the system of Galois coverings of the open or closed subscheme U of X associated to the extension F_∞/F with Galois group G which is unramified over U. Let further Λ be an adic $\mathbb{Z}_p$-algebra and $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.*

(1) *Let Λ' be another adic $\mathbb{Z}_p$-algebra and let $P^\bullet$ be a complex of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules. Then there exists a natural weak equivalence*

$$\Psi_{P[[G]]^{\delta\bullet}} f_! f^* \mathcal{F}^\bullet \cong f_! f^* \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet$$

(2) *Let $F'_\infty \subset F_\infty$ be a subfield such that F'_∞/F is a Galois extension with Galois group G' and let $f': U_{F'_\infty} \to U$ denote the corresponding system of Galois coverings. Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G']]} f_! f^* \mathcal{F}^\bullet \cong (f')_! (f')^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

(3) *Let F'/F be a finite extension inside F_∞/F, let $f_{F'}: U_{F'} \to U$ denote the associated étale covering of U and let $g: U_{F_\infty} \to U'_F$ be the restriction of the system of coverings f to $U_{F'}$. Write $G' \subset G$ for the corresponding open subgroup and view $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then there exists a natural weak equivalence*

$$\Psi_{\Lambda[[G]]} f_! f^* \mathcal{F}^\bullet \cong f_{F'*} (g_! g^*) f^*_{F'} \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

(4) *With the notation of (3), let $\mathcal{G}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$ and view $\Lambda[[G]]$ as a $\Lambda[[G]]$-$\Lambda[[G']]$-bimodule. Then there exists a natural weak equivalence*

$$\Psi_{\Lambda[[G]]} f_{F'*} g_! g^* \mathcal{G}^\bullet \cong f_! f^* (f_{F'*} \mathcal{G}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]])$.

Proof. Part (1) – (3) are proved in [Wit14, Prop. 6.5, 6.7]. We prove (4). First, note that for any $F'' \subset F_\infty$ containing F' and any $I \in \mathfrak{I}_\Lambda$ the canonical map

$$g_{F''!} g^*_{F''} (\Lambda/I)_{U_{F'}} \to f^*_{F'} f_{F''!} f^*_{F''} (\Lambda/I)_U$$

induces an isomorphism

$$\Lambda/I[\mathrm{Gal}(F''/F)] \otimes_{\Lambda/I[\mathrm{Gal}(F''/F')]} g_{F''!} g^*_{F''} (\Lambda/I)_{U_{F'}} \cong f^*_{F'} f_{F''!} f^*_{F''} (\Lambda/I)_U.$$

Hence,

$$\Psi_{\Lambda[[G]]}(g_! g^* \Lambda_{U_{F'}}) \cong f^*_{F'} f_! f^* \Lambda_U$$

in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda[[G]])$. We further recall that

$$f_! f^* f_{F'*} \mathcal{G}^\bullet \cong \Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet$$

in the notation of [Wit14, Prop. 6.3]. The projection formula then implies

$$\begin{aligned}\Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet &\cong f_{F'*}(\Psi_{f^*_{F'} f_! f^* \Lambda_U}(\mathcal{G}^\bullet)) \\ &\cong f_{F'*}(\Psi_{\Psi_{\Lambda[[G]]}(g_! g^* \Lambda_{U_{F'}})}(\mathcal{G}^\bullet)) \\ &\cong f_{F'*}(\Psi_{\Lambda[[G]]}(g_! g^* \mathcal{G}^\bullet))\end{aligned}$$

as desired. □

To understand Part (1) of this proposition, note that if ρ is a representation of G on a finitely generated and projective Λ-module and P_ρ is the corresponding Λ-$\mathbb{Z}_p[[G]]$-bimodule as in Example 2.7, then

$$\mathcal{M}(\rho) \coloneqq \Psi_{P_\rho} f_! f^* (\mathbb{Z}_p)_U \tag{7.1}$$

is simply the smooth sheaf of Λ-modules on U associated to ρ [Wit14, Prop. 6.8]. In general,

$$\Psi_{\tilde{P}^\bullet} \mathcal{F}^\bullet \coloneqq \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet \tag{7.2}$$

should be understood as the derived tensor product over Λ of the complex of sheaves associated to $P^\bullet$ and the complex $\mathcal{F}^\bullet$.

Let F_{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of F and let M be the maximal abelian p-extension of F_{cyc} unramified outside the places over p. Assume that F is a totally real field. By the validity of the weak Leopoldt conjecture for F_{cyc}, the Galois group $\mathrm{Gal}(M/F_{\mathrm{cyc}})$ is a finitely generated torsion module of projective dimension less or equal 1 over the classical Iwaswa algebra $\mathbb{Z}_p[[\mathrm{Gal}(F_{\mathrm{cyc}}/F)]]$ [NSW00, Thm. 11.3.2]. Like in [Kak13], we will assume the vanishing of its Iwasawa μ-invariant in the following sense:

Assumption 7.4. For every totally real field F, the Galois group over F_{cyc} of the maximal abelian p-extension of F_{cyc} unramified outside the places over p is a finitely generated $\mathbb{Z}_p$-module.

In particular, for any totally real field F and any finite set Σ of places of F containing the places over p, the Galois group over F_{cyc} of the maximal abelian p-extension of F_{cyc} unramified outside Σ is also a finitely generated $\mathbb{Z}_p$-module, noting that no finite place is completely decomposed in F_{cyc}/F [NSW00, Cor. 11.3.6]. We also observe that the Galois group $\mathrm{Gal}(F_\Sigma^{(p)}/F_{\mathrm{cyc}})$ of the maximal p-extension of F unramified outside Σ is then a free pro-p-group topologically generated by finitely many elements [NSW00, Thm. 11.3.7].

Definition 7.5. Let F be a number field. An extension F_∞/F inside $\overline{F}$ is called *admissible* if

(1) F_∞/F is Galois and unramified outside a finite set of places,
(2) F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc},
(3) $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ contains an open pro-p subgroup that is topologically finitely generated.

Definition 7.6. Let F be a totally real number field and F_∞/F an admissible extension.

(1) We call F_∞/F *really admissible* if F_∞ is totally real.
(2) We call F_∞/F *CM-admissible* if F_∞ is totally imaginary and there exists an involution $\iota \in \mathrm{Gal}(F_\infty/F)$ such that the fixed field F_∞^ι of ι is totally real.

The notion of really admissible extensions is slightly weaker than the notion of admissible extension used in [Kak13, Def. 2.1]: We do not need to require $\mathrm{Gal}(F_\infty/F)$ to be a p-adic Lie group. For example, as a result of the preceeding discussion, we see that we could choose $F_\infty = F_\Sigma^{(p)}$ for some finite set of places Σ of F containing the places above p.

If F_∞/F is an admissible extension, we let $G \coloneqq \mathrm{Gal}(F_\infty/F)$ denote its Galois group and set $H \coloneqq \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$, $\Gamma \coloneqq \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. We may then choose a continuous splitting $\Gamma \to G$ to identify G with the corresponding semidirect product $G = H \rtimes \Gamma$.

If a really admissible extension F_∞/F is unramfied over the open dense subscheme $U = W$ of X, $\Lambda = \mathbb{Z}_p$ and $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U(1)$, then

$$\varprojlim_{I \in \mathfrak{I}_{\mathbb{Z}_p[[G]]}} \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_p)_U(1))[-3]$$

is by Artin-Verdier duality and comparison of étale and Galois cohomology quasi-isomorphic to the complex $C(F_\infty/F)$ featuring in the main conjecture [Kak13, Thm. 2.11]. In particular,

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_p)_U(1))$$

is in fact an object of $\mathbf{PDG}^{\mathrm{cont}, w_H}(\mathbb{Z}_p[[G]])$. We will generalise this statement in the next section.

Note that for a CM-admissible extension F_∞/F, the automorphism ι is uniquely determined and commutes with every other field automorphism of F_∞. Consequently, F_∞^ι/F is Galois and a hence, a really admissible extension. As usual, we write

$$e_- \coloneqq \frac{1-\iota}{2}, \quad e_+ \coloneqq \frac{1+\iota}{2} \in \Lambda[[G]].$$

for the corresponding idempotents.

8. The S-torsion property

Assume that F_∞/F is an admissible extension that is unramified over the open dense subscheme U of $X = \operatorname{Spec} \mathcal{O}_F$ and that $k\colon U \to W$ is the open immersion into another open dense subscheme of X. Note that p must be invertible on U, because the cyclotomic extension F_{cyc}/F is ramified over all places over p. We also fix an adic $\mathbb{Z}_p$-algebra Λ. Our purpose is to prove:

Theorem 8.1. *Assume that F_∞/F is really admissible and that p is invertible on W. Let $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$ be a complex of Λ-adic sheaves smooth at ∞. If Assumption 7.4 is satisfied, then the complexes*

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1)), \quad \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

In the course of the proof, we will also need to consider the following local variant, whose validity is independent of Assumption 7.4.

Theorem 8.2. *Assume that F_∞/F is an admissible extension with $k\colon U \to W$ as above. Let $i\colon \Sigma \to W$ denote a closed subscheme of W and assume that p is invertible on Σ. For any complex of Λ-adic sheaves $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$*

$$\mathrm{R}\,\Gamma(\Sigma, i^* \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

is in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

Using [Wit14, Prop. 4.8] we may at once reduce to the case that Λ is a finite semisimple $\mathbb{Z}_p$-algebra and that $F_\infty/F_{\mathrm{cyc}}$ is a finite extension. It then suffices to show that the complexes appearing in the above propositions have finite cohomology groups. We may then replace $\mathcal{F}^\bullet$ by a quasi-isomorphic strictly perfect complex. Using stupid truncation and induction on the length of the strictly perfect complex we may assume that $\mathcal{F}$ is in fact a flat and constructible sheaf (unramified over ∞). Note further that the cohomology groups

$$\mathrm{H}^n_c(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) = \varprojlim_{F\subset K\subset F_\infty} \mathrm{H}^n(W_K, \mathrm{R}\, k_* f_K^* \mathcal{F}(1)),$$
$$\mathrm{H}^n(\Sigma, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet) = \varprojlim_{F\subset K\subset F_\infty} \mathrm{H}^n(\Sigma_K, i^* \mathrm{R}\, k_* f_K^* \mathcal{F})$$

do not change if we replace F by a finite extension of F inside F_∞. So, we may assume that $F_\infty = F_{\mathrm{cyc}}$ and that no place in Σ splits in F_∞/F. Further, we may reduce to the case that Σ consists of a single place x. In particular, x does not split or ramify in F_∞/F and x does not lie above p.

