Riemann's existence theorem and the $K(\pi, 1)$-property of rings of integers

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Let k be a number field, S a finite set of nonarchimedean primes of k and p a prime number. We assume that p is odd or that k is totally imaginary. Let $k_S(p)$ be the maximal p-extension of k unramified outside S and $G_S(p) = Gal(k_S(p)|k)$. In geometric terms, we have

$$G_S(p) \cong \pi_1((Spec(\mathcal{O}_k)\backslash S)_{et}^{(p)}),$$

where $(Spec(\mathcal{O}_k)\backslash S)_{et}^{(p)}$ is the p-completion of the étale homotopy type of the scheme $Spec(\mathcal{O}_k)\backslash S$. If S contains the set S_p of primes dividing p (the *wild case*), then $G_S(p)$ has cohomological dimension less or equal to 2. Furthermore, if $T \supseteq S \supseteq S_p$ are sets of primes of k, then the canonical homomorphisms

$$\phi_{T,S} : \underset{\mathfrak{p}\in(T\backslash S)(k_S(p))}{\ast} T_{\mathfrak{p}}(k(p)|k) \longrightarrow G(k_T(p)|k_S(p))$$

of the free pro-p product of the groups $T_{\mathfrak{p}}(k(p)|k)$ into $G(k_T(p)|k_S(p))$; here $T_{\mathfrak{p}}(k(p)|k)$ is the inertia subgroup of the decomposition group $G_{\mathfrak{p}}(k(p)|k) \cong G(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$, where $k_{\mathfrak{p}}$ is the completion of k with respect to the prime $\mathfrak{p}$. We say that Riemann's existence theorem holds for k, S, T.

In the *tame case*, i.e. $S \cap S_p = \varnothing$, and in the *mixed case*, i.e. $\varnothing \neq S \cap S_p \not\subseteq S_p$, until recently not much was known about the group $G_S(p)$: In the tame case $G_S(p)$ is a finitely presented pro-p-group (Koch), which can be infinite (Golod-Šafarevič), and which is a fab-group, i.e. U^{ab} is finite for each open subgroup $U \subseteq G_S(p)$.

In 2005, Labute considered the case $k = \mathbb{Q}$ and found finite sets S of prime numbers (called strictly circular sets) with $p \notin S$ such that $G_S(p)$ has cohomological dimension 2. In [S2] A. Schmidt also considered the tame case: he showed that for a number filed k, which does not contain the group of p-th roots of unity and whose p-part of its ideal class group is trivial, there always exists a finite set T of primes with $T \cap S_p = \varnothing$, such that $(Spec(\mathcal{O}_k)\backslash(S \cup T))_{et}^{(p)}$ is a $K(\pi,1)$ for p, i.e. the higher étale homotopy groups of $(Spec(\mathcal{O}_k)\backslash(S \cup T))_{et}^{(p)}$ vanish; in particular, $\mathrm{cd}_p\, G_{S\cup T}(p) \le 2$.

In this paper we will study the relationship of the $K(\pi,1)$-property of the scheme $Spec(\mathcal{O}_k)\backslash S$ and Riemann's existence theorem for sets $T \supseteq S$, where S is an arbitrary finite set of nonarchimedean primes. We extend results of [5] in the following way (see also [6]):

Theorem. *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let $T \supseteq S$ be finite sets of nonarchimedean primes of k. Assume that $(k_S(p))_{\mathfrak{p}} \neq k_{\mathfrak{p}}$ for all $\mathfrak{p} \in (T\backslash S) \cap S_p$. Then we have the following assertions are equivalent:*

(i) *$Spec(\mathcal{O}_k)\backslash S$ is a $K(\pi,1)$ for p and $(k_S(p))_{\mathfrak{q}} \neq k_{\mathfrak{q}}$ for all $\mathfrak{q} \in (T\backslash(S \cup S_p))_{\min}$.*

(ii) *$Spec(\mathcal{O}_k)\backslash T$ is a $K(\pi,1)$ for p and*

$$\mathop{\ast}_{\mathfrak{p}\in T\backslash S(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_T(p)|k_S(p)).$$

Using this theorem and results of [5], we will show that not only in the tame case but also in the mixed case one can find finite sets S of primes such that $\mathrm{cd}_p\, G_S(p) \le 2$.

