Sets of Completely Decomposed Primes in Extensions of Number Fields

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Let p be a prime number and let $k(p)$ be the maximal p-extension of a number field k. If T is a set of primes of k, then $k^T(p)$ denotes the maximal p-extension of k which is completely decomposed at T. Assuming that T is finite, the canonical homomorphism

$$\phi^T(p) : \mathop{\ast}_{\mathfrak{p} \in T(k^T(p))} G_{\mathfrak{P}}(k(p)|k) \longrightarrow G(k(p)|k^T(p))$$

of the free pro-p-product of the decomposition groups $G_{\mathfrak{P}}(k(p)|k)$ into the Galois group $G(k(p)|k^T(p))$ is an isomorphism, see [3] theorem (10.5.8); here the prime $\mathfrak{P}$ is an arbitrary extension of $\mathfrak{p}$ to $k(p)$.

In the profinite case, i.e. considering the maximal Galois extension k^T which is completely decomposed at the finite set T, there exist suitable extensions $\mathfrak{P}|\mathfrak{p}$ such that ϕ^T is an isomorphism of profinite groups. If $T = S_\infty$ is the set archimedean primes, this is a result of Fried-Haran-Völklein [1] and in general it is proven by Pop [4].

The fact that $\phi^T(p)$ is an isomorphism if T is finite implies very strong properties for the extension $k^T(p)|k$. In particular, the (strict) cohomological dimension of the Galois group $G(k^T(p)|k)$ is equal to 2 (if $p = 2$ one has to require that k is totally imaginary). Furthermore, we get for the corresponding local extensions that

$$(*) \qquad (k^T(p))_{\mathfrak{P}} = k_{\mathfrak{p}}(p) \quad \text{for all primes} \quad \mathfrak{P}|\mathfrak{p},\ \mathfrak{p} \notin T,$$

i.e. $k^T(p)$ realizes the maximal p-extension $k_{\mathfrak{p}}(p)$ of the local fields $k_{\mathfrak{p}}$ for all primes not in T. In particular, the set T is equal to the set $D(k^T(p)|k)$ of all primes of k which decomposed completely in the extension $k^T(p)|k$. We will say that T is *saturated* if it has this property and we call the stronger property $(*)$ that T is *strongly saturated*. If the Dirichlet density $\delta(T)$ is positive, then T is saturated if and only if T is the set of completely decomposed primes of a *finite* Galois extension of k.

If T is an arbitrary set of primes of k, we call the set $\hat{T} = D(k^T(p)|k)$ the *saturation* of T. The most interesting case is that T is an infinite set of primes of density zero. If T is saturated, then the extension $k^T(p)|k$ is infinite. We will see that there exist infinite sets T of primes such that $\delta(\hat{T}) > \delta(T) = 0$, and also sets T such that $\delta(\hat{T}) = \delta(T) = 0$. An important example is the following.

Theorem 1: *Let p be an odd prime number and let k be a CM-field containing the group μ_p of all p-th roots of unity, with maximal totally real subfield k^+, i.e. $k = k^+(\mu_p)$ is totally imaginary and $[k : k^+] = 2$. Let $S_p = \{\mathfrak{p}|p\}$ and*

$$T = \{\mathfrak{p} \mid \mathfrak{p} \cap k^+ \text{ is inert in } k|k^+\}.$$

Then $T \cup S_p$ is strongly saturated.

Moreover, in the example above we get that the Galois group

$$G((k^T)_{nr}(p)|k^T(p))$$

of the maximal unramified p-extension of $(k^T)_{nr}(p)$ of $k^T(p)$ is a free pro-p-group. This will follow from a more general theorem which deals with a generalization of the notion of saturated sets.

A set $T = T(k)$ is called *stably saturated* if the sets $T(k')$ are saturated for every finite Galois extension $k'|k$ inside $k^T(p)$. These sets are necessarily of density 0 if they are not equal to set $\mathcal{P}$ of all primes. Obviously, strongly staturated sets are stably saturated. We have the following theorem.

Theorem 2: *Let $T \neq \mathcal{P}$ be a stably saturated set of primes of a number field k. Then the canonical map*

$$\phi^T(p) : \mathop{\ast}_{\mathfrak{P} \in T(k^T(p))} (G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k)) \xrightarrow{\sim} G(k(p)/k^T(p))$$

is an isomorphism.

Here $\ast_{\mathfrak{P} \in T(k^T(p))}(G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k))$ denotes the free corestricted pro-p-product of the decomposition groups $G_{\mathfrak{P}}(k(p)|k)$ with respect to the inertia groups $T_{\mathfrak{P}}(k(p)|k)$, see [2].

Corollary: *In the situation of theorem 1 let $\tilde{T} = T \cup S_p$. Then*

$$\mathop{\ast}_{\mathfrak{P} \in \tilde{T}(k^{\tilde{T}}(p))} (G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k)) \xrightarrow{\sim} G(k(p)/k^{\tilde{T}}(p)).$$

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1 Saturated sets of primes of a number field

We start with some remarks on complete lattices. Let

$$\mathcal{A} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{B}$$

be maps between complete lattices $(\mathcal{A}, \subseteq)$ and $(\mathcal{B}, \subseteq)$ with the following properties:

I. φ and ψ are order-reversing,

II. $A \subseteq \psi\varphi(A)$ and $B \subseteq \varphi\psi(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we define the *saturation*

$$\hat{A} := \psi\varphi(A) \text{ and } \hat{B} := \varphi\psi(B),$$

and call $A \in \mathcal{A}$ resp. $B \in \mathcal{B}$ to be *saturated* if $A = \hat{A}$ resp. $B = \hat{B}$. We put

$$\mathcal{A}_{sat} = \{A \in \mathcal{A} \,|\, A \text{ is saturated}\}, \qquad \mathcal{B}_{sat} = \{B \in \mathcal{B} \,|\, B \text{ is saturated}\},$$

and we have the following properties:

(i) $A_1 \subseteq A_2$ implies $\hat{A}_1 \subseteq \hat{A}_2$ and $B_1 \subseteq B_2$ implies $\hat{B}_1 \subseteq \hat{B}_2$.

(ii) $\varphi\psi\varphi = \varphi$, $\psi\varphi\psi = \psi$, $\hat{\hat{A}} = \hat{A}$, $\hat{\hat{B}} = \hat{B}$.