We prove Theorem 8.2 in the case that $x \in U$. Under the above assumptions on x, there exists a quasi-isomorphism

$$\mathrm{R}\,\Gamma(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) \cong \mathrm{R}\,\Gamma(x, i^* f_! f^* \mathcal{F}) \cong \mathrm{R}\,\Gamma(x, g_! g^* i^* \mathcal{F})$$

where $g\colon x_\infty \to x$ is the unique $\mathbb{Z}_p$-extension of x. We can now refer directly to [Wit14, Thm. 8.1] or identify

$$\mathrm{H}^n(x, g_! g^* i^* \mathcal{F}) = \mathrm{H}^n(\mathrm{Gal}_{k(x)}, M \otimes_{\mathbb{F}_p} \mathbb{F}_p[[\Gamma]]^\sharp)$$

with $\mathrm{Gal}_{k(x)}$ the absolute Galois group of the residue field $k(x)$ of x, M the stalk of $\mathcal{F}$ in a geometric point over x and $\mathbb{F}_p[[\Gamma]]^\sharp$ being the $\mathrm{Gal}_{k(x)}$-module $\mathbb{F}_p[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{k(x)}$ acting by right multiplication with the image of σ^{-1} in Γ. It is then clear that the only nonvanishing cohomology group is $H^1(\mathrm{Gal}_{k(x)}, M \otimes_{\mathbb{F}_p} \mathbb{F}_p[[\Gamma]]^\sharp)$, of order bounded by the order of M.

Now we prove Theorem 8.2 in the case that $x \in W - U$. By Lemma 6.3 and the smooth base change theorem there exists a chain of quasi-isomorphisms

$$\mathrm{R}\,\Gamma(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet) \cong \mathrm{R}\,\Gamma(\mathrm{Spec}\, F_x, h_! h^* \eta_x^* \mathcal{F}),$$

where F_x is the local field in x with valuation ring $\mathcal{O}_{F_x}$, $\eta_x\colon \mathrm{Spec}\, F_x \to U$ is the map to the generic point of U, and $h\colon \mathrm{Spec}(F_x)_{\mathrm{cyc}} \to \mathrm{Spec}\, F_x$ is the unique $\mathbb{Z}_p$-extension of F_x inside $\overline{F}_x$. We may now identify

$$\mathrm{H}^n(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet) = \mathrm{H}^n(\mathrm{Gal}_{F_x}, M \otimes_{\mathbb{F}_p} \mathbb{F}_p[[\Gamma]]^\sharp)$$

with Gal_{F_x} the absolute Galois group of the local field F_x in x, M the finite Gal_{F_x}-module corresponding to $\eta_x^* \mathcal{F}$ and $\mathbb{F}_p[[\Gamma]]^\sharp$ being the Gal_{F_x}-module $\mathbb{F}_p[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{F_x}$ acting by right multiplication with the image of σ^{-1} in Γ. The finiteness of the cohomology group on the righthand side is well-known: We can use local duality to identify it with the Pontryagin dual of

$$\mathrm{H}^{2-n}(\mathrm{Gal}_{(F_x)_{\mathrm{cyc}}}, M^*)$$

where M^* is the first Tate twist of the Pontryagin dual of M.

Finally, we prove Theorem 8.1. Assume that $F\infty/F$ is really admissible, that $\mathcal{F}$ is smooth at ∞, and that p is invertible on W. We begin with the case of étale cohomology with proper support. Letting $i\colon \Sigma \to W$ denote the complement of U in W, we have the exact excision sequence

$$0 \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, k_! k^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, i_* i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to 0$$

and quasi-isomorphisms

$$\mathrm{R}\Gamma_c(W, k_! k^* \mathrm{R} k_* f_! f^* \mathcal{F}(1)) \cong \mathrm{R}\Gamma_c(U, f_! f^* \mathcal{F}(1)),$$
$$\mathrm{R}\Gamma_c(W, i_* i^* \mathrm{R} k_* f_! f^* \mathcal{F}(1)) \cong \mathrm{R}\Gamma(\Sigma, i^* \mathrm{R} k_* f_! f^* \mathcal{F}(1)).$$

By Theorem 8.2, we may thus reduce to the case $W = U$. Furthermore, we may shrink U ad libitum. Hence, we may assume that $\mathcal{F}$ is locally constant on U and smooth at ∞. Consequently, there exists a finite Galois extension F'/F such that F' is totally real, $g_{F'}: U_{F'} \to U$ is étale and $g_{F'}^* \mathcal{F}$ is constant. Then F'_{cyc}/F is an admissible extension and

$$\rho \coloneqq g_{F'}^* \mathcal{F}(U_{F'})$$

may be viewed as a continuous representation of $G = \mathrm{Gal}(F'_{\mathrm{cyc}}/F)$ on a finitely generated, projective Λ-module. Write $g: U_{F'_{\mathrm{cyc}}} \to U$ for the corresponding system of coverings of U and observe that there exists a quasi-isomorphism

$$\Phi_\rho(\mathrm{R}\Gamma_c(U, g_! g^* (\mathbb{Z}_p)_U(1))) \cong \mathrm{R}\Gamma_c(U, f_! f^* \mathcal{F}(1))$$

with Φ_ρ being defined by (2.3) [Wit14, Prop. 5.9, 6.3, 6.5, 6.7]. Since Φ_ρ takes complexes in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\mathbb{Z}_p[[G]])$ to complexes in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[\Gamma]])$, it remains to show that the cohomology groups $\mathrm{H}^n_c(U, g_! g^* (\mathbb{Z}_p)_U(1))$ are finitely generated as $\mathbb{Z}_p$-modules. Now

$$\mathrm{H}^n_c(U, g_! g^* (\mathbb{Z}_p)_U(1)) = \begin{cases} 0 & \text{if } n \neq 2, 3, \\ \mathrm{Gal}(M/F'_{\mathrm{cyc}}) & \text{if } n = 2, \\ \mathbb{Z}_p & \text{if } n = 3, \end{cases}$$

with M denoting the maximal abelian p-extension of F'_{cyc} unramified over U [Kak13, p. 548]. At this point, we make use of our Assumption 7.4 on the vanishing of the μ-invariant to finish the proof for the first complex.

We now turn to the second complex. If we had a good notion of a Λ-dual for complexes in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ at our disposal, then we could reduce this case immediately to the case that we have treated before, using a duality statement of the type given in [FK06, §1.6.2]. Since this theory has not yet been developed, our argument is a little more involed. We still assume that Λ is a finite ring. Write $\Sigma = W - U$, $V = U \cup (X - W)$ and $j: U \to V$, $\ell: V \to X$, $i: \Sigma \to X$ for the natural immersions. As mentioned in Remark 7.2, the exists a weak equivalence

$$\mathrm{R}\Gamma_c(V, \mathrm{R} j_* f_! f^* \mathcal{F}) \cong \mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}).$$

Moreover, there is an exact sequence

$$0 \to \ell_! \mathrm{R} j_* f_! f^* \mathcal{F} \to \mathrm{R}(\ell \circ j)_* f_! f^* \mathcal{F} \to i_* i^* \mathrm{R} j_* f_! f^* \mathcal{F} \to 0$$

Using Theorem 8.2 we may thus reduce to the case that $V = X$, $W = U$ and $\mathcal{F}$ locally constant on U and smooth at ∞. Recall that we assume that $F_\infty/F = F_{\mathrm{cyc}}/F$ is the cyclotomic extension. Write F_n for the intermediate fields of F_{cyc}/F. Write

$$\mathcal{F}^\vee = \mathcal{H}om_U(\mathcal{F}, \mathbb{Q}_p/\mathbb{Z}_p)$$

for the Pontryagin dual of the locally constant sheaf $\mathcal{F}$ and

$$A^\vee = \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

for the Pontryagin dual of any compact $\mathbb{Z}_p$-module A. By Artin-Verdier duality [Mil06, Thm. II.3.1] there exists an isomorphism

$$\mathrm{H}^s(U, f_! f^* \mathcal{F})^\vee \cong \varinjlim_n \mathrm{H}^{3-s}_c(U_{F_n}, f_{U_{F_n}}^* \mathcal{F}^\vee(1)) = \mathrm{H}^{3-s}_c(U_{F_{\mathrm{cyc}}}, f^* \mathcal{F}^\vee(1)).$$

By what we have already proved above, $\mathrm{R}\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\vee(1))$ is a complex in the category $\mathbf{PDG}^{\mathrm{cont},w_1}(U, \Lambda[[\Gamma]])$. Hence, we may apply Proposition 3.1 to conclude that

$$\mathrm{H}_c^{3-s}(U_{F_{\mathrm{cyc}}}, f^* \mathcal{F}^\vee(1)) \cong \mathrm{H}_c^{4-s}(U, f_* f^* \mathcal{F}^\vee(1))$$

is a finite group for all integers s. This implies that $\mathrm{H}^s(U, f_! f^* \mathcal{F})$ is a finite group and hence, $\mathrm{R}\Gamma(U, f_! f^* \mathcal{F})$ is in $\mathbf{PDG}^{\mathrm{cont},w_1}(U, \Lambda[[\Gamma]])$, as claimed. This finishes the proof of Theorem 8.1.

Corollary 8.3. *Assume that F_∞/F is CM-admissible and that p is invertible on W. Let $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$ be a complex of Λ-adic sheaves smooth at ∞. If Assumption 7.4 is satisfied, then the complexes*

$$e_+ \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}k_* f_! f^* \mathcal{F}^\bullet(1)), \qquad e_- \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}k_* f_! f^* \mathcal{F}^\bullet),$$
$$e_+ \mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet), \qquad e_- \mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet(1))$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Proof. Without loss of generality, we may enlarge F_∞ by adjoining the p-th roots of unity. In particular, the cyclotomic character

$$\varepsilon_{\mathrm{cyc}\,F} \colon \mathrm{Gal}_F \to \mathbb{Z}_p^\times, \quad g\zeta = \zeta^{\varepsilon_{\mathrm{cyc}}(g)}, \quad g \in \mathrm{Gal}_F, \zeta \in \mu_{p^\infty}$$

factors through $G = \mathrm{Gal}(F_\infty/F)$. Set $G_+ \coloneqq \mathrm{Gal}(F_\infty^\iota/F)$ and write g_+ for the image of $g \in G$ in G_+. We then obtain for every odd $n \in \mathbb{Z}$ a ring isomorphism

$$\Lambda[[G]] \to \Lambda[[G_+]] \times \Lambda[[G_+]], \qquad G \ni g \mapsto (g_+, \varepsilon_{\mathrm{cyc}}(g)^n g_+).$$

The projections onto the two components corresponds to the decomposition of $\Lambda[[G]]$ with respect to e_+ and e_-.