1 Free product decomposition

We introduce some notation. If p is a fixed prime number and G a pro-p group, then $H^i(G)$ denotes the cohomology group $H^i(G, \mathbb{Z}/p\mathbb{Z})$ and we put $h^i(G) = \dim_{\mathbb{F}_p} H^i(G)$. Furthermore,

$$\chi(G) = \sum_i (-1)^i h^i(G) \quad \text{and} \quad \chi_n(G) = \sum_{i=0}^{n} (-1)^i h^i(G)$$

denotes the Euler-Poincaré characteristic and partial Euler-Poincaré characteristic of G, respectively. If $K|k$ is a Galois p-extension with Galois group $G(K|k)$, we sometimes write $H^i(K|k)$ for $H^i(G(K|k))$.

Let k is a number field with absolute Galois group by G_k. If p is a prime number, then $k(p)$ is the maximal p-extension of k with Galois group $G_k(p) = G(k(p)|k)$. If $K|k$ is a Galois p-extension with Galois group $G(K|k)$, we sometimes write $H^i(K|k)$ for $H^i(G(K|k))$.

By S_∞, $S_{\mathbb{R}}$ and $S_{\mathbb{C}}$ we denote the sets of archimedean, real and complex primes of k and put $r_1(k) = \#S_{\mathbb{R}}$ and $r_2(k) = \#S_{\mathbb{C}}$, respectively. We consider the extension $\mathbb{C}|\mathbb{R}$ as ramified. If p is a prime number, then S_p is the set of all primes of K above p.

If $\mathfrak{p}$ is a prime k, then $k_\mathfrak{p}$ is the completion of k with respect to $\mathfrak{p}$ with absolute Galois group $G_{k_\mathfrak{p}}$, and $U_\mathfrak{p}$ denotes is group of units.

If $K|k$ is a Galois extension, then we denote the decomposition group and inertia group of the Galois group $G(K|k)$ with respect to $\mathfrak{p}$ by $G_\mathfrak{p}(K|k)$ and $T_\mathfrak{p}(K|k)$, respectively. We write $G_\mathfrak{p} = G_\mathfrak{p}(k) = G_\mathfrak{p}(k(p)|k) \cong G(k_\mathfrak{p}(p)|k_\mathfrak{p})$ and $T_\mathfrak{p} = T_\mathfrak{p}(k) = T_\mathfrak{p}(k(p)|k) \cong T(k_\mathfrak{p}(p)|k_\mathfrak{p})$; then $G_\mathfrak{p}/T_\mathfrak{p} = G(k_\mathfrak{p}^{nr}(p)|k_\mathfrak{p})$, where $k_\mathfrak{p}^{nr}(p)$ is the maximal unramified p-extension of $k_\mathfrak{p}$.

If $S = S(k)$ is a set of primes and $k'|k$ an algebraic extension of k, then we denote the set of primes of k' consisting of all prolongations of S by $S(k')$. Furthermore,

k_S	is the maximal extension of k which is unramified outside S,
$k_S(p)$	is the maximal p-extension of k which is unramified outside S,

and by $G_S = G_S(k)$ and $G_S(p) = G_S(k)(p)$ we denote the Galois groups $G(k_S|k)$ and $G(k_S(p)|k)$, respectively.

For an arbitrary set S of primes of k we define the Šafarevič-Tate groups $Ш^i(G_S(p)) = Ш^i(G_S(p), \mathbb{Z}/p\mathbb{Z})$ and the groups $\mathrm{coker}^i(G_S(p))$ by the exactness of the sequences

$$0 \longrightarrow Ш^i(G_S(p)) \longrightarrow H^i(G_S(p)) \longrightarrow \prod_{\mathfrak{p}\in S} H^i(G_\mathfrak{p}) \longrightarrow \mathrm{coker}^i(G_S(p)) \longrightarrow 0\,.$$

Let

$$V_S(k) = \ker\left(k^\times/k^{\times p} \longrightarrow \prod_{\mathfrak{p}\in S} k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p} \times \prod_{\mathfrak{p}\notin S} k_\mathfrak{p}^\times/U_\mathfrak{p}k_\mathfrak{p}^{\times p}\right),$$

and $Б_S(k) = V_S(k)^\vee$. Observe that when $\mu_p \subseteq k$

$$\begin{aligned} Б_S(k) &= \ker(H^1(G_S(p),\mu_p) \to \textstyle\prod_{\mathfrak{p}\in S} H^1(G_\mathfrak{p},\mu_p))^\vee \\ &= (Cl_S(k)/p)(-1)\,. \end{aligned}$$