(iii) $\mathcal{B}_{sat}$ is the image of $\mathcal{A}$ under φ and $\mathcal{A}_{sat}$ is the image of $\mathcal{B}$ under ψ, i.e. $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{B}_{sat}$ and $\psi : \mathcal{B} \twoheadrightarrow \mathcal{A}_{sat}$, and ψ and φ induce bijections

$$\mathcal{A}_{sat} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{B}_{sat}.$$

(iv)
$$\varphi(\bigcup_i A_i) = \bigcap_i \varphi(A_i), \qquad \varphi(\bigcap_i \psi(B_i)) = \widehat{\bigcup_i B_i}.$$

In particular, the infimum of saturated elements is again saturated. The verification of these statements is straightforward using the properties I and II of the maps φ and ψ. We define the equivalence relations

$$A_1 \sim A_2 \quad :\Leftrightarrow \quad \varphi(A_1) = \varphi(A_2) \qquad \text{for} \quad A_1, A_2 \in \mathcal{A}$$

and

$$B_1 \sim B_2 \quad :\Leftrightarrow \quad \psi(B_1) = \psi(B_2) \qquad \text{for} \quad B_1, B_2 \in \mathcal{B},$$

and denote the classes by $[X]$. Obviously, $X \sim \hat{X}$ by (ii) and for $Y \in [X]$ we have $Y \subseteq \hat{X}$, and so $\hat{X}$ is the unique maximal element of $[X]$. Furthermore, the surjections $\psi\varphi : \mathcal{A} \twoheadrightarrow \mathcal{A}_{sat}$ and $\varphi\psi : \mathcal{B} \twoheadrightarrow \mathcal{B}_{sat}$ induce bijections

$$\psi\varphi : (\mathcal{A}/_\sim) \xrightarrow{\sim} \mathcal{A}_{sat}, \quad [A] \mapsto \hat{A}, \quad \text{and} \quad \varphi\psi : (\mathcal{B}/_\sim) \xrightarrow{\sim} \mathcal{B}_{sat}, \quad [B] \mapsto \hat{B}.$$

Now let K be a number field. We use the following notation: S_∞, $S_\mathbb{R}$ and $S_\mathbb{C}$ are the sets of archimedean, real and complex primes of K, respectively, $\mathcal{P}$ is the set of all primes of K, and if p is a prime number, then S_p is the set of all primes of K above p.

If $\mathfrak{p}$ is a prime of K, then $K_\mathfrak{p}$ is the completion of K with respect to $\mathfrak{p}$. If $L|K$ is a Galois extension, then we denote the decomposition group and inertia group of the Galois group $G(L|K)$ with respect to $\mathfrak{p}$ by $G_\mathfrak{p}(L|K)$ and $T_\mathfrak{p}(L|K)$, respectively.

For a set S of primes of K, let $\delta(S) = \delta_K(S)$ be its Dirichlet density. If $S = S(K)$ is a set of primes and $K'|K$ an algebraic extension of K, then we denote the set of primes of K' consisting of all prolongations of S by $S(K')$.

If $\mathfrak{c}$ is a class of finite groups which is closed under taking subgroups, homomorphic images and group extensions, then $K(\mathfrak{c})$ is the maximal pro-$\mathfrak{c}$-extension of K, and in particular if p is a prime number, $K(p)$ denotes the maximal p-extension of K. By abuse of notation, we denote the maximal pro-$\mathfrak{c}$ extension of K which is completely decomposed at T by $K^T(\mathfrak{c})$, and $K^T(p)$ is the maximal p-extension of K inside K^T.

Let

$$\mathcal{E}_K = \{L \mid L \text{ is a Galois extension of } K\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{T \mid T \text{ is a set of primes of } K\} = \mathcal{S}_K$$

where

$\varphi(L) = D(L\|K)$	is the set of primes which are completely decomposed in $L\|K$,
$\psi(T) = K^T$	is the maximal Galois extension of K which is completely decomposed at T.

Obviously, the maps φ and ψ are order-reversing and

$$L \subseteq \psi\varphi(L) = K^{D(L|K)} =: \hat{L} \quad \text{and} \quad T \subseteq \varphi\psi(T) = D(K^T|K) =: \hat{T}.$$

Furthermore, we define the equivalence relations

$$L_1 \sim L_2 \quad :\Leftrightarrow \quad D(L_1|K) = D(L_2|K) \qquad \text{for } L_1, L_2 \in \mathcal{E}_K$$

and

$$T_1 \sim T_2 \quad :\Leftrightarrow \quad K^{T_1} = K^{T_2} \qquad \text{for } T_1, T_2 \in \mathcal{S}_K.$$

Definition 1.1 *The extension $L \in \mathcal{E}_K$ is called* **saturated** *if $\hat{L} = L$, i.e.*

$$L = K^{D(L|K)},$$

and the set $T \in \mathcal{S}_K$ is called **saturated** *if $\hat{T} = T$, i.e.*

$$T = D(K^T|K).$$

We strengthen the notion of saturated sets in the following way:

Definition 1.2 *Let T be a set of primes of K.*

(i) *The set $T = T(K)$ is called* **stably saturated** *if the sets $T(K') \in \mathcal{S}_{K'}$ are saturated for every finite Galois extension $K'|K$ inside K^T.*

(ii) *We call T to be* **strongly saturated** *if*

$$(K^T)_{\mathfrak{P}} = \bar{K}_{\mathfrak{p}} \quad \textit{for all primes} \quad \mathfrak{P}|\mathfrak{p},\ \mathfrak{p} \notin T,$$

where $\bar{K}_{\mathfrak{p}}$ is the algebraic closure of $K_{\mathfrak{p}}$.

Remark 1: If we consider the set $\mathcal{E}_K(\mathfrak{c}) = \{L \mid L \text{ is a pro-}\mathfrak{c}\text{-extension of } K\}$, we define a set $T \in \mathcal{S}_K$ to be $\mathfrak{c}$-saturated if $T = D(K^T(\mathfrak{c})|K)$, and an extension $L \in \mathcal{E}_K(\mathfrak{c})$ is called **$\mathfrak{c}$-saturated** if $L = K^{D(L|K)}(\mathfrak{c})$. Analogously, we define stably $\mathfrak{c}$-saturated and strongly $\mathfrak{c}$-saturated, e.g. T is strongly $\mathfrak{c}$-saturated if $(K^T(\mathfrak{c}))_{\mathfrak{P}} = K_{\mathfrak{p}}(\mathfrak{c})$ for all primes $\mathfrak{P}|\mathfrak{p}$, $\mathfrak{p} \notin T$.

We say, a prime of K is **redundant** (or more precise, $\mathfrak{c}$-redundant) if is totally decomposed in every extension inside $K(\mathfrak{c})$. Obviously, redundant primes are necessarily archimedean primes and the complex primes are always redundant. But also real primes might be redundant if we restrict to pro-$\mathfrak{c}$-extensions, e.g. if we consider p-extensions where p is an odd prime number. Therefore we make the following

Convention: In the following all considered primes are not redundant and $\mathcal{S}_K = \mathcal{S}_K(\mathfrak{c})$ consists only of sets of non-redundant primes for the extension $K(\mathfrak{c})|K$.