We will construct the corresponding decomposition of $\mathbf{A}(\Lambda[[G]])$, where $\mathbf{A} \in \{\mathbf{PDG}^{\mathrm{cont}}, {}_{w_H}\mathbf{PDG}^{\mathrm{cont}}, \mathbf{PDG}^{\mathrm{cont},w_H}\}$. Write $T_{\varepsilon_{\mathrm{cyc}}{}^n}$ for the Λ-$\Lambda[[G]]$-bimodule Λ with $g \in G$ acting by $\varepsilon_{\mathrm{cyc}}{}^n(g^{-1})$ from the right and $T_{\varepsilon_{\mathrm{cyc}}{}^n}[[G]]^\delta$ for the $\Lambda[[G]]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]] \otimes_\Lambda T_{\varepsilon_{\mathrm{cyc}}{}^n}$ with the diagonal right action of G. According to Example 2.5, we obtain Waldhausen exact functors

$$\Psi_{T_{\varepsilon_{\mathrm{cyc}}{}^n}[[G]]^\delta} \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]]).$$

Moreover, considering $\Lambda[[G_+]]$ as a $\Lambda[[G_+]]$-$\Lambda[[G]]$-bimodule or as a $\Lambda[[G]]$-$\Lambda[[G_+]]$-bimodule, we obtain Waldhausen exact functors

$$\Psi_{\Lambda[[G_+]]} \colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G_+]]), \qquad \Psi_{\Lambda[[G_+]]} \colon \mathbf{A}(\Lambda[[G_+]]) \to \mathbf{A}(\Lambda[[G]]).$$

Note that the composition

$$\Psi_{\Lambda[[G_+]]} \circ \Psi_{\Lambda[[G_+]]} \colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]])$$

is just the projection onto the e_+-component, whereas the projection onte the e_--component may be written as

$$\Psi_{T_{\varepsilon_{\mathrm{cyc}}{}^n}[[G]]^\delta} \circ \Psi_{\Lambda[[G_+]]} \circ \Psi_{\Lambda[[G_+]]} \circ \Psi_{T_{\varepsilon_{\mathrm{cyc}}{}^{-n}}[[G]]^\delta} \colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]])$$

for every odd $n \in \mathbb{Z}$. We further note that

$$\Psi_{T_{\varepsilon_{\mathrm{cyc}}{}^n}[[G]]^\delta}(f_! f^* \mathcal{F}) = f_! f^* \mathcal{F}(n).$$

With this description, the claim of the corollary is an immediate consequence of Theorem 8.1 □

8.1. **Calculation of the cohomology.** We retain the notation from the beginning of the previous section. Our objective is to investigate the cohomology of the complexes

$$\mathrm{R}\Gamma_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}(1)), \quad \mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}), \quad \mathrm{R}\Gamma(\Sigma, i^* \mathrm{R}k_* f_! f^* \mathcal{F})$$

for a constructible adic sheaf $\mathcal{F}$, i. e. a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ concentrated in degree 0. We will not use these results in the proof of our main conjecture in Section 12, but they might help to tie the connection to classical objects in Iwasawa theory. In particular, as a central result of this section, we want to establish the link with the p-adic realisations of Picard-1-motives considered in [GP15].

The following two propositions are direct consequences of Proposition 3.1, Theorem 8.2, and Corollary 8.3.

Proposition 8.4. *Let F_∞/F be any admissible extension unramified over U. Assume that $i: x \to W$ is a closed point not lying over p. Then*

$$\mathrm{H}^s(x, i^* \mathrm{R}k_* f_! f^* \mathcal{F}) \cong \varprojlim_{F'} \mathrm{H}^{s-1}(x_{F'_{\mathrm{cyc}}}, i_* \mathrm{R}k_* \mathcal{F})$$

where F' runs through the finite subextensions of F_∞/F and $x_{F'_{\mathrm{cyc}}}$ denotes the finite set of places of F'_{cyc} lying over x. In particular,

$$\mathrm{H}^s(x, i^* \mathrm{R}k_* f_! f^* \mathcal{F}) = 0$$

for $s \neq 1$ if $x \in U$ and for $s \neq 1, 2$ if $x \in W - U$.

Proposition 8.5. *Let F be totally real and F_∞/F be a CM-admissible extension unramified over U. Assume that p is invertible on W and that $\mathcal{F}$ is smooth at ∞. If Assumption 7.4 is valid, then*

$$e_+ \mathrm{H}^s_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}(1)) \cong \varprojlim_{F'} e_+ \mathrm{H}^{s-1}_c(W_{F'_{\mathrm{cyc}}}, \mathrm{R}k_* \mathcal{F}(1))$$
$$e_- \mathrm{H}^s_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}) \cong \varprojlim_{F'} e_- \mathrm{H}^{s-1}_c(W_{F'_{\mathrm{cyc}}}, \mathrm{R}k_* \mathcal{F})$$
$$e_+ \mathrm{H}^s(W, k_! f_! f^* \mathcal{F}) \cong \varprojlim_{F'} e_+ \mathrm{H}^{s-1}(W_{F'_{\mathrm{cyc}}}, k_! \mathcal{F})$$
$$e_- \mathrm{H}^s(W, k_! f_! f^* \mathcal{F}) \cong \varprojlim_{F'} e_- \mathrm{H}^{s-1}(W_{F'_{\mathrm{cyc}}}, k_! \mathcal{F})$$

where F' runs through the finite subextensions of F_∞/F. In particular,

1. *$e_+ \mathrm{H}^s_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}(1)) = e_- \mathrm{H}^s_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}) = 0$ for $s \neq 2$ if $U \neq W$ and for $s \neq 2, 3$ if $W = U$.*
2. *$e_+ \mathrm{H}^s(W, k_! f_! f^* \mathcal{F}) = e_- \mathrm{H}^s(W, k_! f_! f^* \mathcal{F}(1)) = 0$ for $s \neq 2$ if $U \neq W$ or if $U = W$ and $F_\infty/F_{\mathrm{cyc}}$ is infinite and for $s \neq 1, 2$ if $U = W$ and F_∞/F is finite.*

In particular, we obtain the following corollary.

Corollary 8.6. *Let F be totally real and F_∞/F be a CM-admissible extension unramified over U. Assume that p is invertible on $W \neq U$ and that $\mathcal{F}$ is smooth at ∞. If Assumption 7.4 is valid, then*

$$e_+ \mathrm{H}^2_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}(1)), \qquad e_- \mathrm{H}^2_c(W, \mathrm{R}k_* f_! f^* \mathcal{F}),$$
$$e_+ \mathrm{H}^2(W, k_! f_! f^* \mathcal{F}), \qquad e_- \mathrm{H}^2(W, k_! f_! f^* \mathcal{F}(1))$$

are finitely generated and projective as $\Lambda[[H]]$-modules and have strictly perfect resolutions of length 1 as $\Lambda[[G]]$-modules. In particular, we may consider their *classes in* $\mathrm{K}_0(\Lambda[[G]],\Lambda[[G]]_S)$ *and obtain*

$$[e_+\,\mathrm{H}^2_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}(1))] = [e_+\,\mathrm{R}\,\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}(1))],$$
$$[e_-\,\mathrm{H}^2_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F})] = [e_-\,\mathrm{R}\,\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F})],$$
$$[e_+\,\mathrm{H}^2(W,k_!f_!f^*\mathcal{F})] = [e_+\,\mathrm{R}\,\Gamma(W,j_!f_!f^*\mathcal{F})],$$
$$[e_-\,\mathrm{H}^2(W,k_!f_!f^*\mathcal{F}(1))] = [e_-\,\mathrm{R}\,\Gamma(W,j_!f_!f^*\mathcal{F}(1))],$$

Proof. We give the argument for $X \coloneqq e_+\,\mathrm{H}^2(W,k_!f_!f^*\mathcal{F})$; the proof of the other cases is essentially the same. The $\Lambda[[G]]$-module X is the only nonvanishing cohomology group of the perfect complex of $\Lambda[[G]]$-modules

$$P^\bullet \coloneqq \varprojlim_{I\in\mathfrak{I}_{\Lambda[[G]]}} e_+\,\mathrm{R}\,\Gamma(W,j_!(f_!f^*\mathcal{F})_I).$$

Since for any simple $\Lambda[[G]]$-module M,

$$M\otimes^{\mathbb{L}}_{\Lambda[[G]]}P^\bullet \cong e_+\,\mathrm{R}\,\Gamma(W,j_!M\otimes_{\Lambda[[G]]/\operatorname{Jac}(\Lambda[[G]])}(f_!f^*\mathcal{F})_{\operatorname{Jac}(\Lambda[[G]])})$$

has no cohomology except in degrees 1 and 2, we conclude that there exists a strictly perfect complex of $\Lambda[[G]]$-modules concentrated in degrees -1 and 0 and quasi-isomorphic to X.

To show that X is a projective $\Lambda[[H]]$-module, it is sufficient to show that for any simple $\Lambda[[H]]$-module M, the complex $M\otimes^{\mathbb{L}}_{\Lambda[[H]]}P^\bullet$ has no cohomology in degree 1. We may assume that Λ and H are finite. Write $g\colon U_{F_\infty}\to U_{F_{\mathrm{cyc}}}$ for the Galois covering of $U_{F_{\mathrm{cyc}}}$ with Galois group H. Then

$$M\otimes^{\mathbb{L}}_{\Lambda[[H]]}P^\bullet[1] \cong e_+\,\mathrm{R}\,\Gamma(W_{F_{\mathrm{cyc}}},j_!M\otimes_{\Lambda[H]}g_!g^*\mathcal{F})$$

and

$$e_+\,\mathrm{H}^0(W_{F_{\mathrm{cyc}}},j_!M\otimes_{\Lambda[H]}g_!g^*\mathcal{F}) = 0$$

as desired.

We then have

$$[X] = [P^\bullet] = [e_+\,\mathrm{R}\,\Gamma(W,j_!f_!f^*\mathcal{F})]$$

in $\mathrm{K}_0(\Lambda[[G]],\Lambda[[G]]_S)$. □

9. Noncommutative Euler factors

Assume as before that F_∞/F is an admissible extension of a number field F which is unramified over a dense open subscheme U of X and write $f\colon U_{F_\infty}\to U$ for the system of Galois coverings of U corresponding to F_∞/F. Let W be another dense open subscheme of X containing U, but not the places over p and let $k\colon U\to W$ denote the corresponding open immersion. We consider a complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. As the complexes

$$\mathrm{R}\,\Gamma(x,i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ for $i\colon x\to W$ a closed point, we conclude that the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)\xrightarrow{\mathrm{id}-\mathfrak{F}_x}\mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)$$

is in fact a weak equivalence in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Hence, it gives rise to an element in $\mathrm{K}_1(\Lambda[[G]]_S)$.

Definition 9.1. The noncommutative Euler factor of $\mathrm{R}\,k_*\mathcal{F}^\bullet$ at x, denoted by

$$\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet),$$

is the inverse of the class of the above weak equivalence in $\mathrm{K}_1(\Lambda[[G]]_S)$.

Note that $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$ is independent of our specific choice of a geometric point above x. Indeed, by (6.1) and Relation (R5) in [Wit14, Def. A.4], we conclude that the classes $[\mathrm{id} - \mathfrak{F}_x]$ and $[\mathrm{id} - \mathfrak{F}'_x]$ agree in $\mathrm{K}_1(\Lambda[[G]]_S)$. Moreover, $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$ does not change if we enlarge W by adding points not lying over p or shrink U by removing a finite set of points different from x.

Proposition 9.2. *The noncommutative Euler factor is a characteristic element for* $\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$*:*

$$\partial \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) = -[\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S)$*.*

Proof. The complex $\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$ is weakly equivalent to the cone of the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\mathrm{id} - \mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

shifted by one. Hence, the result follows from the explicit description of ∂ given in [Wit14, Thm. A.5]. □

Definition 9.3. For a generator $\gamma \in \Gamma$, we define the local modification factor at x to be the element

$$M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) \coloneqq \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) s_\gamma([\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet)]).$$

in $\mathrm{K}_1(\Lambda[[G]])$.

We obtain the following transformation properties.