Furthermore, we set

$$\delta = \begin{cases} 1, & \mu_p \subseteq k, \\ 0, & \mu_p \nsubseteq k, \end{cases} \quad \text{and} \quad \delta_\mathfrak{p} = \begin{cases} 1, & \mu_p \subseteq k_\mathfrak{p}, \\ 0, & \mu_p \nsubseteq k_\mathfrak{p}. \end{cases}$$

The following primes cannot ramify in a p-extension, and are therefore redundant in S:

1. Complex primes.
2. Real primes if $p \neq 2$.
3. Primes $\mathfrak{p} \nmid p$ with $N(\mathfrak{p}) \not\equiv 1 \bmod p$.

Removing all these redundant places from S, we obtain a subset $S_{\min} \subseteq S$ which has the property that

$$G_S(p) = G_{S_{\min}}(p).$$

We need some results on the cohomology of a free product in the following case, see [3] chap.IV: Let $T = \varprojlim_\lambda \bar{T}_\lambda$, where the sets $\bar{T}_\lambda = T_\lambda \cup \{*_\lambda\}$ are the one-point compactifications of discrete sets T_λ. Let $\mathcal{G} = \varprojlim_\lambda \mathcal{G}_\lambda$ be the projective limit of bundles $\mathcal{G}_\lambda = \dot{\bigcup}_{t_\lambda \in T_\lambda} G_{t_\lambda} \ \dot{\bigcup} \{*_\lambda\}$, and let $G_t = \varprojlim_\lambda G_{t_\lambda}$. Let A be an abelian torsion group considered as a trivial G-module where $G = \mathop{\ast}\limits_T \mathcal{G}$. Then there are isomorphisms

$$H^i(G, A) = \varinjlim_\lambda \bigoplus_{T_\lambda} H^i(G_{t_\lambda}, A), \quad i \geq 0.$$

We will use the notation

$$\bigoplus_T{}' H^i(G_t, A) := \varinjlim_\lambda \bigoplus_{T_\lambda} H^i(G_{t_\lambda}, A).$$

We need the following

Lemma 1.1 *Let*

$$\begin{tikzcd} 1 \arrow[r] & \mathscr{H}_\lambda \arrow[r] \arrow[d, hook] & \mathscr{G}_\lambda \arrow[r] \arrow[d, hook] & G_\lambda \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathscr{H} \arrow[r] & \mathscr{G} \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

be an exact and commutative diagram of pro-p-groups and assume that $\mathscr{H}$ is a free pro-p-product of the form

$$\mathop{\ast}_{\lambda \in S} \mathop{\ast}_{\sigma \in G|G_\lambda} \mathscr{H}_\lambda^\sigma \xrightarrow{\ \sim\ } \mathscr{H},$$

where S is a profinite set, $\mathscr{H}_\lambda^\sigma$ is a closed subgroup of $\mathscr{H}$, which is conjugated to $\mathscr{H}_\lambda$ under an arbitrary extension of σ to $\mathscr{G}$, and $G|G_\lambda$ is a complete system of representatives of G_λ in G. Assume that $cd_p\ \mathscr{H}_\lambda \leq 1$ and $cd_p\ G_\lambda \leq 1$ for all $\lambda \in S$. Then there is an exact sequence

$$0 \to H^1(G, A) \to H^1(\mathscr{G}, A) \to \bigoplus_\lambda{}' H^1(\mathscr{H}_\lambda, A)^{G_\lambda}$$

$$\to H^2(G, A) \to H^2(\mathscr{G}, A) \to \bigoplus_\lambda{}' H^2(\mathscr{G}_\lambda, A) \to H^3(G, A) \to H^3(\mathscr{G}, A) \to 0,$$

where A is a torsion group (considered as a $\mathscr{G}$-module with trivial action), and

