# Extensions of Profinite Duality Groups

by Alexander Schmidt and Kay Wingberg

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Let $G$ be a profinite group and let $p$ be a prime number. By $Mod_p(G)$ we denote the category of discrete $p$-primary $G$-modules. For $A \in Mod_p(G)$ and $i \geq 0$, let

$$D_i(G,A) = \varinjlim_{U} H^i(U,A)^*,$$

where $^*$ is $\mathrm{Hom}(-,\mathbb{Q}_p/\mathbb{Z}_p)$, the direct limit is taken over all open subgroups $U$ of $G$ and the transition maps are the duals of the corestriction maps. $D_i(G,A)$ is a discrete $G$-module in a natural way. Assume that $n = cd_p\, G$ is finite. Then the $G$-module

$$I(G) = \varinjlim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^\nu\mathbb{Z})$$

is called the **dualizing module** of $G$ at $p$. Its importance lies in the functorial isomorphism

$$H^n(G,A)^* \cong \mathrm{Hom}_G(A, I(G))$$

for all $A \in Mod_p(G)$. This isomorphism is induced by the cup-products $(V \subseteq U)$

$$H^n(G,A)^* \times {}_{p^\nu}A^U \longrightarrow H^n(V,\mathbb{Z}/p^\nu\mathbb{Z})^*,\ (\phi,a) \longmapsto \big(\alpha \mapsto \phi(cor\,{}_G^V(\alpha \cup a))\big)$$

by passing to the limit over $\nu$ and $V$, and then over $U$. The identity-map of $I(G)$ gives rise to the homomorphism

$$tr : H^n(G, I(G)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p\,,$$

called the **trace map**.

The profinite group $G$ is called a **duality group at $p$ of dimension $n$** if for all $i \in \mathbb{Z}$ and all finite $G$-modules $A \in Mod_p(G)$, the cup-product and the trace map

$$H^i(G, \mathrm{Hom}(A, I(G)) \times H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, I(G)) \xrightarrow{tr} \mathbb{Q}_p/\mathbb{Z}_p$$

yield an isomorphism

$$H^i(G, \mathrm{Hom}(A, I(G))) \cong H^{n-i}(G, A)^*.$$

**Remark:** In [Ve], J.-L. Verdier used the name **strict Cohen-Macaulay at *p*** for what we call a profinite duality group at $p$ here. In [Pl], A. Pletch defined $D_p^n$-groups (and called them duality groups at $p$ of dimension $n$). The $D_p^n$-groups of Pletch are exactly the duality groups at $p$ (in our sense) which, in addition, satisfy the following finiteness condition:

$FC(G,p)$: *$H^i(G, A)$ is finite for all finite $A \in Mod_p(G)$ and for all $i \geq 0$.*

Since any finite, discrete $G$-module is trivialized by an open subgroup $U$ of $G$, condition $FC(G,p)$ can also be rephrased in the form:

$FC(G,p)$: *$H^i(U, \mathbb{Z}/p\mathbb{Z})$ is finite for all open subgroups $U$ of $G$ and all $i \geq 0$.*

By a duality theorem due to J. Tate, see [Ta] Thm. 3 or [Ve] Prop. 4.3 or [NSW] (3.4.6), a profinite group $G$ of cohomological $p$-dimension $n$ is a duality group at $p$ if and only if

$$D_i(G, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } 0 \leq i < n.$$

As a consequence we see that every open subgroup of a duality group at $p$ is a duality group at $p$ as well (of the same cohomological dimension), and if an open subgroup of $G$ is a duality group at $p$ and $cd_p G < \infty$, then $G$ is a duality group at $p$ of the same cohomological dimension (use [NSW] (3.3.5)(ii)). Furthermore, any profinite group of cohomological $p$-dimension 1 is a duality group at $p$.

We call a profinite group $G$ **virtually a duality group at $\boldsymbol{p}$ of (virtual) dimension $\boldsymbol{vcd_p G = n}$** if an open subgroup $U$ of $G$ is a duality group at $p$ of dimension $n$.

The objective of this paper is to give a proof of Theorem 1 below, which states that the class of duality groups is closed under group extensions $1 \to H \to G \to G/H \to 1$ if the kernel satisfies $FC(H,p)$. Weaker forms of Theorem 1 were first proved by A. Pletch (for $D_p^n$-groups, see [Pl][1]) and by the second author (for Poincaré groups, see [Wi]).

[1] The proof given by Pletch in [Pl] is only correct for pro-$p$-groups as the author assumes that finitely generated projective modules over the complete group ring $\mathbb{Z}_p[\![G]\!]$ are free.