Proposition 9.4. *With* $k: U \to W$ *as above, let* Λ *be any adic* $\mathbb{Z}_p$*-algebra and let* $\mathcal{F}^\bullet$ *be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$*.*

1. *Let* Λ' *be another adic* $\mathbb{Z}_p$*-algebra. For any complex* $P^\bullet$ *of* Λ'*-*$\Lambda[[G]]$*-bimodules which is strictly perfect as complex of* Λ'*-modules we have*
$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$
in $\mathrm{K}_1(\Lambda'[[G]]_S)$ *and*
$$\Psi_{P[[G]]^{\delta\bullet}}(M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$
in $\mathrm{K}_1(\Lambda'[[G]])$*.*
2. *Let* F'_∞/F *be an admissible subextension of* F_∞/F *with Galois group* G'*. Then*
$$\Psi_{\Lambda[[G']]}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{L}_{F'_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$$
in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*
$$\Psi_{\Lambda[[G']]}(M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = M_{F'_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$$
in $\mathrm{K}_1(\Lambda[[G']])$*.*
3. *Let* F'/F *be a finite extension inside* F_∞/F*. Set* $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$*. Write* $f_{F'}: U_{F'} \to U$ *for the corresponding étale covering and* $x_{F'}$ *for the fibre in* $\operatorname{Spec} \mathcal{O}_{F'}$ *above* x*. Let* $G' \subset G$ *be the Galois group of the admissible extension* F_∞/F' *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G']]$*-*$\Lambda[[G]]$*-bimodule. Then*
$$\Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* f_{F'}^* \mathcal{F}^\bullet)$$
in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*
$$\Psi_{\Lambda[[G]]}\big(M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} M_{F_\infty/F',\gamma^r}(y, \mathrm{R}\,k_* f_{F'}^* \mathcal{F}^\bullet)$$
in $\mathrm{K}_1(\Lambda[[G']])$*.*

(4) *With the notation of* (3), *assume that* $\mathcal{G}^\bullet$ *is a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U_{F'},\Lambda)$ *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G]]$-$\Lambda[[G']]$-*bimodule. Then*

$$\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* \mathcal{G}^\bullet)\big) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* f_{F'\,*}\mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$ *and*

$$\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(M_{F_\infty/F',\gamma^r}(y, \mathrm{R}\,k_* \mathcal{G}^\bullet)\big) = M_{F_\infty/F',\gamma}(x, \mathrm{R}\,k_* f_{F'\,*}\mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]])$.

Proof. Note that the functor Ψ commutes up to weak equivalences with $\mathrm{R}\,\Gamma$, i^*, and $\mathrm{R}\,k_*$ [Wit08, 5.5.7] and apply Proposition 7.3 and Proposition 4.1. Part (1) and (2) are direct consequences.

For Part (3), we additionally need the same reasoning as in the proof of [Wit14, Thm. 8.4.(3)] to verify that for any $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'},\Lambda)$

$$[\mathrm{id} - \mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(y\times_x \hat{x}, \mathrm{R}\,k_* g_! g^* \mathcal{G}^\bullet)] = [\mathrm{id} - \mathfrak{F}_y \circlearrowleft \mathrm{R}\,\Gamma(\hat{y}, \mathrm{R}\,k_* g_! g^* f_{F'}^* \mathcal{G}^\bullet)] \tag{9.1}$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$. Here, $g\colon U_{F_\infty}\to U_{F'}$ denotes the system of coverings induced by f. This implies the formula for $\Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big)$. Moreover, we have a weak equivalence

$$\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \cong \mathrm{R}\,\Gamma(x_{F'}, \mathrm{R}\,k_* g_! g^* f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$. In particular,

$$s_{\gamma^r}([\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)]) = \prod_{y\in x_{F'}} s_{\gamma^r}([\mathrm{R}\,\Gamma(y, \mathrm{R}\,k_* g_! g^* f_{F'}^*\mathcal{F}^\bullet)])$$

from which the formula for $\Psi_{\Lambda[[G]]}\big(M_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big)$ follows.

For Part (4) we use (9.1) to show

$$\begin{aligned}\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_*\mathcal{G}^\bullet)\big) &= \\ &= \Psi_{\Lambda[[G]]}([\mathrm{id} - \mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(x_{F'}\times_x \hat{x}, \mathrm{R}\,k_* g_! g^*\mathcal{G}^\bullet)]^{-1}) \\ &= [\mathrm{id} - \mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,k_* f_! f^* f_{F'\,*}\mathcal{G}^\bullet)]^{-1} \\ &= \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* f_{F'\,*}\mathcal{G}^\bullet).\end{aligned}$$

On the other hand, we also have

$$\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x_{F'}, \mathrm{R}\,k_* g_! g^*\mathcal{G}^\bullet) \cong \mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^* f_{F'\,*}\mathcal{G}^\bullet),$$

thence the formula for the local modification factors. □

10. Euler factors for the cyclotomic extension

In the case $F_\infty = F_{\mathrm{cyc}}$, we can give a different description of $\mathcal{L}_{F_\infty/F}(x,\mathcal{F}^\bullet)$. We will undergo the effort to allow arbitrary adic $\mathbb{Z}_p$-algebras Λ as coefficient rings, but in the end, we will use the results only in the case that Λ is the valuation ring in a finite extension of $\mathbb{Q}_p$. If one restricts to this case, some of the technical constructions that follow may be skipped.

Let $\Lambda[t]$ be the polynomial ring over Λ in the indeterminate t that is assumed to commute with the elements of Λ. In Section 13 we define a Waldhausen category $w_t\mathbf{P}(\Lambda[t])$: The objects are perfect complexes of $\Lambda[t]$-modules and cofibrations are injective morphism of complexes such that the cokernel is again perfect. A weak equivalence is a morphism $f\colon P^\bullet\to Q^\bullet$ of perfect complexes of $\Lambda[t]$-modules such that $\Lambda\otimes^{\mathbb{L}}_{\Lambda[t]} f$ is a quasi-isomorphism of complexes of Λ-modules. Here, Λ is considered as a Λ-$\Lambda[t]$-bimodule via the augmentation map and $\Lambda\otimes^{\mathbb{L}}_{\Lambda[t]}\cdot$ denotes the total derived tensor product as functor between the derived categories.

If Λ is noetherian, then the subset

$$S_t \coloneqq \{f(t) \in \Lambda[t] \mid f(0) \in \Lambda^\times\} \subset \Lambda[t]$$

is a left and right denominator set, the localisation $\Lambda[t]_{S_t}$ is semilocal and $\Lambda[t] \to \Lambda[t]_{S_t}$ induces an isomorphism

$$\mathrm{K}_1(w_t\mathbf{P}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t})$$

(Proposition 13.1). For commutative adic rings, which are always noetherian [War93, Cor. 36.35], we may further identify

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \cong \Lambda[t]^\times_{S_t}$$

via the determinant map. In general, S_t is not a left or right denominator set. We then take

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \coloneqq \mathrm{K}_1(w_t\mathbf{P}(\Lambda[t]))$$

as a definition.

For any adic $\mathbb{Z}_p$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_p$, the ring homomorphism

$$\mathrm{ev}_\gamma\colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

induces a homomorphism

$$\mathrm{ev}_\gamma\colon \mathrm{K}_1(\Lambda[t]_{S_t}) \to \mathrm{K}_1(\Lambda[[\Gamma]]_S)$$

(Proposition 13.2). In the noetherian case, the proof boils down to a verification that $\mathrm{ev}_\gamma(S_t) \subset S$.

Definition 10.1. For $\mathcal{F}^\bullet = (\mathcal{F}^\bullet_I)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ we define

$$L(x, \mathrm{R}\,k_*\mathcal{F}^\bullet, t) \coloneqq [P^\bullet \xrightarrow{\mathrm{id}-t\mathfrak{F}_x} P^\bullet]^{-1} \in \mathrm{K}_1(\Lambda[t]_{S_t})$$

where

$$P^\bullet = \Lambda[t] \otimes_\Lambda \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet).$$

For any $1 \neq \gamma \in \Gamma$, we write $L(x, \mathrm{R}\,k_*\mathcal{F}^\bullet, \gamma)$ for the image of $L(x, \mathrm{R}\,k_*\mathcal{F}^\bullet, t)$ under

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\mathrm{ev}_\gamma} \mathrm{K}_1(\Lambda[[\Gamma]]_S).$$

Since the endomorphism $\mathrm{id} - t\mathfrak{F}_x$ is canonical, it follows easily from the relations in [Wit14, Def. A.4] that $L(x, \mathcal{F}^\bullet, t)$ does only depend on the weak equivalence class of $\mathcal{F}^\bullet$ and is multiplicative on exact sequences. So, it defines a homomorphism

$$L(x, \mathrm{R}\,k_*(-), t)\colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)) \to \mathrm{K}_1(\Lambda[t]_{S_t}).$$

Proposition 10.2. *Let $\gamma_x \in \Gamma$ be the image of $\mathfrak{F}_x$ in Γ. Then*

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) = L(x, \mathrm{R}\,k_*\mathcal{F}^\bullet, \gamma_x^{-1}).$$

Proof. Since p is invertible on W, the extension F_{cyc}/F is unramified over W. By the smooth base change theorem applied to the étale morphism $f_K\colon W_K \to W$ for each finite subextension K/F of F_{cyc}/F and the quasicompact morphism $k\colon U \to W$ there exists a weak equivalence

$$f_! f^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet \cong \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(W,\Lambda)$. Hence, we may assume $x \in U = W$.

For any finite subextension K/F in F_{cyc}/F write x_K for the set of places of K lying over x and $g\colon x_{F_{\mathrm{cyc}}} \to x$ for the corresponding system of Galois covers. (We note that this system might not be admissible in the sense of [Wit14, Def. 2.6] for any base field $\mathbb{F} \subset k(x)$: for example if $F = \mathbb{Q}$ and $x = (\ell)$ with $\ell \neq p$ splitting in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.) By the proper base change theorem there exists a weak equivalence

$$i^* f_! f^*\mathcal{F}^\bullet \cong g_! g^* i^*\mathcal{F}^\bullet.$$

By our choice of the embedding $\overline{F} \subset \overline{F}_x$, we have a compatible system of morpisms $\operatorname{Spec} \overline{k(x)} \to x_K$ for each $K \subset F_{\mathrm{cyc}}$ and hence, distinguished isomorphisms

$$\alpha\colon \mathbb{Z}[\mathrm{Gal}(K/F)] \otimes_{\mathbb{Z}} \mathcal{M}_{\hat{x}} \to (g_{K!} g_K^* \mathcal{M})_{\hat{x}}$$

for the stalk $\mathcal{M}_{\hat{x}}$ in $\hat{x}$ of any étale sheaf $\mathcal{M}$ on x. The action of the Frobenius $\mathfrak{F}_x$ on the righthand side corresponds to the operation of $\cdot\gamma_x^{-1} \otimes \mathfrak{F}_x$ on the lefthand side. By compatibility, we may extend α to an isomorphism

$$\alpha\colon \Psi_{\Lambda[[\Gamma]]} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathcal{F}^\bullet) \cong \mathrm{R}\,\Gamma(\hat{x}, g_! g^* i^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma]])$. Hence,

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) = [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Psi_{\Lambda[[\Gamma]]} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathcal{F}^\bullet)]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. Furthermore, we may choose a strictly perfect complex of Λ-modules $P^\bullet$ with an endomorphism f and a quasi-isomorphism

$$\beta\colon P^\bullet \to \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathcal{F}^\bullet)$$

under which f and $\mathfrak{F}_x$ are compatible up to chain homotopy [Wit08, Lemma 3.3.2]. Hence, using [Wit08, Lemma 3.1.6], we may conclude

$$[\mathrm{id} - tf \circlearrowright \Lambda[t]_{S_t} \otimes_\Lambda P^\bullet]^{-1} = L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, t)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, \gamma_x^{-1}) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. □

We will make this construction a little more explicit in the case that $\mathcal{F} = (\mathcal{F}_I)_{I \in \mathfrak{I}_\Lambda}$ is a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ concentrated in degree 0, i. e. for each $I \in \mathfrak{I}_\Lambda$, the object $\mathcal{F}_I$ is a constructible flat étale sheaf of Λ/I-modules such that $\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I = \mathcal{F}_J$ for $I \subset J$.