(i) *$cd_p\ \mathscr{G} \leq 2$ implies $cd_p\ G \leq 3$,*

(ii) *$cd_p\ G \leq 2$ implies $cd_p\ \mathscr{G} \leq 2$.*

Proof: Using the results on the cohomology of free products, see [3] chap.IV, we obtain

$$H^i(G, H^j(\mathscr{H}, A)) \cong {\bigoplus_{\lambda\in S}}' H^i(G_\lambda, H^j(\mathscr{H}_\lambda, A)), \qquad j \geq 1.$$

These groups can be non-trivial only for $i = 0, 1$ and $j = 1$. Furthermore, we have

$$H^1(G_\lambda, H^1(\mathscr{H}_\lambda, A)) \cong H^2(\mathscr{G}_\lambda, A).$$

Since $cd_p\,\mathscr{H} \leq 1$, the Hochschild-Serre spectral sequence gives the result. $\square$

Corollary 1.2 *Let k be number field and p prime number. Assume that k is totally imaginary if $p = 2$. Let $T \supseteq S$ be non-empty sets of primes of k. Assume that $S_p \subseteq T$. Assume further that we have a free product decomposition*

$$\mathop{\ast}_{\mathfrak{p}\in(T\backslash S)(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_T(p)|k_S(p)),$$

and that $(k_S(p))_{\mathfrak{p}} = k_{\mathfrak{p}}^{nr}(p)$ for all $\mathfrak{p} \in (T\backslash S)_{\min}$. Then

$$cd_p\ G(k_S(p)|k) \leq 2.$$

Proof: Since

$$cd_p\, T_{\mathfrak{p}}(k) = 1, \quad cd_p\, G_{\mathfrak{p}}(k)/T_{\mathfrak{p}}(k) = 1, \quad cd_p\, G(k_T(p)|k) \leq 2,$$

we obtain from lemma (1.1), that the vertical left sequence in the commutative diagram

$$\begin{tikzcd}
 & \bigoplus_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}}(k)) \arrow[d, hook] \arrow[dr, two heads] & \\
H^2(G(k_T(p)|k)) \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p}\in T} H^2(G_{\mathfrak{p}}(k)) \arrow[r, two heads, "\Sigma"] \arrow[d] & H^0(G(k_T|k), \mu_p)^\vee \\
\bigoplus_{\mathfrak{p}\in T\backslash S} H^2(G_{\mathfrak{p}}(k)) \arrow[r, equal] \arrow[d, two heads] & \bigoplus_{\mathfrak{p}\in T\backslash S} H^2(G_{\mathfrak{p}}(k)) & \\
H^3(G(k_S(p)|k)). & &
\end{tikzcd}$$

is exact. By the theorem of Poitou-Tate, see [3] (8.6.13), the horizontal sequence is exact. We obtain $H^3(G(k_S(p)|k)) = 0$, hence $cd_p\ G(k_S(p)|k) \leq\ 2$. $\square$

Proposition 1.3 *Let p be a prime number and let k be the number field.*

(i) *For an arbitrary set S of primes of k there is a canonical exact and commutative diagram*

$$\begin{tikzcd}
H^1(G(k(p)|k)) \arrow[r] \arrow[d, equal] & \bigoplus\limits_{\mathfrak{p}\notin S} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)} \arrow[r] & Б_S(k) \arrow[r] & 0 \\
H^1(G(k(p)|k)) \arrow[r] & H^1(G(k(p)|k_S(p)))^{G_S(p)} \arrow[r] \arrow[u, hook] & Ш^2(G_S(p)) \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

(ii) *Let $T \supseteq S$ be sets of primes of k. Assume that*

$$\varinjlim_{k'\subseteq k_S(p)} Б_S(k') = 0\,,$$

where k' runs through the finite extensions of k inside $k_S(p)$. Then the canonical map

$$H^1(G(k_T(p)|k_S(p))) \xrightarrow{\sim} {\bigoplus_{\mathfrak{p}\in T\setminus S(k_S(p))}}' H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k_S(p))}$$

is an isomorphism.

(iii) *Let $T \supseteq S \supseteq S_p \cup S_\infty$ be sets of primes of k. Then the canonical map*

$$H^1(G(k_T(p)|k_S(p))) \xrightarrow{\sim} {\bigoplus_{\mathfrak{p}\in T\setminus S(k_S(p))}}' H^1(T_\mathfrak{p}(k))$$

is an isomorphism.