Immediately from the definitions (and the convention) above we get

Lemma 1.3 *Let $T \in \mathcal{S}_K$.*

(i) *T is saturated if and only if $(K^T)_{\mathfrak{p}} \neq K_{\mathfrak{p}}$ for all $\mathfrak{p} \notin T$.*

(ii) *T is stably saturated if and only if $(K^T)_{\mathfrak{p}}|K_{\mathfrak{p}}$ is an infinite extension for all $\mathfrak{p} \notin T$.*

From the general remarks on partial ordered sets we see that there are bijections

$$(\mathcal{E}_K)_{sat} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} (\mathcal{S}_K)_{sat}$$

and we have the following

Lemma 1.4

(i) *$T_1 \subseteq T_2$ implies $\hat{T}_1 \subseteq \hat{T}_2$ and $L_1 \subseteq L_2$ implies $\hat{L}_1 \subseteq \hat{L}_2$.*

(ii) *$K^T = K^{\hat{T}}$ and $D(L|K) = D(\hat{L}|K)$.*

(iii) *$D(\prod_i L_i|K) = \bigcap_i D(L_i|K)$ and $K^{\bigcup_i T_i} = \bigcap_i K^{T_i}$.*

(iv) *$D(\bigcap_i K^{T_i}|K) = \widehat{\bigcup_i T_i}$ and $K^{\bigcap_i D(L_i|K)} = \widehat{\prod_i L_i}$.*

Theorem 1.5

(i) *If T is a finite set of primes of K, then T is strongly saturated.*

(ii) *If L is a finite Galois extension of K, then L is saturated.*

Proof: (i) Let p be a prime number and let $L|K$ be a finite Galois extension inside K^T. Let $\mathfrak{p}_0 \notin T$, $\mathfrak{P}_0$ a fixed extension of $\mathfrak{p}_0$ to K^T and $\overline{\mathfrak{P}}_0$ the restriction of $\mathfrak{P}_0$ to L. By the theorem of Grunwald/Wang (see [3], theorem (9.2.2)) the canonical homomorphism

$$H^1(L, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(L_{\overline{\mathfrak{P}}_0}, \mathbb{Z}/p\mathbb{Z}) \oplus \bigoplus_{\mathfrak{P} \in T(L)} H^1(L_{\mathfrak{P}}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. In particular, for every $\alpha_{\overline{\mathfrak{P}}_0} \in H^1(L_{\overline{\mathfrak{P}}_0}, \mathbb{Z}/p\mathbb{Z}))$ there exists an element $\beta \in H^1(L, \mathbb{Z}/p\mathbb{Z})$ which is mapped to $(\alpha_{\overline{\mathfrak{P}}_0}, 0, \dots, 0)$. But β lies in the subgroup $H^1(K^T|L, \mathbb{Z}/p\mathbb{Z})$ of $H^1(L, \mathbb{Z}/p\mathbb{Z})$. Therefore

$$H^1(K^T|L, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(L_{\overline{\mathfrak{P}}_0}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Varying L and p, it follows that the completion of K^T with respect to the prime $\mathfrak{P}_0$, $\mathfrak{P}_0|\mathfrak{p}_0$ and $\mathfrak{p}_0 \notin T$, is equal to the algebraic closure of $K_{\mathfrak{p}_0}$ (since $G(\overline{K}_{\mathfrak{p}_0}|K_{\mathfrak{p}_0})$ is pro-solvable).

(ii) Let L' be a finite Galois extension of K with $L \subseteq L' \subseteq K^{D(L|K)}$. Since $D(L|K) \subseteq D(L'|K)$, we obtain for the densities of these sets the inequality $\delta(D(L|K)) \leq \delta(D(L'|K))$, and so, by Čebotarev's density theorem,

$$[L' : K] = \delta(D(L'|K))^{-1} \leq \delta(D(L|K))^{-1} = [L : K].$$

This shows that $L' = L$ and so $L = K^{D(L|K)}$. □

Now we consider sets of primes which are infinite and of density equal to 0.

Proposition 1.6 *For a set T of primes we have*

$$\delta(\hat{T}) = 0 \quad \Leftrightarrow \quad K^T|K \text{ is an infinite extension.}$$

Proof: Since $\hat{T} = D(K^T|K)$, it follows that $\delta(\hat{T}) > 0$ if $K^T|K$ is a finite extension. Conversely, assume that $K^T|K$ is infinite and let L be a finite Galois extension of K inside K^T. Then

$$\frac{1}{[L:K]} = \delta(D(L|K)) \geq \delta(\hat{T}),$$

and so $\delta(\hat{T}) = 0$. □

Remark 2: From the proposition above and lemma (1.3)(ii) it follows that a stably saturated set of primes $T \neq \mathcal{P}$ has necessarily density equal to 0.

Remark 3: Let $n \in \mathbb{N}$. Then there exist infinite sets T of primes such that

$$\frac{1}{n} = \delta(\hat{T}) > \delta(T) = 0.$$

Example 1: Let $L|K$ be a Galois extension of the number field K such that $[L:K] = n$. The set $\mathcal{E}_{L|K}^{fin}$ of proper finite extensions of L being Galois over K is countable, say $\mathcal{E}_{L|K}^{fin} = \{L_i,\ i \in \mathbb{N}\}$. We choose for every $i \in \mathbb{N}$ an element $\sigma_i \in G(L_i|L)$, $\sigma_i \neq 1$, and a prime $\mathfrak{p}_i$ of K which is unramified in $L_i|K$ having a Frobenius $\left(\frac{L_i|K}{\mathfrak{P}_i}\right) = \sigma_i$, $\mathfrak{P}_i|\mathfrak{p}_i$. Since there exist infinitely many such primes, there is a $\mathfrak{p}_i$ such that $N_{K|\mathbb{Q}}\mathfrak{p}_i \geq i^2$. Then

$$\sum_{i=1}^{\infty} \frac{1}{N_{K|\mathbb{Q}}\mathfrak{p}_i} \leq \sum_{i=1}^{\infty} \frac{1}{i^2}$$

converges, and so the set $T = \{\mathfrak{p}_i,\ i \in \mathbb{N}\}$ has density equal to 0. Furthermore we have

$$K^T = L.$$

Indeed, since every $\mathfrak{p} \in T$ is completely decomposed in L, we have $L \subseteq K^T$ and a finite Galois extension $E|K$, $L \subsetneq E \subseteq K^T$ would be a field L_i for some $i \in \mathbb{N}$ and so $\mathfrak{p}_i \in T$ would not be completely decomposed in $E|K$. Therefore $\hat{T} = D(L|K)$ and $\delta(\hat{T}) = \frac{1}{n} > 0$.

Remark 4: There exist infinite sets T of primes such that

$$\delta(\hat{T}) = \delta(T) = 0.$$

Example 2: Assume that K is a number field such that
(i) K is not totally real,
(ii) there exists a proper subfield E of K such that $K|E$ is a cyclic extension.