If $x \in U$, then there is an obvious weak equivalence

$$(\mathcal{F}_{I,\hat{x}})_{I \in \mathfrak{I}_\Lambda} \cong \mathrm{R}\,\Gamma(\hat{x}, i_* \mathrm{R}\,k_* \mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ compatible with the operation of the Frobenius $\mathfrak{F}_x$ on both sides. Moreover,

$$M \coloneqq \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathcal{F}_{I,\hat{x}}$$

is a finitely generated, projective Λ-module such that

$$\Lambda/I \otimes_\Lambda M = \mathcal{F}_{I,\hat{x}}.$$

Hence, we have

$$L(x, \mathrm{R}\,k_* \mathcal{F}, t) = [\Lambda[t] \otimes_\Lambda M \xrightarrow{\mathrm{id} - t\mathfrak{F}_x} \Lambda[t] \otimes_\Lambda M]^{-1} \tag{10.1}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_* \mathcal{F}) = [\Lambda[[\Gamma]] \otimes_\Lambda M \xrightarrow{\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x} \Lambda[[\Gamma]] \otimes_\Lambda M]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. In particular, if Λ is commutative, then the isomorphism

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\det} \Lambda[t]_{S_t}^\times$$

sends $L(x, \mathrm{R}\,k_* \mathcal{F}, t)$ to the inverse of the reverse characteristic polynomial of the geometric Frobenius operation on M.

If $x \in W - U$, we consider the perfect complex of Λ-modules

$$P^\bullet \coloneqq \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(F_x^{\mathrm{nr}}, \eta_x^* \mathcal{F}_I).$$

We may then use Lemma 6.3 to identify $L(x,\mathcal{F},t)$ with the class

$$[\Lambda[T]\otimes_\Lambda P^\bullet \xrightarrow{\mathrm{id}-t\mathfrak{F}_x} \Lambda[t]\otimes_\Lambda P^\bullet]^{-1}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$. By construction, $P^\bullet$ may also be canonically identified with the homogenous continuous cochain complex

$$X^\bullet_{\mathrm{cts}}(\mathrm{Gal}(\overline{F}_x/F_x),N)^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}$$

(in the notation of [NSW00, Ch. II, §7]) of the compact $\mathrm{Gal}(\overline{F}_x/F_x)$-module N corresponding to $\eta_x^*\mathcal{F}$. Let F_x^{tr} be the maximal tame extension of F_x inside $\overline{F}_x$ and set $N_{\mathrm{tr}} \coloneqq N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{tr}})}$. Since x does not lie over p, N_{tr} is a direct summand of the finitely generated and projective Λ-module N. The inflation map provides a quasi-isomorphism

$$X^\bullet_{\mathrm{cts}}(\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x),N_{\mathrm{tr}})^{\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x^{\mathrm{nr}})} \to P^\bullet.$$

Let τ be a topological generator of $\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x^{\mathrm{nr}})$ and $\varphi\in\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x)$ a lift of the geometric Frobenius $\mathfrak{F}_x$. Then τ and φ generate $\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x)$ topologically and

$$\varphi\tau\varphi^{-1} = \tau^{-q}$$

with $q=q_x$ the number of elements of $k(x)$ [NSW00, Thm. 7.5.3]. We define a strictly perfect complex $D_x^\bullet(N)$ of Λ-modules with an action of $\mathfrak{F}_x$ as follows: For $k\neq 0,1$ we set $D_x^k(N)\coloneqq 0$. As Λ-modules we have $D_x^0(N)=D_x^1(N)=N_{\mathrm{tr}}$ and the differential is given by $\mathrm{id}-\tau$. The geometric Frobenius $\mathfrak{F}_x$ acts on $D_x^0(N)$ via φ and on $D_x^1(N)$ via

$$\varphi\left(\frac{1-\tau^q}{1-\tau}\right)\in\Lambda[[\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x)]]^\times.$$

There is a quasi-isomorphism compatible with the $\mathfrak{F}_x$-operation

$$\alpha\colon D_x^\bullet(N)\to X^\bullet_{\mathrm{cts}}(\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x),N_{\mathrm{tr}})^{\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x^{\mathrm{nr}})}$$

given by

$$\alpha(n)\colon G\to N_{\mathrm{tr}},\qquad \tau^a\varphi^b\mapsto\tau^a n \qquad \text{for } n\in D^0(N),$$

$$\alpha(n)\colon G\times G\to N_{\mathrm{tr}},\qquad (\tau^a\varphi^b,\tau^c\varphi^d)\mapsto\frac{\tau^c-\tau^a}{1-\tau}n \qquad \text{for } n\in D^1(N).$$

Note that

$$\frac{\tau^c-\tau^a}{1-\tau}=\tau^c\sum_{n=1}^{\infty}\binom{a-c}{n}(\tau-1)^{n-1}$$

is a well-defined element of $\Lambda[[\mathrm{Gal}(F_x^{\mathrm{tr}}/F_x)]]$ for any $a,c\in\mathbb{Z}_p$.

We conclude that for $x\in W-U$,

$$\begin{aligned}L(x,\mathrm{R}\,k_*\mathcal{F},t) =&[\Lambda[t]\otimes_\Lambda D^0(N)\xrightarrow{1-t\mathfrak{F}_x}\Lambda[t]\otimes_\Lambda D^0(N)]^{-1}\\ &[\Lambda[t]\otimes_\Lambda D^1(N)\xrightarrow{1-t\mathfrak{F}_x}\Lambda[t]\otimes_\Lambda D^1(N)]\end{aligned}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\begin{aligned}\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\mathrm{R}\,k_*\mathcal{F}) =&[\Lambda[[\Gamma]]\otimes_\Lambda D^0(N)\xrightarrow{1-\gamma_x^{-1}\otimes\mathfrak{F}_x}\Lambda[[\Gamma]]\otimes_\Lambda D^0(N)]^{-1}\\ &[\Lambda[[\Gamma]]\otimes_\Lambda D^1(N)\xrightarrow{1-\gamma_x^{-1}\otimes\mathfrak{F}_x}\Lambda[[\Gamma]]\otimes_\Lambda D^1(N)].\end{aligned}$$

If the order of the image of $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ in the automorphism group of M is prime to p, (for example if $\mathcal{F}$ is smooth at x) then

$$N_{\mathrm{nr}}\coloneqq N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}$$

is a direct summand of N_{tr} and hence, also finitely generated and projective as Λ-module. Our formula then simplifies to

$$L(x, \mathrm{R}\, k_* \mathcal{F}, t) = [\Lambda[t] \otimes_\Lambda N_{\mathrm{nr}} \xrightarrow{1 - t\mathfrak{F}_x} \Lambda[t] \otimes_\Lambda N_{\mathrm{nr}}]^{-1} \\ [\Lambda[t] \otimes_\Lambda N_{\mathrm{nr}} \xrightarrow{1 - qt\mathfrak{F}_x} \Lambda[t] \otimes_\Lambda N_{\mathrm{nr}}]. \tag{10.2}$$

Hence, if Λ is commutative, then $\det L(x, \mathcal{F}, t)$ is the quotient of the reverse characteristic polynomials of the geometric Frobenius operations on $N_{\mathrm{nr}}(-1)$ and N_{nr}.

11. Artin representations

From now on, we let F denote a totally real number field. Consider an Artin representation $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_n(\mathcal{O}_C)$ (i. e. with open kernel) over the valuation ring $\mathcal{O}_C$ of a finite extension field C of $\mathbb{Q}_p$ inside a fixed algebraic closure $\overline{\mathbb{Q}}_p$. Assume for simplicity that ρ is unramified over W: For each $x \in W$, $\rho \restriction_{\mathrm{Gal}(\overline{F}_x / F_x^{\mathrm{nr}})}$ is trivial. Then ρ corresponds to the smooth $\mathcal{O}$-sheaf $\mathcal{M}(\rho)$ on $U \subset W$ defined by (7.1) and therefore, to an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \mathcal{O}_C)$.

The augmentation map $\varphi\colon \mathcal{O}_C[[\Gamma]] \to \mathcal{O}_C$ extends to a map

$$\varphi\colon \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S) \cong \mathcal{O}_C[[\Gamma]]_S^\times \to \mathbb{P}^1(C).$$

Indeed, let $\frac{a}{s} \in \mathcal{O}_C[[\Gamma]]_S^\times$. Since $\mathcal{O}_C[[\Gamma]]$ is a unique factorisation domain and the augmentation ideal is a principal prime ideal, we may assume that not both a and s are contained in the augmentation ideal. Hence, we obtain a well-defined element

$$\varphi(\frac{a}{s}) \coloneqq [\varphi(a) : \varphi(s)] \in \mathbb{P}^1(C) = C \cup \{\infty\}$$

Note that this map agrees with φ' in [Kak13, §2.4]. From (10.1) and (10.2) we conclude

$$\varphi(\mathcal{L}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n))) = \\ \begin{cases} [1 : \det(1 - \rho(\mathfrak{F}_x) q_x^{-n})] & \text{if } x \in U, \\ [\det(1 - \rho(\mathfrak{F}_x) q_x^{1-n}) : \det(1 - \rho(\mathfrak{F}_x) q_x^{-n})] & \text{if } x \in W - U, \end{cases}$$

where q_x denotes the number of elements of the residue field $k(x)$. We have $\det(1 - \rho(\mathfrak{F}_x) q_x^{-n}) = 0$ if and only if $n = 0$ and $\rho \restriction_{\mathrm{Gal}(F_x^{\mathrm{nr}}/F_x)}$ contains the trivial representation as a subrepresentation.

Set $\Sigma \coloneqq \mathrm{Spec}(\mathcal{O}_F) - W$, $T \coloneqq W - U$ and let $\alpha\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$ be an embedding of $\overline{\mathbb{Q}}_p$ into the complex numbers. We can then associate to the complex Artin representation $\alpha \circ \rho$ the classical Σ-truncated T-modified Artin L-function

$$L_{\Sigma,T}(\alpha \circ \rho, s) \coloneqq \prod_{x \in W} \det(1 - \alpha \circ \rho(\mathfrak{F}_x) q_x^{-s})^{-1} \prod_{x \in T} \det(1 - \alpha \circ \rho(\mathfrak{F}_x) q_x^{1-s}).$$

Note that we follow the geometric convention of using the geometric Frobenius in the definition of the Artin L-function as in [CL73]. With this convention, we have

$$L_{\Sigma,T}(\alpha \circ \rho, n) = \prod_{x \in W} \alpha(\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n))))$$

for all $n \in \mathbb{Z}$, $n > 1$.