Proof: Let $T_S = G(k(p)|k_S(p))$. We consider the group extension

$$1 \longrightarrow T_S \longrightarrow G_k(p) \longrightarrow G_S(p) \longrightarrow 1\,.$$

From the commutative exact diagram

$$\begin{tikzcd}
H^1(G_k(p)) \arrow[r] & H^1(T_S)^{G_S(p)} \arrow[r] & H^2(G_S(p)) \arrow[r] \arrow[d] & H^2(G_k(p)) \arrow[d, hook] \\
 & & \bigoplus\limits_{\mathfrak{p}\in S} H^2(G_\mathfrak{p}(k)) \arrow[r, hook] & \bigoplus\limits_{\mathfrak{p}} H^2(G_\mathfrak{p}(k)),
\end{tikzcd}$$

where the right-hand vertical map is injective by [3](9.1.10) and (10.4.8), we obtain the exact sequence

$$H^1(G_k(p)) \longrightarrow H^1(T_S)^{G_S(p)} \longrightarrow Ш^2(G_S(p)) \longrightarrow 0\,.$$

Furthermore, we consider the commutative exact diagram

$$\begin{tikzcd}
H^1(T_S)^{G_S(p)} \arrow[r, hook] & \bigoplus\limits_{\mathfrak{p}\notin S} H^1(T_\mathfrak{p})^{G_\mathfrak{p}} & & \\
H^1(G_k(p)) \arrow[u] \arrow[r, hook] & \prod\limits_{\mathfrak{p}} H^1(G_\mathfrak{p}) \arrow[u, two heads] \arrow[r, two heads] & H^1(G_k,\mu_p)^\vee \arrow[d, equal] & \\
 & \prod\limits_{\mathfrak{p}\in S} H^1(G_\mathfrak{p}) \times \prod\limits_{\mathfrak{p}\notin S} H^1_{nr}(G_\mathfrak{p}) \arrow[u, hook] \arrow[r] & H^1(G_k,\mu_p)^\vee \arrow[r, two heads] & Б_S(k)\,.
\end{tikzcd}$$

The row in the middle is exact by the Poitou-Tate theorem, see [3] (8.6.10) and (9.1.10), and the upper map is injective by definition of the group T_S. The exactness of the bottom row follows from the definition of $Б_S(k) = (V_S(k))^\vee$ and from $H^1_{nr}(G_\mathfrak{p})^\vee = k_\mathfrak{p}^\times/U_\mathfrak{p}k_\mathfrak{p}^{\times p}$. This diagram and the exact sequence above imply that the commutative diagram

$$\begin{tikzcd}
H^1(G_k(p)) \arrow[r] \arrow[d, equal] & \bigoplus\limits_{\mathfrak{p}\notin S} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)} \arrow[r] & Б_S(k) \arrow[r] & 0 \\
H^1(G_k(p)) \arrow[r] & H^1(T_S)^{G_S(p)} \arrow[u] \arrow[r] & Ш^2(G_S(p)) \arrow[r] & 0
\end{tikzcd}$$

is exact. This finishes the proof of (i).

Now let $T \supseteq S$ be sets of primes of k. Using (i) and passing to limit, we obtain

$$H^1(G(k(p)|k_S(p))) \xrightarrow{\sim} {\bigoplus_{\mathfrak{p}\notin S(k_S(p))}}' H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k_S(p))},$$

as $\varinjlim_{k'\subseteq k_S(p)} Б_S(k') = 0$ by assumption. From this assumption follows that $\varinjlim_{k'\subseteq k_S(p)} Б_T(k') = 0$, as $Б_S(k')$ surjects onto $Б_T(k')$. Thus we also obtain an isomorphism

$$H^1(G(k(p)|k_T(p)))^{G(k_T(p)|k_S(p))} \xrightarrow{\sim} {\bigoplus_{\mathfrak{p}\notin T(k_S(p))}}' H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k_S(p))}$$

Now the the exact sequence

$$0 \to H^1(G(k_T(p)|k_S(p))) \to H^1(G(k(p)|k_S(p))) \to H^1(G(k(p)|k_T(p)))^{G(k_T(p)|k_S(p))}.$$

implies assertion (ii).