We denote the Galois group $G(K|E)$ by $\varDelta$. Let

$$T_0 = \{\mathfrak{p} \,|\, \mathfrak{p} \cap E \text{ is inert in } K|E\}.$$

Then T_0 is an infinite set of primes of K of density equal to 0. Let p be a prime number not dividing $[K : E]$ whose extensions to E are completely decomposed or totally ramified in $K|E$. Let K_∞ resp. E_∞ be the compositum of all $\mathbb{Z}_p$-extension of K resp. E. Then

$$G(K_\infty|K) = \mathbb{Z}_p^{r_2(K)+1+\delta_K}, \qquad G(E_\infty|E) = \mathbb{Z}_p^{r_2(E)+1+\delta_E},$$

where r_2 denotes the number of complex places and δ is the so-called Leopoldt defect. We have a decomposition of $\mathbb{Z}_p[\varDelta]$-modules

$$G(K_\infty|K) \cong G(E_\infty|E) \oplus M,$$

where M is a $\varDelta$-module with $M^\varDelta = 0$ and $r = \mathrm{rang}_{\mathbb{Z}_p} M \geq r_2(K) - r_2(E) > 0$, since $\delta_K \geq \delta_E$. Let L be the subfield of K_∞ corresponding to $G(K_\infty|K)^\varDelta \cong G(E_\infty|E)$, i.e.

$$G(L|K) \cong M$$

and so $G(L|K)^\varDelta = 0$. Observe that L is a Galois extension of E. Now let $\mathfrak{p} \in T_0$. For the decomposition group with respect to $\mathfrak{p}$ we have the split exact sequence

$$0 \longrightarrow G_\mathfrak{p}(L|K) \longrightarrow G_\mathfrak{p}(L|E) \longrightarrow \varDelta_\mathfrak{p} \longrightarrow 0.$$

If $G_\mathfrak{p}(L|K)$ would be non-trivial, then $G_\mathfrak{p}(L|E)$ is not abelian since $\varDelta_\mathfrak{p} = \varDelta \neq 0$ acts non-trivially on $G_\mathfrak{p}(L|K) \subseteq G(L|K)$. But $\mathfrak{p}$ lies not above p and so it is unramified in $L|K$, thus unramified in $L|E$ and so $G_\mathfrak{p}(L|E)$ is cyclic. Therefore all primes in T_0 are completely decomposed in the infinite extension $L|K$. Let

$$T = D(L|K).$$

Then T is an infinite saturated set (containing T_0), i.e. $T = \hat{T}$, and since K^T is an infinite extension (containing L), it follows from proposition (1.6) that $\delta(\hat{T}) = 0$.

Furthermore, if $S_\mathbb{R} \subseteq T$, then

$$(K^T)_\mathfrak{p}|K_\mathfrak{p} \quad \text{is an infinite extension if } \mathfrak{p} \notin T,$$

hence T is stably saturated. This follows from the fact, that for a prime $\mathfrak{p} \notin D(L|K)$ the non-trivial decomposition group $G_\mathfrak{p}(L|K)$ has to be a infinite subgroup of $G(L|K) \cong \mathbb{Z}_p^r$.

An example for the situation above is a CM-field K with maximal totally real subfield K^+. If $T_0(K^+) = \{\mathfrak{p} \,|\, \mathfrak{p} \text{ is inert in } K\}$, then $T_0 = T_0(K)$ is infinite of density equal to zero. Let K_{ac} be the anti-cyclotomic $\mathbb{Z}_p$-extension of K, $p \neq 2$, and $T = D(K_{ac}|K)$. Then $T_0 \subseteq T = \hat{T}$ and so $K^T|K$ is an infinite extension and therefore $\delta(\hat{T}) = 0$.

Now we consider the cardinality of an equivalence class $[T]$ of a set of primes T of K. By definition, the saturation $\hat{T}$ of T is the unique maximal element of $[T]$ (with respect to the inclusion). But the example 1 of remark 3 shows that there might be infinitely many different (even pairwise disjoint) minimal elements in the class $[T]$.

Proposition 1.7

(i) *If L is a finite Galois extension of K, then*

$$\#[L] = 1 \quad \textit{and} \quad \#[D(L|K)] = \infty.$$

(ii) *If T is a finite set of primes, then*

$$\#[T] = 1 \quad \textit{and} \quad \#[K^T] = \infty.$$

Proof: Let $L' \in [L]$. Then $L' \subseteq \hat{L} = L$ by theorem (1.5)(ii). Thus $L'|K$ is finite and so

$$L' = K^{D(L'|K)} = K^{D(L|K)} = L.$$

In order to prove the second assertion of (i), let L be a finite Galois extension of K and let $T \subseteq D(L|K)$ be a subset of density equal to zero. Then

$$L = K^{D(L|K)} = K^{D(L|K)\backslash T},$$

hence every subset $D(L|K)\backslash T$ with $\delta(T) = 0$ is contained in $[D(L|K)]$.

By theorem (1.5)(i), every subset of T is (strongly) saturated. Thus the first assertion of (ii) follows. For the second let $S = T \cup \{\mathfrak{p}\}$, where $\mathfrak{p}$ is some prime not in T. By [3](10.5.8) we have for the Galois group of the extension $K^T(p)|K^S(p)$ the isomorphism

$$\mathop{\ast}_{\mathfrak{p} \in S\backslash T(K^S(p))} G(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \xrightarrow{\sim} G(K^T(p)|K^S(p)),$$

where p is some prime number and $E(p)|K$ denotes the maximal p-extension inside a Galois extension $E|K$. In particular, the extension $K^T|K^S$ is infinite. Since every Galois extension $L|K$ with $K^S \subsetneq L \subseteq K^T$ is contained in $[K^T]$, the second assertion of (ii) follows. $\square$

2 The maximal p-extension $k^T(p)$ of k

In the following we will consider p-extensions and by a saturated set of primes we always mean a p-saturated set (see remark 1 of the first section).

Let S, T be sets of primes of a (not necessarily finite) number field K and let $\mathfrak{p}$ be a prime of K. Let $K(p)$ be the maximal p-extension of K. Mostly we will drop the notion (p). So let

K_{nr} is the maximal unramified p-extension of K,

K_S is the maximal p-extension of K which is unramified outside S,

K^T is the maximal p-extension of K which is completely decomposed at T,

K_S^T is the maximal p-extension of K which is unramified outside S and completely decomposed at T,

$G_{\mathfrak{p}}(K) = G_{\mathfrak{p}}(K(p)|K) \cong G(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})$ is the decomposition group,

$T_{\mathfrak{p}}(K) = T_{\mathfrak{p}}(K(p)|K) \cong T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})$ is the inertia group with respect to $\mathfrak{p}$.