By [CL73, Cor 1.4] there exists for each $n \in \mathbb{Z}$, $n < 0$ a well defined number $L_{\Sigma,T}(\rho, n) \in C$ such that

$$\alpha(L_{\Sigma,T}(\rho, n)) = L_{\Sigma,T}(\alpha \circ \rho, n) \in \mathbb{C}$$

Consequently,

$$L_{\Sigma',T'}(\rho, n) = L_{\Sigma,T}(\rho, n) \prod_{x \in \Sigma' \cup T' - \Sigma \cup T} \varphi(\mathcal{L}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n)))^{-1}$$

if $\Sigma \subset \Sigma'$ and $T \subset T'$ with disjoint subsets Σ' and T' of $X = \operatorname{Spec} \mathcal{O}_F$ such that ρ is unramified over $X - \Sigma'$ and all the primes over p are contained in Σ'.

Let $\kappa_F \colon \mathrm{Gal}_F \to \mathbb{Z}_p^\times$ denote the cylotomic character such that

$$\sigma(\zeta) = \zeta^{\kappa_F(\sigma)}$$

for every $\sigma \in \mathrm{Gal}_F$ and $\zeta \in \mu_{p^\infty}$. Further, we write $\omega_F \colon \mathrm{Gal}_F \to \mu_{p-1}$ for the Teichmüller character, i. e. the composition of κ_F with the projection $\mathbb{Z}_p^\times \to \mu_{p-1}$. Finally, we set $\epsilon_F \coloneqq \kappa_F \omega^{-1}$ and note that ϵ_F factors through $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$.

Assume that ρ factors through the Galois group of a totally real field. Then $\mathcal{M}(\rho)$ is smooth at ∞. Under Assumption 7.4 it follows from [Gre83] and from the validity of the classical main conjecture that there exists unique elements

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\, k_* \mathcal{M}(\rho)(1)) \in \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S)$$

such that for every $n > 0$

$$\varphi(\Phi_{\epsilon_F^{-n}}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\, k_* \mathcal{M}(\rho)(1)))) = L_{\Sigma,T}(\rho\omega_F^n, 1-n). \tag{11.1}$$

Beware that Greenberg uses the arithmetic convention for L-functions.

Definition 11.1. Let $\gamma \in \Gamma$ be a generator. We define the global modification factors of $\mathrm{R}\, k_* \mathcal{M}(\rho)(1)$ to be

$$\begin{aligned} M_{F_{\mathrm{cyc}}/F,\gamma}(W, \mathrm{R}\, k_* \mathcal{M}(\rho)(1)) \coloneqq & \mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\, k_* \mathcal{M}(\rho)(1)) \\ & s_\gamma([\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \mathcal{M}(\rho)(1))]) \end{aligned}$$

12. Noncommutative L-functions for Λ-adic sheaves

Throughout this section, Assumption 7.4 is in effect. We recall the main theorem of [Kak13].

Theorem 12.1. *Let $U \subset \operatorname{Spec} \mathcal{O}_F$ be a dense open subscheme with complement Σ and assume that p is invertible on U. Assume that F_∞/F is a really admissible extension which is unramfied over U and that $G = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group. Then there exists elements $\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[G]]_S)$, unique up to elements in $\mathrm{SK}_1(\mathbb{Z}_p[[G]])$, such that*

(1)
$$\partial \mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_p)_U(1))],$$
(2) *For any Artin representation ρ factoring through G*
$$\Phi_\rho(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))$$

Proof. This is [Kak13, Thm. 2.11] translated into our notations. Recall that our $\Phi_{\rho\kappa_F^{-n}}$ corresponds to $\Phi_{\check{\rho}\kappa_F^n}$ in the notation of the cited article. Moreover, Kakde uses the arithmetic convention in the definition of L-values. Further, note that the p-adic L-function $\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))$ is uniquely determined by the values $\varphi(\Phi_{\epsilon_F^n}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))))$ for $n < 0$ and $n \equiv 0 \mod p-1$. Finally, Kakde's complex $\mathcal{C}(F_\infty/F)$ corresponds to $\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_p)_U(1))$ shifted by 3 and therefore, the images of the two complexes under ∂ differ by a sign, but at the same time, his definition of ∂ differs by a sign from ours. □

We will improve this theorem as follows. Let $\Xi = \Xi_F$ be the set of pairs (U, F_∞) such that $U \subset \operatorname{Spec} \mathcal{O}_F$ is a dense open subscheme with p invertible on U and F_∞/F is a really admissible extension unramified over U.

Theorem 12.2. *Let $\gamma \in \Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a generator. There exists a unique family of elements*

$$\left(M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))\right)_{(U,F_\infty) \in \Xi}$$

such that

(1) $M_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$,
(2) *if* $U \subset U'$ *with complement* Σ *and* $(U,F_\infty),(U',F_\infty) \in \Xi$ *then*

$$M_{F_\infty/F,\gamma}(U',(\mathbb{Z}_p)_{U'}(1)) = M_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1)) \prod_{x\in\Sigma} M_{F_\infty/F,\gamma}(x,(\mathbb{Z}_p)_{U'}(1)),$$

(3) *if* $(U,F_\infty),(U,F'_\infty) \in \Xi$ *and* $F'_\infty \subset F_\infty$ *is a subfield, then*

$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(M_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1))) = M_{F'_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1)),$$

(4) *if* $(U,F_\infty) \in \Xi$ *and* $\rho\colon \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)$ *is an Artin representation, then*

$$\Phi_\rho(M_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1))) = M_{F_{\mathrm{cyc}}/F,\gamma}(U,\mathcal{M}(\rho)(1)).$$

Proof. Uniqueness: Assume that $m_k(U,F_\infty)$, $k=1,2$ are two families with the listed properties. Then

$$d(F_\infty) \coloneqq m_2(U,F_\infty)^{-1} m_1(U,F_\infty)$$

does not depend on U.

Let $(U,F_\infty) \in \Xi$ be any pair such that $F_\infty/F_{\mathrm{cyc}}$ is finite and write $f\colon U_{F_\infty} \to U$ for the system of coverings of U associated to F_∞/F. Then the elements

$$m_i(U,F_\infty) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))])$$

both agree with $\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1))$ modulo $\mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$. Hence,

$$d(F_\infty) \in \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]).$$

By Corollary 5.4, we may find a pair $(U',F'_\infty) \in \Xi$ such that F'_∞/F_∞ is finite, $U' \subset U$, and

$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}\colon \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]) \to \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$$

is the zero map. We conclude $d(F_\infty) = 1$ for all (U,F_∞) with $F_\infty/F_{\mathrm{cyc}}$ finite. Now for any really admissible extension F_∞/F,

$$\mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]) = \varprojlim_{F'_\infty} \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$$

where F'_∞ runs through the really admissible subextensions of F_∞/F with $F'_\infty/F_{\mathrm{cyc}}$ finite. We conclude $d(F_\infty) = 1$ in general.

Existence: For $(U,F_\infty) \in \Xi$ with $F_\infty/F_{\mathrm{cyc}}$ finite, choose (U',F'_∞) as above and any $m \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$ such that

$$m s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]) \equiv \mathcal{L}_{F'_\infty/F}(U',(\mathbb{Z}_p)_{U'}(1)) \mod \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$$

Define

$$M_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1)) \coloneqq \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}(m) \prod_{x\in U-U'} M_{F_\infty/F,\gamma}(x,(\mathbb{Z}_p)_U(1)).$$

□

Corollary 12.3. *There exists a unique family of elements*

$$\left(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1))\right)_{(U,F_\infty)\in\Xi}$$

such that

(1) $\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]_S)$,
(2) *if* $(U,F_\infty) \in \Xi$ *and* $f\colon U_{F_\infty} \to U$ *denotes the associated system of coverings, then*

$$\partial \mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]$$

(3) *if* $U' \subset U$ *with complement* Σ *and* $(U', F_\infty), (U, F_\infty) \in \Xi$ *then*
$$\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) = \mathcal{L}_{F_\infty/F}(U', (\mathbb{Z}_p)_{U'}(1)) \prod_{x\in\Sigma} \mathcal{L}_{F_\infty/F}(x, (\mathbb{Z}_p)_U(1)),$$
(4) *if* $(U, F_\infty), (U, F'_\infty) \in \Xi$ *and* $F'_\infty \subset F_\infty$ *is a subfield, then*
$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F'_\infty/F}(U, (\mathbb{Z}_p)_U(1)),$$
(5) *if* $(U, F_\infty) \in \Xi$ *and* $\rho\colon \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)$ *is an Artin representation, then*
$$\Phi_\rho(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1)).$$

Proof. Fix a generator $\gamma \in \Gamma$ and set
$$\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) := M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]).$$
If $(\ell(U, F_\infty))_{(U,F_\infty)\in\Xi}$ is a second family with the listed properties, then
$$\ell(U, F_\infty) s_\gamma([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]) = M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$$
by the uniqueness of $M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$. □

Let $\Theta = \Theta_F$ be the set of tripels (U, F_∞, Λ) such that $U \subset \mathrm{Spec}\,\mathcal{O}_F$ is a dense open subscheme with p invertible on U, F_∞/F is a really admissible extension unramified over U and Λ is an adic $\mathbb{Z}_p$-algebra.

Theorem 12.4. *Let* $\gamma \in \Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ *be a generator. There exists a unique family of homomorphisms*
$$\left(M_{F_\infty/F,\gamma}(U, (-)(1))\colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])\right)_{(U,F_\infty,\Lambda)\in\Theta}$$
such that

(1) *for any* $(U, F_\infty, \mathbb{Z}_p) \in \Theta$, $M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$ *is the element constructed in Theorem 12.2,*

(2) *if* $j\colon U' \to U$ *is an open immersion and* $(U', F_\infty, \Lambda), (U, F_\infty, \Lambda) \in \Theta$ *then*
$$M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = M_{F_\infty/F,\gamma}(U', j^*\mathcal{F}^\bullet(1)) \prod_{x\in U-U'} M_{F_\infty/F,\gamma}(x, \mathcal{F}^\bullet(1)),$$
for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

(3) *if* $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda) \in \Theta$ *and* $F'_\infty \subset F_\infty$ *is a subfield, then*
$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = M_{F'_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)),$$
for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

(4) *if* $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda') \in \Theta$ *and* $P^\bullet$ *is a complex of* Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-*bimodules, strictly perfect as complex of* Λ'-*modules then*
$$\Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta}(M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = M_{F_\infty/F,\gamma}(U, \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1))$$
for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

Proof. Applying (4) to the Λ/I-$\Lambda[[G]]$-bimodule $\Lambda/I[[G]]$ for any open two-sided ideal I of Λ and using
$$\mathrm{K}_1(\Lambda[[G]]) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{K}_1(\Lambda/I[[G]]),$$
we conclude that it is sufficient to consider triples $(U, F_\infty, \Lambda) \in \Theta$ with Λ a finite ring. So, let Λ be finite. Since $M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ depends only on the class of $\mathcal{F}^\bullet$ in $\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda))$, we may assume that $\mathcal{F}^\bullet$ is a bounded complex of flat constructible étale sheaves of Λ-modules. Using (2) we may shrink U until $\mathcal{F}^\bullet$ is a complex of locally constant étale sheaves. Hence, there exists a $(U, F'_\infty, \Lambda) \in \Theta$ such that F_∞/F is a subextension of F'_∞/F and such that the restriction of $\mathcal{F}^\bullet$ to U_K for some finite subextension K/F of F'_∞/F is a complex of constant sheaves. By (3),

we may replace F_∞ by F'_∞. We may then find a complex of Λ-$\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$-bimodules $P^\bullet$, strictly perfect as complex of Λ modules, such that

$$\Psi_{P^\bullet} f_! f^*(\mathbb{Z}_p)_U(1) \cong \mathcal{F}^\bullet(1)$$

[Wit14, Prop. 6.8]. By (4), the only possible definition of $M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ is

$$M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = \Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta}(M_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))).$$