If $S_\infty \cup S_p \subseteq S$, then we have an isomorphism of finite groups

$$Ш^2(G_S(p)) \cong Б_S(k)$$

by [3] (10.4.8) and (8.6.9). Therefore the map

$$H^1(G(k(p)|k_S(p)))^{G_S(k)(p)} \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \notin S(k)} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)},$$

is an isomorphism. Passing to the limit and observing that $G_\mathfrak{p}(k_S(p)) = T_\mathfrak{p}(k)$ for $\mathfrak{p} \notin S$ as $k_S(p)$ contains the cyclotomic $\mathbb{Z}_p$-extension, we obtain

$$H^1(G(k(p)|k_S(p))) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \notin S(k_S(p))}{}' H^1(T_\mathfrak{p}(k)).$$

By the same argument as in (ii), the last assertion follows. $\square$

Proposition 1.4 *Let p be a prime number, k a the number field and $T \supseteq S$ sets of primes of k. Assume that*

(i) $\varinjlim_{k' \subseteq k_S(p)} Б_S(k') = 0,$

(ii) *the local extensions $(k_S(p))_\mathfrak{p}|k_\mathfrak{p}$ are infinite for all $\mathfrak{p} \in T_{\min} \backslash S_\infty$, and, if $p = 2$, then $(k_S(2))_\mathfrak{p} = \mathbb{C}$ for all $\mathfrak{p} \in S \cap S_\infty$.*

Then there is a free product decomposition

$$\mathop{\ast}_{\mathfrak{p} \in T \backslash S(k_S(p))} T_\mathfrak{p}(k) \xrightarrow{\sim} G(k_T(p)|k_S(p)).$$

Proof: We may assume that $T = T_{\min}$. Since $(k_S(p))_\mathfrak{p}|k_\mathfrak{p}$ is infinite for a prime $\mathfrak{p} \in T\backslash(S \cup S_\infty)$, the field $k_S(p)_\mathfrak{p}$ is the maximal unramified p-extension of $k_\mathfrak{p}$. Using proposition (1.3)(ii), it follows that

$$H^1(G(k_T(p)|k_S(p))) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in T \backslash S(k_S(p))}{}' H^1(T_\mathfrak{p}(k)).$$

Now we consider the exact sequence

$$0 \longrightarrow Ш^2(G_T(k')(p)) \longrightarrow H^2(G(k_T(p)|k')) \longrightarrow \bigoplus_{\mathfrak{p} \in T(k')} H^2(G_\mathfrak{p}(k')),$$

where k' is a finite extension of k inside $k_S(p)$. Passing to the limit, we obtain

$$0 \longrightarrow \varinjlim_{k' \subseteq k_S(p)} Ш^2(G_T(k')(p)) \longrightarrow H^2(G(k_T(p)|k_S(p))) \longrightarrow \bigoplus_{\mathfrak{p} \in T(k_S(p))}{}' H^2(G_{\mathfrak{p}}(k_S(p))).$$

By proposition (1.3)(i), we have an injection

$$Ш^2(G_T(k')(p)) \hookrightarrow Б_T(k'),$$

and the group on the right-hand side is an homomorphic image of $Б_S(k')$. Since $\varinjlim_{k' \subseteq k_S(p)} Б_S(k')$ is trivial by assumption, it follows that

$$\varinjlim_{k' \subseteq k_S(p)} Ш^2(G_T(k')(p)) = 0.$$

Furthermore, $H^2(G_{\mathfrak{p}}(k_S(p))) \cong H^2(G(k_{\mathfrak{p}}(p)|k_S(p)_{\mathfrak{p}})) = 0$ for all $\mathfrak{p} \in T \backslash S_\infty$ as $k_S(p)_{\mathfrak{p}}|k_{\mathfrak{p}}$ is infinite, see [3] (7.1.8)(i), (7.5.8). It follows that

$$H^2(G(k_T(p)|k_S(p))) \longrightarrow \bigoplus_{\mathfrak{p} \in (S_\infty \cap (T \backslash S))(k_S(p))}{}' H^2(T_{\mathfrak{p}}(k)) = \bigoplus_{\mathfrak{p} \in T \backslash S(k_S(p)}{}' H^2(T_{\mathfrak{p}}(k))$$

is injective. Thus we proved that

$$H^i(G(k_T(p)|k_S(p))) \longrightarrow H^i(\mathop{\ast}_{\mathfrak{p} \in T \backslash S(k_S(p))} T_{\mathfrak{p}}(k))$$

is an isomorphism for $i = 1$ and injective for $i = 2$. By [3](1.6.15), the desired result follows. $\square$