In the following k will always denote a finite number field. If $K|k$ is an infinite extension of number fields and S a set of primes of k, then $S(K)$ denotes the profinite space

$$S(K) = \varprojlim_{k'} S(k') \cup \{*_{k'}\}$$

where k' runs through the finite subextensions of $K|k$ and $S(k') \cup \{*_{k'}\}$ is the one-point compactification of the discrete set $S(k')$ of primes of k' lying above S. According to [3] (10.5.8) and (10.5.10) we have

Theorem 2.1 *If $R' \subseteq R \subseteq S \subseteq S'$ are sets of primes of k such that $\delta(S) = 1$ and R is finite, then the canonical homomorphism*

$$\phi_{S',S}^{R',R} : \mathop{\ast}_{\mathfrak{p} \in S' \setminus S(k_S^R)} T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) * \mathop{\ast}_{\mathfrak{p} \in R \setminus R'(k_S^R)} G(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \longrightarrow G(k_{S'}^{R'}|k_S^R)$$

is an isomorphism. Furthermore we have the following assertions concerning the (strict) cohomological dimension: Assume that k is totally imaginary if $p = 2$, then

$$cd_p\ G(k_S^R|k) = scd_p\ G(k_S^R|k) = 2.$$

If $L|K$ is a Galois extension we write $H^i(L|K, A)$ for $H^i(G(L|K), A)$, and for a pro-p group G we put $H^i(G) = H^i(G, \mathbb{Z}/p\mathbb{Z})$. For a p-extension $K|k$ we will use the notation

$$\bigoplus_{\mathfrak{p}\in T(K)}{}' H^i(G_{\mathfrak{p}}(k(p)|K)) := \varinjlim_{k'} \bigoplus_{\mathfrak{p}\in T(k')} H^i(G_{\mathfrak{p}}(k(p)|k')$$

where k' runs through all finite subextensions of $K|k$.

Proposition 2.2 *Let p be a prime number and let T be a set of primes of a number field k of density $\delta(T)=0$. Then the following assertions are equivalent:*

(i) *T is stably saturated.*

(ii) *$cd_p\ G_{\mathfrak{p}}(k(p)|k^T)\le 1$ for all $\mathfrak{p}\notin T$.*

(iii) *The canonical map*

$$H^2(k(p)|k^T) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in T(k^T)}{}' H^2(G_{\mathfrak{p}}(k))$$

is an isomorphism.

(iv) *The group $G((k^T)_S|k^T)$ is free for every S with $\delta(S)=1$, $S\cap T=\varnothing$.*

Proof: (i)⇔(ii): By lemma (1.3) the set T is stably saturated if and only if the extensions $(k^T)_{\mathfrak{p}}|k_{\mathfrak{p}}$ are infinite for all primes $\mathfrak{p}\notin T$, i.e. if and only if $cd_p\ G_{\mathfrak{p}}(k(p)|k^T)\le 1$, see [3] (7.1.8)(i).

In order to prove (ii)⇔(iii), first observe that $k^T|k$ is an infinite extension as $\delta(T)=0$. Using the Poitou-Tate theorem, see [3] (8.6.10), (10.4.8), and the Hasse principle, loc. cit. (9.1.16), and passing to the limit over all finite extensions k' inside $k^T|k$, we obtain the isomorphism

$$H^2(k(p)|k^T) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in T(k^T)}{}' H^2(G_{\mathfrak{p}}(k)) \oplus \bigoplus_{\mathfrak{p}\notin T(k^T)}{}' H^2(G_{\mathfrak{p}}(k(p)|k^T)),$$

since $\varinjlim_{k'} H^0(G(k(p)|k'),\mu_p)^\vee = 0$. This shows that (iii) is equivalent to

$$H^2(G_{\mathfrak{p}}(k(p)|k^T)) = 0 \text{ for all } \mathfrak{p}\notin T(k^T),$$

and so we get (ii)⇔(iii).

From the Poitou-Tate exaxt sequences and the Hasse principle for the extensions $k(p)|k$ and $k_S|k$ (using $\delta(S)=1$) we get the exact sequence

$$0\longrightarrow H^2(k_S|k)\longrightarrow H^2(k(p)|k)\longrightarrow \bigoplus_{\mathfrak{p}\notin S} H^2(G_{\mathfrak{p}}(k))\longrightarrow 0.$$

Passing to the limit from k to k^T, we obtain the exact sequence

$$0 \to H^2((k^T)_S|k^T) \to H^2(k(p)|k^T) \to \bigoplus_{\mathfrak{p}\notin S(k^T)}' H^2(G_\mathfrak{p}(k^T)) \to 0.$$

Assuming (ii), we have $H^2(G_\mathfrak{p}(k(p)|k^T)) = 0$ for all $\mathfrak{p} \notin T(k^T)$, and so

$$\bigoplus_{\mathfrak{p}\notin S(k^T)}' H^2(G_\mathfrak{p}(k(p)|k^T)) = \bigoplus_{\mathfrak{p}\in T(k^T)}' H^2(G_\mathfrak{p}(k(p)|k^T)) = \bigoplus_{\mathfrak{p}\in T(k^T)}' H^2(G_\mathfrak{p}(k))$$

for every S with $S \cap T = \varnothing$. It follows that $H^2((k^T)_S|k^T) = 0$, i.e. assertion (iv) holds.

Finally, if $H^2((k^T)_{\bar{T}}|k^T) = H^2(k_{\bar{T}}|k^T) = 0$ for $\bar{T} = \mathcal{P}\backslash T$, then, using again the exact sequence above, assertion (iii) holds. $\square$

There is another situation in which the pro-p group $G((k^T)_S|k^T)$ is free.

Theorem 2.3 *Let p be a prime number and let T and S are sets of primes of a number field k, where $T \cup S \neq \mathcal{P}$ is strongly saturated and $T \cap S = \varnothing$. Then the following holds:*

(i) $(k^T)_S = k_{\bar{T}}$ *and* $\mathrm{cd}_p\, G((k^T)_S|k^T) \leq 1$.

(ii) *If $\delta(S) = 0$, then $\mathrm{cd}_p\, G((k^T)_{nr}|k^T) \leq 1$, i.e. the Galois group of the maximal unramified p-extension $(k^T)_{nr}|k^T$ is a free pro-p-group.*

In particular, if $T \neq \mathcal{P}$ is a strongly saturated set, then

$$\mathrm{cd}_p\, G((k^T)_{nr}|k^T) \leq 1.$$

Proof: Since $T \cup S \neq \mathcal{P}$ is stably saturated, we have $\delta(T) \leq \delta(T \cup S) = 0$, hence $\delta(\bar{T}) = 1$, where $\bar{T} = \mathcal{P}\backslash T$. It follows from proposition (2.2) that the group $G(k_{\bar{T}}|k^T)$ is free. Since $T \cup S$ is strongly saturated, the extension $k^{T\cup S}$ realizes the local extensions for all primes in $\overline{T \cup S} = \bar{T}\backslash S$, and so the extension k^T has this property. Therefore $k_{\bar{T}}|k^T$ is completely decomposed by $\overline{T \cup S}$, hence $k_{\bar{T}} = (k^T)_{\bar{T}} = (k^T)_S$. This proves (i).