□

Proposition 12.5. *Assume that γ, γ' are two generators of Γ. Then*

$$\frac{M_{F_\infty/F,\gamma}}{M_{F_\infty/F,\gamma'}}(U, \mathcal{F}^\bullet(1)) = \frac{s_{\gamma'}}{s_\gamma}([\mathrm{R}\,\Gamma_\mathrm{c}(U, f_! f^* \mathcal{F}^\bullet(1))])$$

for any $(U, F_\infty, \Lambda) \in \Theta$ and any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

Proof. By definition, this is true for the local modification factors and by Corollary 12.3 it is true for $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U$. Hence,

$$M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = \frac{s_{\gamma'}}{s_\gamma}([\mathrm{R}\,\Gamma_\mathrm{c}(U, f_! f^* \mathcal{F}^\bullet(1))]) M_{F_\infty/F,\gamma'}(U, \mathcal{F}^\bullet(1))$$

by Theorem 12.4. □

Theorem 12.6. *Let F'/F be a finite extension of totally real fields. Set $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$ and let $\gamma \in \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a generator. Assume that $(U, F_\infty, \Lambda) \in \Theta_F$ with $F' \subset F_\infty$ and write $f_{F'}: U_{F'} \to U$ for the associated covering. Then*

(1) *for every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$,*

$$M_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^* \mathcal{F}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]} M_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)),$$

(2) *for every $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'}, \Lambda)$,*

$$M_{F_\infty/F,\gamma}(U, f_{F'*} \mathcal{G}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]} M_{F_\infty/F',\gamma^r}(U_{F'}, \mathcal{G}^\bullet(1)).$$

Proof. We first note that for any complex $P^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, there exists an obvious isomorphism of complexes of $\Lambda'[[\mathrm{Gal}(F_\infty/F')]]$-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules

$$\Lambda'[[\mathrm{Gal}(F_\infty/F)]] \otimes_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta \cong P^\bullet[[\mathrm{Gal}(F_\infty/F')]]^\delta \otimes_{\Lambda[[\mathrm{Gal}(F_\infty/F')]]} \Lambda[[\mathrm{Gal}(F_\infty/F)]].$$

Hence,

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta} = \Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F')]]^\delta} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]} \tag{12.1}$$

as homomorphisms from $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ to $\mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F')]])$. Likewise, for a complex $Q^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F')]]$-bimodules, strictly perfect as complex of Λ'-modules, we have an equality

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{Q^\bullet[[\mathrm{Gal}(F_\infty/F')]]^\delta} = \Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}. \tag{12.2}$$

in $\mathrm{Hom}(\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F')]]), \mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F)]]))$.

In particular, we may reduce to the case of finite $\mathbb{Z}_p$-algebras Λ. By Proposition 9.4.(4) we may then shrink U until $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ may be assumed to be strictly perfect complexes of locally constant étale sheaves. Using the identities (12.1) and (12.2) again, we may reduce to the case $\Lambda = \mathbb{Z}_p$ and $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U$, $\mathcal{G}^\bullet = (\mathbb{Z}_p)_{U_{F'}}$. Moreover, we may assume that $F_\infty/F'_{\mathrm{cyc}}$ is a finite extension.

Let $g: U_{F_\infty} \to U_{F'}$ denote the restriction of $f: U_{F_\infty} \to U$. Write

$$M = \mathbb{Z}_p[\mathrm{Gal}(F_\infty/F') \backslash \mathrm{Gal}(F_\infty/F)]$$

for the $\mathbb{Z}_p$-$\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$-bimodule freely generated as $\mathbb{Z}_p$-module by the right cosets $\mathrm{Gal}(F_\infty/F)\sigma$ for $\sigma \in \mathrm{Gal}(F_\infty/F)$ and on which $\tau \in \mathrm{Gal}(F_\infty/F)$ operates by right multiplication. From Proposition 7.3 we conclude

$$\begin{aligned}\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}\,\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1)) &\cong \mathrm{R}\,\Gamma_c(U, f_{F'*} g_! g^* f_{F'}^*(\mathbb{Z}_p)_U(1)) \\ &\cong \mathrm{R}\,\Gamma_c(U_{F'}, g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1)),\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}\,\mathrm{R}\,\Gamma_c(U_{F'}, g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1))& \\ \cong \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}\,&\mathrm{R}\,\Gamma_c(U, f_{F'*} g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1)) \\ \cong \mathrm{R}\,\Gamma_c(U,& f_! f^* f_{F'*}(\mathbb{Z}_p)_U(1)) \\ \cong \Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\,&\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1)).\end{aligned}$$

Additionally, we note that

$$M_{F_\infty/F,\gamma}(U, f_{F'*}(\mathbb{Z}_p)_U(1)) = \Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta} M_{\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$$

by Theorem 12.4.

Hence, according to Proposition 4.1, it is sufficient to verify

(12.3)
$$\begin{aligned}\mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_{U_{F'}}(1)) &= \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)), \\ \Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) &= \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_U(1)).\end{aligned}$$

Let $C/\mathbb{Q}_p$ be a finite field extension and

$$\begin{aligned}\rho'&: \mathrm{Gal}(F_\infty/F') \to \mathrm{Gl}_n(\mathcal{O}_C) \\ \rho&: \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)\end{aligned}$$

be Artin representations. Write

$$\begin{aligned}\varphi_F&: \mathcal{O}_C[[\mathrm{Gal}(F_{\mathrm{cyc}}/F)]] \to \mathcal{O}_C \\ \varphi_{F'}&: \mathcal{O}_C[[\mathrm{Gal}(F'_{\mathrm{cyc}}/F')]] \to \mathcal{O}_C\end{aligned}$$

for the augmentation maps. We denote by $\mathrm{Ind}_{F'}^F \rho'$ and $\mathrm{Res}_F^{F'} \rho$ the induced an restricted representations, respectively.

Then for every $n \in \mathbb{Z}$

$$\varphi_{F'} \circ \Phi_{\rho'\epsilon_{F'}^n} \circ \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]} = \varphi_F \circ \Phi_{\mathrm{Ind}_{F'}^F \rho'\epsilon_{F'}^n} = \varphi_F \circ \Phi_{\epsilon_F^n \mathrm{Ind}_{F'}^F \rho'}$$

as maps from $\mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]_S)$ to $\mathbb{P}^1(C)$ and

$$\varphi_F \circ \Phi_{\rho\epsilon_F^n} \circ \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]} = \varphi_{F'} \circ \Phi_{\mathrm{Res}_F^{F'} \rho\epsilon_F^n} = \varphi_{F'} \circ \Phi_{\epsilon_{F'}^n \mathrm{Res}_F^{F'} \rho}$$

as maps from $\mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]])$ to $\mathbb{P}^1(C)$. From (11.1) and the transformation properties of the complex Artin L-functions with respect to inflation and restriction we conclude that for $n < -1$ and $\Sigma = \mathrm{Spec}\,\mathcal{O}_F - U$

$$\begin{aligned}\varphi_{F'} \circ \Phi_{\rho'\epsilon_{F'}^n}(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) &= \\ &= L_{\Sigma,\varnothing}(\omega_F^{-n}\,\mathrm{Ind}_{F'}^F \rho', 1+n) \\ &= L_{\Sigma_{F'},\varnothing}(\rho'\omega_{F'}^{-n}, 1+n) \\ &= \varphi_{F'} \circ \Phi_{\rho'}(\mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_{U_{F'}}(1))), \\ \varphi_F \circ \Phi_\rho(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_{U_{F'}}(1)) &= \\ &= L_{\Sigma_{F'},\varnothing}(\omega_{F'}^{-n}\,\mathrm{Res}_F^{F'} \rho, 1+n) \\ &= L_{\Sigma,\varnothing}(\omega_F^{-n}\,\mathrm{Ind}_F^{F'}\,\mathrm{Res}_F^{F'} \rho, 1+n) \\ &= \varphi_F \circ \Phi_{\mathrm{Ind}_F^{F'}\mathrm{Res}_F^{F'} \rho}(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)))) \\ &= \varphi_F \circ \Phi_\rho(\Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))).\end{aligned}$$

This implies that the two sides of equation (12.3) agree up to elements of the groups $\mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]])$ and $\mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$, respectively. Using Corollary 5.4 we find a suitable extension F'_∞/F_∞ such that the possible difference in (12.3) for F'_∞ vanishes if we take the images under

$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]}\colon \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F')]]) \to \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]) \quad \text{and}$$
$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}\colon \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]) \to \mathrm{SK}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]),$$

respectively. □

Definition 12.7. Let F be a totally real field, $k\colon U \to W$ be an open immersion of open dense subschemes of $\operatorname{Spec} \mathcal{O}_F$ such that p is invertible on W, and Λ be an adic $\mathbb{Z}_p$-algebra. Fix a generator $\gamma \in \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, and any really admissible extension F_∞/F unramified over U, we set

$$M_{F_\infty/F,\gamma}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) \coloneqq M_{F_\infty/F',\gamma}(U, \mathcal{F}^\bullet(1)) \prod_{x \in W-U} M_{F_\infty/F',\gamma}(x, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ and

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) \coloneqq M_{F_\infty/F,\gamma}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet(1))])$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]_S)$.

Note that we do not assume that F_∞/F is unramified over W. If it is unramified over W, then

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet(1)) = \mathrm{R}\,\Gamma_{\mathrm{c}}(W, f_! f^* \,\mathrm{R}\, k_* \mathcal{F}^\bullet(1))$$

and the two possible definitions of $M_{F_\infty/F,\gamma}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))$ agree. Moreover, by Proposition 12.5, $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))$ does not depend on the choice of γ.