**Theorem 1.** *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

*be an exact sequence of profinite groups such that condition $FC(H,p)$ is satisfied. Then the following assertions hold:*

(i) *If $G$ is a duality group at $p$, then $H$ is a duality group at $p$ and $G/H$ is virtually a duality group at $p$.*

(ii) *If $H$ and $G/H$ are duality groups at $p$, then $G$ is a duality group at $p$.*

*Moreover, in both cases we have:*

$$cd_p\, G = cd_p\, H + vcd_p\, G/H,$$

*and there is a canonical $G$-isomorphism*

$$I(G)^\vee \cong I(H)^\vee \,\hat{\otimes}_{\mathbb{Z}_p}\, I(G/H)^\vee,$$

*where $^\vee$ is the Pontryagin dual and $\hat{\otimes}_{\mathbb{Z}_p}$ is the tensor-product in the category of compact $\mathbb{Z}_p$-modules.*

**Remark:** The assumption $FC(H,p)$ is necessary, as the following examples show:

1. Let $G$ be the free pro-$p$-group on two generators $x, y$ and let $H \subset G$ be the normal subgroup generated by $x$. Then $H$ is free of infinite rank, $G/H$ is free of rank one and $1 \to H \to G \to G/H \to 1$ is an exact sequence in which all three groups are duality groups of dimension one.

2. Let $D$ be a duality group at $p$ of dimension 2, $F$ a duality group at $p$ of dimension 1 and $G = F * D$ their free product. The kernel of the projection $G \twoheadrightarrow D$ has cohomological $p$-dimension 1, hence is a duality group a $p$ of dimension 1. The group $G$ has cohomological $p$-dimension 2 but is is not a duality group at $p$.

In the proof of Theorem 1, we make use of the following

**Proposition 2.** *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

*be an exact sequence of profinite groups. Assume that FC(H,p) holds. Then there is a spectral sequence of homological type*

$$E^2_{ij} = D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \Longrightarrow D_{i+j}(G, \mathbb{Z}/p\mathbb{Z}).$$

*Proof.* Let $g$ run through the open normal subgroups of $G$. Then $gH/H \cong g/g\cap H$ runs through the open normal subgroups of $G/H$. For a $G$-module $A \in Mod_p(G)$, we consider the Hochschild-Serre spectral sequence

$$E(g, g\cap H, A): \; E_2^{ij}(g, g\cap H, A) = H^i(g/g\cap H, H^j(g\cap H, A)) \Longrightarrow H^{i+j}(g, A).$$

If $g' \subseteq g$ is another open normal subgroup of $G$, then the corestriction yields a morphism

$$cor : E(g', g' \cap H, A) \longrightarrow E(g, g \cap H, A)$$

of spectral sequences. The map

$$E_2^{ij}(g', g' \cap H, A) \longrightarrow E_2^{ij}(g, g \cap H, A)$$

is the composite of the maps

$$H^i\big(g'/g' \cap H, H^j(g' \cap H, A)\big) \overset{cor\,{}^{g'\cap H}_{g\cap H}}{\longrightarrow} H^i\big(g'/g' \cap H, H^j(g \cap H, A)\big)$$

$$\overset{cor\,{}^{g'/g'\cap H}_{g/g\cap H}}{\longrightarrow} H^i\big(g/g \cap H, H^j(g \cap H, A)\big)$$

and the map between the limit terms is the corestriction

$$cor\,{}^{g'}_{g} : H^{i+j}(g', A) \longrightarrow H^{i+j}(g, A).$$

For $2 \leq r \leq \infty$ we set

$$E_{ij}^2 = D_{ij}^r(G, H, A) := \varinjlim_{g} E_r^{ij}(g, g \cap H, A)^*.$$

As taking duals and direct limits are exact operations, the terms $D_{ij}^r(G, H, A)$, $2 \leq r \leq \infty$, establish a homological spectral sequence which converges to $D_n(G, A)$. If $h$ runs through the open subgroups of $H$ which are normal in $G$, then the cohomology groups $H^j(h, A)$ are $G$-modules in a natural way. If $g$ is open in $G$ with $g \cap H \subseteq h$, then these groups are $g/g \cap H$-modules. We see that

$$D_{ij}^2(G, H, A) = \varinjlim_{\substack{h\subseteq H \\ h \trianglelefteq G}} \; \varinjlim_{\substack{g\subseteq G \\ g\cap H\subseteq h}} \; H^i(g/g \cap H, H^j(h, A))^*,$$

where for both limits the transition maps are (induced by) $cor^*$. In order to conclude the proof of the proposition, it remains to construct isomorphisms

$$D_{ij}^2(G, H, \mathbb{Z}/p\mathbb{Z}) \cong D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z})$$

for all $i$ and $j$. To this end note that all occurring abelian groups are $\mathbb{F}_p$-vector spaces, so that $^*$ is $\mathrm{Hom}(-,\mathbb{F}_p)$. Further note that for vector spaces $V, W$ over a field $k$ the homomorphism

$$V^* \otimes W^* \longrightarrow (V \otimes W)^*, \ \phi \otimes \psi \longmapsto \big(v \otimes w \mapsto \phi(v)\psi(w)\big)$$