Now assume that $\delta(S) = 0$, hence $\delta(\bar{T}\backslash S) = 1$. Let $K|k$ be a finite extension inside k^T. Using the Poitou-Tate theorem and the Hasse principle, we see that the canonical map

$$H^2(K_{\bar{T}\backslash S}|K) \longrightarrow H^2(k_{\bar{T}}|K)$$

is injective. Passing to the limit, it follows that

$$H^2((k^T)_{\bar{T}\backslash S}|k^T) \longrightarrow H^2(k_{\bar{T}}|k^T)$$

is injective. Since $(k^T)_{\bar{T}\backslash S} = (k^T)_{nr}$, the desired result follows from (i). $\square$

By theorem (1.5)(i) finite sets are strongly saturated. Now we will show that there are also infinite strongly saturated sets.

Theorem 2.4 *Let p be an odd prime number and let k be a CM-field containing the group μ_p of all p-th roots of unity, with maximal totally real subfield k^+, i.e. $k = k^+(\mu_p)$ is totally imaginary and $[k : k^+] = 2$. Let*

$$T = \{\mathfrak{p} \mid \mathfrak{p} \cap k^+ \textit{ is inert in } k|k^+\}.$$

Then the set $T_0 = T \cup S_p$ of primes of k is strongly saturated. Furthermore, the Galois group $G((k^{T_0})_{nr}|k^{T_0})$ is a free pro-p-group.

Proof: Let

$$S = \{\mathfrak{p} \mid \mathfrak{p} \cap k^+ \text{ is decomposed in } k|k^+\}$$

and $S_1 = \mathcal{P}\backslash T_0 = S\backslash S_p$. Let S_0 be a subset of S_1 invariant under the action of $G(k|k^+)$ such that $V = S_1\backslash S_0$ is finite (observe that $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$ is cyclic for $\mathfrak{p} \in V$ as $V \cap S_p = \varnothing$). Let $K|k$ be a finite extension inside k^{T_0} being Galois over k^+ (observe that $k^{T_0}|k^+$ is a Galois extension as T_0 is invariant under $G(k|k^+)$). First we show the following

Claim: There exists an abelian (not necessarily finite) p-extension $L|K$, which is Galois over k^+, central over k, unramified by $T_0(K)$, completely decomposed at S_p and ramified at each prime of $V(K)$:

$$T(k_{\mathfrak{p}}(p)|K_{\mathfrak{p}})_{G_{\mathfrak{p}}(k(p)|k)} \subseteq G_{\mathfrak{p}}(L|K) \text{ for all } \mathfrak{p} \in V(k).$$

Proof : Consider the group extension

$$1 \longrightarrow G(K^{S_p}_{S_1\cup S_p}|K^{S_p}_{S_0\cup S_p}) \longrightarrow G(K^{S_p}_{S_1\cup S_p}|K) \longrightarrow G(K^{S_p}_{S_0\cup S_p}|K) \longrightarrow 1.$$

Since $\delta(S_0) = 1$, we have $H^2(G(K^{S_p}_{S_0\cup S_p}|K), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, see (2.1). In the proof of this claim we write $H^i(E|F)$ for $H^i(G(E|F), \mathbb{Q}_p/\mathbb{Z}_p)$. We obtain an exact sequence

$$0 \to H^1(K^{S_p}_{S_0\cup S_p}|K) \to H^1(K^{S_p}_{S_1\cup S_p}|K) \to H^1(K^{S_p}_{S_1\cup S_p}|K^{S_p}_{S_0\cup S_p})^{G(K^{S_p}_{S_0\cup S_p}|K)} \to 0,$$

and so an exact sequence

$$0 \to H^1(K^{S_p}_{S_0\cup S_p}|K)^{G(K|k)} \to H^1(K^{S_p}_{S_1\cup S_p}|K)^{G(K|k)} \to H^1(K^{S_p}_{S_1\cup S_p}|K^{S_p}_{S_0\cup S_p})^{G(K^{S_p}_{S_0\cup S_p}|k)}$$

$$\to H^1(K|k, H^1(K^{S_p}_{S_0\cup S_p}|K)) \to H^1(K|k, H^1(K^{S_p}_{S_1\cup S_p}|K)).$$

Using again that $H^2(G(K_{\tilde{S}}^{S_p}|K), \mathbb{Q}_p/\mathbb{Z}_p) = 0$ where $\tilde{S} = S_0 \cup S_p$ resp. $\tilde{S} = S_1 \cup S_p$, the Hochschild-Serre spectral sequences

$$E_2^{i,j} = H^i(K|k, H^j(K_{\tilde{S}}^{S_p}|K)) \Rightarrow H^{i+j}(K_{\tilde{S}}^{S_p}|k)$$

show that in the commutative diagram

$$\begin{array}{ccc} H^1(K|k, H^1(K_{S_0\cup S_p}^{S_p}|K)) & \longrightarrow & H^1(K|k, H^1(K_{S_1\cup S_p}^{S_p}|K)) \\ \downarrow{\scriptstyle d_2^{2,1}} & & \downarrow{\scriptstyle d_2^{2,1}} \\ H^3(K|k, H^0(K_{S_0\cup S_p}^{S_p}|K)) & = & H^3(K|k, H^0(K_{S_1\cup S_p}^{S_p}|K)) \end{array}$$

the differentials $d_2^{2,1}$ are isomorphisms. Thus we obtain an exact sequence

$$0 \longrightarrow \big(G(K_{S_1\cup S_p}^{S_p}|K_{S_0\cup S_p}^{S_p})^{ab}\big)_{G(K_{S_0\cup S_p}^{S_p}|k)} \longrightarrow$$

$$\big(G(K_{S_1\cup S_p}^{S_p}|K)^{ab}\big)_{G(K|k)} \longrightarrow \big(G(K_{S_0\cup S_p}^{S_p}|K)^{ab}\big)_{G(K|k)} \longrightarrow 0.$$

Using (2.1), we obtain the isomorphism

$$\prod_{\mathfrak{p}\in V(k)} T(k_{\mathfrak{p}}(p)|K_{\mathfrak{p}})_{G_{\mathfrak{p}}(k(p)|k)} \overset{can}{\tilde{\to}} G(K_{S_1\cup S_p}^{S_p}|K_{S_0\cup S_p}^{S_p})^{ab}_{G(K_{S_0\cup S_p}^{S_p}|k)} \subseteq G(K_{S_1\cup S_p}^{S_p}|K)^{ab}_{G(K|k)}.$$

Thus the abelian extension $L|K$ with $G(L|K) = G(K_{S_1\cup S_p}^{S_p}|K)^{ab}_{G(K|k)}$ has the desired properties and we proved the claim.

As $L|K$ is central over k, $G(k|k^+) =< \sigma >$ acts on $G(L|K)$. Thus $G(L|K) = G(L|K)^+ \oplus G(L|K)^-$, where $G(L|K)^\pm = G(L|K)^{\sigma\pm 1}$. Let $L^\pm$ be defined by $G(L|L^\pm) = G(L|K)^\pm$, i.e. $G(L^\pm|K) = G(L|K)/G(L|K)^\mp$.