Corollary 12.8. *Let F be a totally real field, $k\colon U \to W$ be an open immersion of open dense subschemes of* $\operatorname{Spec} \mathcal{O}_F$ *such that p is invertible on W, and Λ be an adic $\mathbb{Z}_p$-algebra. Fix a $\mathcal{F}^\bullet$ in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$*, and a really admissible extension F_∞/F unramified over U.*

(1) *Write $f\colon U_{F_\infty} \to U$ for the system of coverings associated to F_∞/F. Then*
$$\partial \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet(1))]$$
(2) *If $G^\bullet$ and $F^\bullet$ are weakly equivalent in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$*, then*
$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{G}^\bullet(1)).$$
(3) *If $0 \to \mathcal{F}'^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}''^\bullet \to 0$ is an exact sequence in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$*, then*
$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}'^\bullet(1)) \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}''^\bullet(1)).$$
(4) *If $i\colon x \to U$ is a closed point, then*
$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* i_* i^* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))$$
(5) *If W' is an open dense subscheme of* $\operatorname{Spec} \mathcal{O}_F$ *on which p is invertible and $k'\colon W \to W'$ is an open immersion, then*
$$\mathcal{L}_{F_\infty/F}(W', \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)) \prod_{x \in W'-W} \mathcal{L}_{F_\infty/F}(x, \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1)).$$
(6) *If F'_∞/F is a really admissible subextension of F_∞/F, then*
$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))) = \mathcal{L}_{F'_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)).$$
(7) *If Λ' is another adic $\mathbb{Z}_p$-algebra and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then*
$$\Psi_{P^\bullet[[\mathrm{Gal}(F_\infty/F)]]^\delta}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1)).$$

(8) *If F'/F is a finite extension inside F_∞ and $f_{F'}: U_{F'} \to U$ the associated covering, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1))) = \mathcal{L}_{F_\infty/F'}(W_{F'}, \mathrm{R}\,k_* f_{F'}^* \mathcal{F}^\bullet(1)).$$

(9) *With the notation of* (8), *if $\mathcal{G}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'}, \Lambda)$, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F'}(W_{F'}, \mathrm{R}\,k_*\mathcal{G}^\bullet(1))) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_* f_{F'*}\mathcal{G}^\bullet(1)).$$

13. Appendix: Localisation in polynomial rings

For the moment, let R be any associative ring with 1 and let $R[t]$ be the polynomial ring over R in one indeterminate t that commutes with the elements of R. Write $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$ for the Waldhausen categories of strictly perfect and perfect complexes of $R[t]$-modules. Consider R as a R-$R[t]$-bimodule via the augmentation map

$$R[t] \to R, \qquad t \mapsto 0.$$

We then define full subcategories

$$\mathbf{SP}^{w_t}(R[t]) \coloneqq \{P^\bullet \in \mathbf{SP}(R[t]) \mid R \otimes_{R[t]} P^\bullet \text{ is acyclic}\},$$

$$\mathbf{P}^{w_t}(R[t]) \coloneqq \{P^\bullet \in \mathbf{P}(R[t]) \mid P^\bullet \text{ is quasi-isomorphic to a complex in } \mathbf{SP}^{w_t}(R[t])\}.$$

These categories are in fact Waldhausen subcategories of $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, respectively, since they are closed under shifts and extensions [Wit08, 3.1.1]. Hence, we can construct new Waldhausen categories $w_t\mathbf{SP}(R[t])$ and $w_t\mathbf{P}(R[t])$ with the same objects, morphisms, and cofibrations as $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, but with weak equivalences being those morphisms with cone in $\mathbf{SP}^{w_t}(R[t])$ and $\mathbf{P}^{w_t}(R[t])$, respectively. By the Approximation Theorem [TT90, 1.9.1], the inclusion functor $w_t\mathbf{SP}(R[t]) \to w_t\mathbf{P}(R[t])$ induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(w_t\mathbf{P}(R[t]))$$

for all $n \geq 0$. It might be reassuring to know that, if R is noetherian, we can identify these K-groups for $n \geq 1$ with the K-groups of a localisation of $R[t]$: Set

$$S_t \coloneqq \{f(t) \in R[t] \mid f(0) \in R^\times\}$$

Proposition 13.1. *Assume that R is a noetherian. Then S_t is a left (and right) denominator set in the sense of* [GW04, Ch. 10] *such that the localisation $R[t]_{S_t}$ exists and is noetherian. Its Jacobson radical* $\mathrm{Jac}(R[t]_{S_t})$ *is generated by the Jacobson radical* $\mathrm{Jac}(R)$ *of R and t. In particular, if R is semilocal, then so is $R[t]_{S_t}$.*

Moreover, the category $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. In particular,

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(R[t]_{S_t})$$

for $n \geq 1$.

Proof. Clearly, the set S_t consists of nonzerodivisors, such that we only need to check the Ore condition:

$$\forall s \in S_t : \forall a \in R[t] : \exists x \in R[t] : \exists y \in S_t : xs = ya.$$

Moreover, we may assume that $s(0) = y(0) = 1$. Write

$$s = 1 - \sum_{i=1}^{\infty} s_i t^i, \quad a = \sum_{i=0}^{\infty} a_i t^i, \quad , x = \sum_{i=0}^{\infty} x_i t^i, \quad y = 1 + \sum_{i=1}^{\infty} y_i t^i.$$

and assume that $s_i = a_{i-1} = 0$ for $i > n$. Comparing coefficients, we obtain the recurrence equation

$$x_i = \sum_{j=0}^{i-1} x_j s_{i-j} + \sum_{j=1}^{i} y_j a_{i-j} + a_i = \sum_{j=1}^{i} y_j b_{i-j} + b_i \tag{13.1}$$

with

$$b_i \coloneqq \sum_{j=0}^{i-1} b_j s_{i-j} + a_i.$$

Write $B_i \coloneqq (b_{i-n+1}, \dots, b_i) \in R^n$ with the convention that $b_i = 0$ for $i < 0$. Then for $i \geq n$

$$B_i = B_{i-1}S = B_{n-1}S^{i-n+1}$$

with

$$S \coloneqq \begin{pmatrix} 0 & \dots & \dots & s_n \\ 1 & \ddots & & s_{n-1} \\ 0 & \ddots & 0 & \vdots \\ \vdots & & 1 & s_1 \end{pmatrix}.$$

Since R was assumed to be noetherian, there exists a $m \geq n$ and $y_n, \dots, y_m \in R$ such that

$$0 = \sum_{j=n}^{m} y_j B_{m-j} + B_m = \sum_{j=n}^{m} y_j B_{i-j} + B_i$$

for all $i \geq m$. Hence, we can find a solution $(x_i, y_i)_{i=0,1,2\dots}$ of equation (13.1) with $x_i = y_i = 0$ for $i > m$ and $y_i = 0$ for $i < n$. This shows that S_t is indeed a left denominator set such that $R[t]_{S_t}$ exists and is noetherian [GW04, Thm. 10.3, Cor. 10.16].

Let $N \subset R[t]$ be the semiprime ideal of $R[t]$ generated by t and the Jacobson ideal $\operatorname{Jac}(R)$ of R. Then S_t is precisely the set of elements of $\Lambda[t]$ which are units modulo N. In particular, the localisation N_{S_t} is a semiprime ideal of $R[t]_{S_t}$ such that

$$R[t]_{S_t}/N_{S_t} = R[t]/N = R/\operatorname{Jac}(R)$$

[GW04, Thm. 10.15, 10.18]. We conclude $\operatorname{Jac}(R[t]_{S_t}) \subset N_{S_t}$. For the other inclusion it suffices to note that for every $s \in S_t$ and every $n \in N$, the element $s + n$ is a unit modulo N.

The Nakayama lemma implies that for any noetherian ring R with Jacobson radical $\operatorname{Jac}(R)$, a strictly perfect complex of R-modules $P^\bullet$ is acyclic if and only if $R/\operatorname{Jac}(R) \otimes_R P^\bullet$ is acyclic. Hence, if $P^\bullet$ is a strictly perfect complex of $R[t]$-modules, then $R \otimes_{R[t]} P^\bullet$ is acyclic if and only if $R[t]_{S_t} \otimes_R P^\bullet$ is acyclic. This shows that $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. From the localisation theorem in [WY92] we conclude that the Waldhausen exact functor

$$w_t\mathbf{SP}(R[t]) \to \mathbf{SP}(R[t]_{S_t}), \qquad P^\bullet \mapsto R[t]_{S_t} \otimes_{R[t]} P^\bullet$$

induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \begin{cases} \mathrm{K}_n(R[t]_{S_t}) & \text{if } n > 0, \\ \operatorname{im}(\mathrm{K}_0(R[t]) \to \mathrm{K}_0(R[t]_{S_t})) & \text{if } n = 0. \end{cases}$$

□

The set S_t fails to be a left denominator set if $R = \mathbb{F}_p\langle\langle x, y\rangle\rangle$ is the power series ring in two noncommuting indeterminates: $a(1 - xt) = by$ has no solution with $a \in R[t]$, $b \in S_t$. Note also that a commutative adic ring is always noetherian [War93, Cor. 36.35]. In this case, S_t is the union of the complements of all maximal ideals of $\Lambda[t]$ containing t and the determinant provides an isomorphism

$$\mathrm{K}_1(w_t\mathbf{SP}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow[\cong]{\det} \Lambda[t]_{S_t}^\times.$$

For any adic $\mathbb{Z}_p$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_p$, we have a ring homomorphism

$$\operatorname{ev}_\gamma \colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

inducing homomorphisms $\mathrm{K}_n(\Lambda[t]) \to \mathrm{K}_n(\Lambda[[\Gamma]])$.

Proposition 13.2. *Assume that* $\gamma \neq 1$. *Then the ring homomorphism* ev_γ *induces homomorphisms*

$$\mathrm{ev}_\gamma\colon \mathrm{K}_n(w_t\mathbf{P}(\Lambda[[t]])) \cong \mathrm{K}_n(w_t\mathbf{SP}(\Lambda[t])) \to \mathrm{K}_n(w_1\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma]]))$$

for all $n \geq 0$.

Proof. It suffices to show that for any complex $P^\bullet$ in $\mathbf{SP}^{w_t}(\Lambda[t])$, the complex

$$Q^\bullet \coloneqq \Lambda[[\Gamma]] \otimes_{\Lambda[t]} P^\bullet$$

is perfect as complex of Λ-modules. We can check this after factoring out the Jacobson radical of Λ [Wit14, Prop. 4.8]. Hence, we may assume that Λ is semisimple, i. e.

$$\Lambda = \prod_{i=1}^{m} M_{n_i}(k_i)$$

where $M_{n_i}(k_i)$ is the algebra of $n_i \times n_i$-matrices over a finite field k_i of characteristic p. By the Morita theorem, the tensor product over Λ with the $\prod_i k_i$-Λ-bimodule

$$\prod_{i=1}^{m} k_i^{n_i}$$

induces equivalences of categories

$$\mathbf{SP}^{w_t}(\Lambda[t]) \to \mathbf{SP}^{w_t}\left(\prod_{i=1}^{n} k_i[t]\right),$$
$$\mathbf{PDG}^{\mathrm{cont},w_1}(\Lambda[[\Gamma]]) \to \mathbf{PDG}^{\mathrm{cont},w_1}(\prod_{i=1}^{n} k_i[[\Gamma]]).$$

Hence, we are reduced to the case

$$\Lambda = \prod_{i=1}^{m} k_i.$$

In this case, the set $S \subset \Lambda[[\Gamma]]$ defined in (2.1) consists of all nonzerodivisors of $\Lambda[[\Gamma]]$, i. e. all elements with nontrivial image in each component $k_i[[\Gamma]]$. Since $\Lambda[[\Gamma]]$ is commutative, this is trivially a left denominator set. Moreover, the complex $Q^\bullet$ is perfect as complex of Λ-modules precisely if its cohomology groups are S-torsion. On the other hand, as a trivial case of Proposition 13.1, we know that S_t is a left denominator set and that the cohomology groups of $P^\bullet$ are S_t-torsion. Since $f(0)$ is a unit in Λ for each $f \in S_t$, the element $f(\gamma)$ has clearly nontrivial image in each component $k_i[[\Gamma]]$. Hence, ev_γ maps S_t to S and $Q^\bullet$ is indeed perfect as complex of Λ-modules. □

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