is an isomorphism provided that $V$ or $W$ is finite-dimensional. Let $h$ be an open subgroup of $H$ which is normal in $G$ and let $g' \subseteq g$ be open subgroups of $G$ such that $g$ acts trivially on the finite group $H^j(h, \mathbb{Z}/p\mathbb{Z})$. Then, by [NSW] (1.5.3)(iv), the diagram

$$\begin{array}{ccc} H^i(g'/g' \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow[\sim]{\cup} & H^i\big(g'/g' \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})\big) \\ \Big\downarrow {\scriptstyle cor \otimes id} & & \Big\downarrow {\scriptstyle cor} \\ H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow[\sim]{\cup} & H^i\big(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})\big) \end{array}$$

commutes. For fixed $h$, we therefore obtain isomorphisms

$$\begin{aligned} & D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \big(\varinjlim_g H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^*\big) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \varinjlim_g \ H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \varinjlim_g \big(H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})\big)^* \\ & \qquad \cong \varinjlim_g H^i\big(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})\big)^*. \end{aligned}$$

Passing to the limit over $h$, we obtain the required isomorphism

$$D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \cong D^2_{ij}(G, H, \mathbb{Z}/p\mathbb{Z}).$$

□

**Corollary 3.** *Under the assumptions of Proposition 2, let $i_0$ and $j_0$ be the smallest integers such that $D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \neq 0$ and $D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0$, respectively. Then $D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \neq 0$.*

*Proof.* The spectral sequence constructed in Proposition 2 induces an isomorphism

$$D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \cong D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$

□

*Proof of Theorem 1.* Assume that $G$ is a duality group at $p$ of dimension $d$. Let $cd_p\, H = m$ and $n = d - m$. Then there exists an open subgroup $H_1$ of $H$ such that $H^m(H_1, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let $G_1$ be an open subgroup of $G$ such that $H_1 = G_1 \cap H$. Then $G_1$ is a duality group at $p$ of dimension $d$, $cd_p\, H_1 = m$ and $G_1/H_1$ is an open subgroup of $G/H$. We consider the exact sequence

$$1 \longrightarrow H_1 \longrightarrow G_1 \longrightarrow G_1/H_1 \longrightarrow 1.$$

As $H^m(H_1, \mathbb{Z}/p\mathbb{Z})$ is finite and nonzero, we have $vcd_p\, G_1/H_1 = n$, see [NSW] (3.3.9). Furthermore, $D_i(G_1, \mathbb{Z}/p\mathbb{Z}) = 0$, $i < n + m$. Using Corollary 3, we see that $D_i(G_1/H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i < n$ and $D_j(H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $j < m$. Thus $G_1/H_1$, hence $G/H$, is virtually a duality group at $p$ of dimension $n$, and $H_1$, and so $H$, is a duality group at $p$ of dimension $m$. This shows (i).

Assume now that $H$ and $G/H$ are duality groups at $p$ of dimension $m$ and $n$. Then, $cd_pG = n + m$ by [NSW] (3.3.8), and in the spectral sequence of Proposition 2 we have $E^2_{ij} = 0$ for $(i, j) \neq (n, m)$. Hence $D_r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \neq n + m$ showing that $G$ is a duality group at $p$ of dimension $n + m$.

In order to prove the assertion about the dualizing modules, let $h$ run through all open subgroups of $H$ which are normal in $G$ and $g$ runs through the open subgroups of $G$. Since $m = cd_p\, H$, the Hochschild-Serre spectral sequence induces isomorphisms

$$H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z}) \cong H^n(g/g \cap H, H^m(g \cap H, \mathbb{Z}/p^\nu\mathbb{Z})),$$

and we obtain

$$\begin{aligned}
I(G) &\cong \varinjlim_{\nu} \varinjlim_{g} H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z})^* \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \varinjlim_{g} H^n\big(g/g \cap H, H^m(h, \mathbb{Z}/p^\nu\mathbb{Z})\big)^* \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \varinjlim_{g,res} H^0\big(g/g \cap H, \mathrm{Hom}\,(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H))\big) \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \mathrm{Hom}(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\
&\cong \mathrm{Hom}_{cts}(\varprojlim_{\nu} \varprojlim_{h} H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\
&\cong \mathrm{Hom}_{cts}\big((\varinjlim_{\nu} \varinjlim_{h} H^m(h, \mathbb{Z}/p^\nu\mathbb{Z})^*)^\vee, I(G/H)\big) \\
&\cong \mathrm{Hom}_{cts}\,(I(H)^\vee, I(G/H)) \cong \big(I(H)^\vee \,\hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee\big)^\vee
\end{aligned}$$

(see [NSW] (5.2.9) for the last isomorphism). This completes the proof of the theorem. $\square$

# References

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Alexander Schmidt, NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Deutschland. email: alexander.schmidt@mathematik.uni-regensburg.de

Kay Wingberg, Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Deutschland. email: wingberg@mathi.uni-heidelberg.de