For $\mathfrak{q} \in T_0 \backslash S_p$ we have the exact sequence

$$1 \longrightarrow G_{\mathfrak{q}}(L^-|k) \longrightarrow G_{\mathfrak{q}}(L^-|k^+) \longrightarrow G_{\mathfrak{q}}(k|k^+) \longrightarrow 1,$$

where $G_{\mathfrak{q}}(L^-|k) = G_{\mathfrak{q}}(L^-|K)$. Suppose that $G_{\mathfrak{q}}(L^-|K) \neq 1$. Since $G(k|k^+) = G_{\mathfrak{q}}(k|k^+)$ acts non-trivially on $G(L^-|K)$, the group $G_{\mathfrak{q}}(L^-|k^+)$ is non-abelian. On the other hand the extension $L^-|k^+$ is unramified at $\mathfrak{q}$. This contradiction shows that all primes of $T_0 \backslash S_p$ are completely decomposed in $L^-|K$, and so in $L^-|k$. Since $L|K$ is completely decomposed at S_p, we obtain $L^- \subseteq k^{T_0}$.

Let $\mathfrak{p} \in V$ and so $\mathfrak{p} \cap k^+$ splits in $k|k^+$. Let $\bar{\mathfrak{p}} = \mathfrak{p}^\sigma$ be the conjugated prime and let $\mathfrak{P}$ be a prolongation of $\mathfrak{p}$ to K and $\bar{\mathfrak{P}} = \mathfrak{P}^\sigma$. By the claim it follows that

$$\big(T(k_{\mathfrak{P}}(p))|K_{\mathfrak{P}})_{G_{\mathfrak{P}}(k(p))|k)} \oplus T(k_{\bar{\mathfrak{P}}}(p))|K_{\bar{\mathfrak{P}}})_{G_{\bar{\mathfrak{P}}}(k(p))|k)}\big)^-$$

injects into $G(L|K)^- \xrightarrow{\sim} G(L^-|K) = G(L|K)/G(L|K)^+$, and so $L^-|K$ is ramified by $\mathfrak{P}$.

Varying the set S_0 and the extension $K|k$, it follows that the extension $k^{T_0}|k$ realizes the ramified part of $k_\mathfrak{p}(p)|k_\mathfrak{p}$ for all $\mathfrak{p} \in S_1 = \mathcal{P}\backslash T_0$. Since $k^{T_0}|k$ is a Galois extension, also the unramified part must be realized, i.e.

$$(k^{T_0})_\mathfrak{P} = k_\mathfrak{p}(p) \quad \text{ for all } \mathfrak{P}|\mathfrak{p},\, \mathfrak{p} \notin T_0$$

(if $k_\mathfrak{P}(p)|(k^{T_0})_\mathfrak{P}$ would have a non-trivial unramified part, then, as the subgroup generated by the Frobenius automorphism is not normal, this extension would also have a ramified part).

The last assertion of the theorem follows from theorem (2.3). $\square$

Let

$$\prod_{t\in T}(A_t, B_t) = \Big\{(a_t)_{t\in T} \in \prod_{t\in T} A_t \mid a_t \in B_t \text{ for almost all } t \in T\Big\}$$

be the restricted product over a discrete set T of abelian locally compact groups A_t with respect to closed subgroups B_t. The topology is given by the subgroups V with

(i) $V \cap A_t$ is open in A_t for all $t \in T$,

(ii) $V \supseteq B_t$ for almost all $t \in T$.

Then we call

$$\prod_{t\in T}^{c}(A_t, B_t) := \varprojlim_{V} \Big(\prod_{t\in T}(A_t, B_t)\Big)/V,$$

the *compactification* of $\prod_{t\in T}(A_t, B_t)$, where V runs through all open subgroups of finite index in $\prod_{t\in T}(A_t, B_t)$. The the canonical map $\prod_{t\in T}(A_t, B_t) \to \prod_{t\in T}^{c}(A_t, B_t)$ has dense image.

We define the *discretization* of $\prod_{t\in T}(A_t, B_t)$ by

$$\prod_{t\in T}^{d}(A_t, B_t) := \varinjlim_{W} W$$

where W runs through the finite subgroups of $\prod_{t\in T}(A_t, B_t)$. If the subgroups B_t of A_t, $t \in T$, are open and compact, then $\prod_{t\in T}(A_t, B_t)$ is locally compact. Using the equality

$$(\prod_{t\in T}(A_t, B_t))^\vee = \prod_{t\in T}(A_t^\vee, (A_t/B_t)^\vee),$$

we obtain

$$\prod_{t\in T}^{d}(A_t, B_t) = (((\prod_{t\in T}(A_t, B_t))^\vee)^c)^\vee \quad \text{ and } \quad \prod_{t\in T}^{c}(A_t, B_t) = (((\prod_{t\in T}(A_t, B_t))^\vee)^d)^\vee,$$

where $^\vee$ denotes the Pontryagin-dual.

Proposition 2.5 *Let T be a set of primes of a number field k with $\delta(T) = 0$. Then*

(i) $G(k(p)/k^T)/G(k(p)/k^T)^* = \varprojlim_K \prod^c_{\mathfrak{P}\in T(K)} (G_{\mathfrak{P}}(k)/G_{\mathfrak{P}}(k)^*, \tilde{T}_{\mathfrak{P}}(k))$,

(ii) $H^1(G(k(p)/k^T)) = \varinjlim_K \prod^d_{\mathfrak{P}\in T(K)} (H^1(G_{\mathfrak{P}}(k)), H^1_{nr}(G_{\mathfrak{P}}(k)))$,

where K runs through the finite subextensions of $k^T|k$ and $\tilde{T}_{\mathfrak{P}}(k)$ is the group $T_{\mathfrak{P}}(k)G_{\mathfrak{P}}(k)^*/G_{\mathfrak{P}}(k)^*$.

Proof: Since the set $\mathcal{P}\backslash T$ has density equal to 1, we get from the Poitou-Tate exact sequence and the Hasse principle the commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\bar{k}|k,\mu_p) \arrow[r] \arrow[d, equal] & \prod_{\mathfrak{p}\notin T} H^1(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mu_p) & & \\
0 \arrow[r] & H^1(\bar{k}|k,\mu_p) \arrow[r] & \prod_{\mathfrak{p}} H^1(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mu_p) \arrow[r] \arrow[u, two heads] & H^1(\bar{k}|k,\mathbb{Z}/p\mathbb{Z})^\vee \arrow[r] & 0 \\
& & \prod_{\mathfrak{p}\in T} H^1(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mu_p) \arrow[r, "\sim"] \arrow[u, hook] & \prod_{\mathfrak{p}\in T} H^1(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mathbb{Z}/p\mathbb{Z})^\vee \arrow[u] &
\end{tikzcd}$$

where we use the local duality theorem

$$H^1(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mu_p) \xrightarrow{\sim} H^1(G_{\mathfrak{p}}(k),\mathbb{Z}/p\mathbb{Z})^\vee = G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*$$

and for $\mathfrak{p}$ not above p or ∞ the isomorphism

$$H^1_{nr}(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}},\mu_p) \xrightarrow{\sim} (H^1(T_{\mathfrak{p}}(k),\mathbb{Z}/p\mathbb{Z})^{G_{\mathfrak{p}}(k)})^\vee \cong T_{\mathfrak{p}}(k)G_{\mathfrak{p}}(k)^*/G_{\mathfrak{p}}(k)^*,$$

see [3] (8.6.10), (9.1.9), (9.1.10), (7.2.15). Thus we get a continuous injection

$$\prod_{\mathfrak{p}\in T(k)} (G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k)) \hookrightarrow G(k(p)/k)/G(k(p)/k)^*$$

and so an injection $\prod^c_{\mathfrak{p}\in T(k)}(G_{\mathfrak{p}}(k)//G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k)) \hookrightarrow G(k(p)/k)/G(k(p)/k)^*$. Passing to the limit, we obtain the injection

$$\varprojlim_K \prod^c_{\mathfrak{P}\in T(K)} (G_{\mathfrak{P}}(k)/G_{\mathfrak{P}}(k)^*, \tilde{T}_{\mathfrak{P}}(k)) \hookrightarrow G(k(p)/k^T)/G(k(p)/k^T)^*$$

which, by definition of the field k^T, is also surjective. Therefore we obtain (i) and (ii) is just the dual assertion. $\square$

Proposition 2.6 *Let p be a prime number and let $T \neq \mathcal{P}$ be a stably saturated set of primes of a number field k. Assume that k is totally imaginary if $p = 2$. Then*

$$cd_p \; G(k^T|k) \leq 2.$$

Proof: We consider the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G/N, H^j(N)) \Rightarrow E^{i+j} = H^{i+j}(G)$$

for an extension $1 \to N \to G \to G/N \to 1$ of pro-p groups. If $\mathrm{cd}_p(G) \leq 2$, $E_2^{11} = 0$ and the edge homomorphism $E^2 \to E_2^{02}$ is surjective, then it follows that $E_2^{30} = 0$, i.e. $\mathrm{cd}_p(G/N) \leq 2$.

Now let $G = G(k(p)|k)$ and $N = G(k(p)|k^T)$. Using (2.2) and the Poitou-Tate theorem, it follows that the canonical map

$$H^2(k(p)|k) \twoheadrightarrow \bigoplus_{\mathfrak{p} \in T(k)} H^2(G_\mathfrak{p}(k)) \xrightarrow{\sim} H^0(k^T|k, H^2(k(p)|k^T))$$

is surjective. Furthermore, using (2.5)(ii), we see that

$$\begin{aligned} H^1(k(p)|k^T) &= \varinjlim_K H^1(\textstyle\prod^c_{\mathfrak{p} \in T(K)} (G_\mathfrak{p}(k)/G_\mathfrak{p}(k)^*, \tilde{T}_\mathfrak{p}(k))) \\ &= \varinjlim_K Coind^{G(K|k)} H^1(\textstyle\prod^c_{\mathfrak{p} \in T(k)} (G_\mathfrak{p}(k)/G_\mathfrak{p}(k)^*, \tilde{T}_\mathfrak{p}(k))) \\ &= Coind^{G(k^T|k)} H^1(\textstyle\prod^c_{\mathfrak{p} \in T(k)} (G_\mathfrak{p}(k)/G_\mathfrak{p}(k)^*, \tilde{T}_\mathfrak{p}(k))) \end{aligned}$$

is a $G(k^T|k)$-coinduced module, hence $H^1(k^T|k, H^1(k(p)|k^T)) = 0$. Therefore we obtain the desired result. $\square$

If T is a set of primes of k, then let $T(k^T) = \varprojlim_K T(K) \cup \{*_K\}$, where K runs through the finite subextensions of k^T/k. The following theorem asserts that the Galois group $G(k(p)/k^T)$, where $T \neq \mathcal{P}$ is stably saturated, is a corestricted free pro-p-product of the family $(G_\mathfrak{P}(k))_{\mathfrak{P} \in T(k^T)}$ of decomposition groups with respect to the continuous family $(T_\mathfrak{P}(k))_{\mathfrak{P} \in T(k^T)}$ of inertia groups, see [2] for the definition.

Theorem 2.7 *Let $T \neq \mathcal{P}$ be a stably saturated set of primes of a number field k. Then the canonical map*

$$\varphi : \mathop{\ast}_{\mathfrak{P} \in T(k^T)} (G_\mathfrak{P}(k), T_\mathfrak{P}(k)) \xrightarrow{\sim} G(k(p)/k^T)$$

is an isomorphism.

Proof: We can apply [2] prop. (4.3) and have to show that the induced maps

$$\varphi_* : H^1(G(k(p)/k^T), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} \varinjlim_K \prod_{T(K)}^{d} (H^1(G_{\mathfrak{P}}(k), \mathbb{Z}/p\mathbb{Z}), H^1_{nr}(G_{\mathfrak{P}}(k), \mathbb{Z}/p\mathbb{Z}))$$

$$\varphi_* : H^2(G(k(p)/k^T), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow \varinjlim_K \prod_{T(K)}^{d} (H^2(G_{\mathfrak{P}}(k), \mathbb{Z}/p\mathbb{Z}), H^2_{nr}(G_{\mathfrak{P}}(k), \mathbb{Z}/p\mathbb{Z}))$$

are bijective resp. injective. Since $H^2_{nr}(G_{\mathfrak{P}}(k)) = 0$, it follows that

$$H^2(G) = \varinjlim_K \prod_{T(K)}^{d} (H^2(G_{\mathfrak{P}}(k), H^2_{nr}(G_{\mathfrak{P}}(k))) = \bigoplus_{\mathfrak{p}\in T(k^T)}{}' H^2(G_{\mathfrak{p}}(k)).$$

Now the result follows from (2.5)(ii) and (2.2). □

In the situation of theorem (2.4) we obtain

Corollary 2.8 *Let p be an odd prime number and let k be a CM-field containing the group μ_p of all p-th roots of unity, with maximal totally real subfield k^+, i.e. $k = k^+(\mu_p)$ is totally imaginary and $[k : k^+] = 2$. Let*

$$T = \{\mathfrak{p} \mid \mathfrak{p} \cap k^+ \text{ is inert in } k|k^+\} \cup S_p\,.$$

Then

$$\mathop{\ast}_{\mathfrak{P}\in T(k^T)} (G_{\mathfrak{P}}(k), T_{\mathfrak{P}}(k)) \xrightarrow{\sim} G(k(p)/k^T).$$

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