2 The $K(\pi, 1)$-property

A locally noetherian scheme Y is called a $K(\pi, 1)$ for a prime number p if the higher homotopy groups of the p-completion $Y_{et}^{(p)}$ of its etale homotopy type Y_{et} vanish, see [5] §2.

Let p a fixed prime number. Let k be a number field and S a finite set of nonarchimedean primes of k. We assume that k is totally imaginary if $p = 2$. For the scheme $X = Spec(\mathcal{O}_k) \backslash S$ we have

$$G_S(p) \cong \pi_1((Spec(\mathcal{O}_k) \backslash S)_{et}^{(p)}),$$

where we omit the base point. We consider the property

$$\mathcal{K}(\mathcal{O}_k, S) : \quad Spec(\mathcal{O}_k) \backslash S \ \text{ is a } K(\pi, 1) \text{ for } p.$$

If S is infinite, one can extend the notion of being a $K(\pi, 1)$ for p in an obvious manner, see [5] §4. In the following we write $H^i_{et}(Spec(\mathcal{O}_k) \backslash S)$ for the group $H^i_{et}(Spec(\mathcal{O}_k) \backslash S, \mathbb{Z}/p\mathbb{Z})$ and $h^i(Spec(\mathcal{O}_k) \backslash S) = \dim_{\mathbb{F}_p} H^i_{et}(Spec(\mathcal{O}_k) \backslash S)$

Proposition 2.1 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let S be a non-empty set of non-archimedean primes of k. Then the following assertions are equivalent:*

(i) *$Spec(\mathcal{O}_k)\backslash S$ is a $K(\pi,1)$ for p.*

(ii) *$cd_p\, G_S(p) \leq 2$ and the canonical map*

$$H^2(G_S(p)) \hookrightarrow H^2_{et}(Spec(\mathcal{O}_k)\backslash S)$$

is surjective.

(iii) *$cd_p\, G_S(p) \leq 2$, $Ш^2(G_S(p)) \xrightarrow{\sim} Б_S(k)$ and $\dim_{\mathbb{F}_p} \operatorname{coker}^2(G_S(p)) = \delta$.*

(iv) *$cd_p\, G_S(p) \leq 2$,* $\quad H^1(G(k_T(p)|k_S(p)))^{G_S(p)} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in T\backslash S} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)}$

for some set T containing $S \cup S_p$ and $\dim_{\mathbb{F}_p} \operatorname{coker}^2(G_S(p)) = \delta$.

If S is finite, then these assertions are equivalent to

(v) *$cd_p\, G_S(p) \leq 2$ and* $\chi(G_S(p)) = r_1(k) + r_2(k) - \sum_{\mathfrak{p}\in S\cap S_p} [k_\mathfrak{p} : \mathbb{Q}_p]$.

Proof: For the equivalence (i)$\Leftrightarrow$ (ii) see [5] cor. 3.5. In order to show (ii)$\Leftrightarrow$ (iii) we only have to consider the commutative and exact diagram

$$\begin{tikzcd}
Ш^2(G_S(p)) \arrow[r, hook] \arrow[d, hook] & H^2(G_S(p)) \arrow[r] \arrow[d, hook] & \prod_{\mathfrak{p}\in S} H^2(G_\mathfrak{p}(k)) \arrow[d, equal] & \\
Б_S(k) \arrow[r, hook] & H^2_{et}(Spec(\mathcal{O}_k)\backslash S) \arrow[r] & \prod_{\mathfrak{p}\in S} H^2(G_\mathfrak{p}(k)) \arrow[r, two heads] & H^3_{et}(Spec(\mathcal{O}_k)),
\end{tikzcd}$$

where $\dim_{\mathbb{F}_p} H^3_{et}(Spec(\mathcal{O}_k))) = \delta$, see [5] thm.3.4 and thm.3.6.

By (1.3)(i), the surjectivity of the map $Ш^2(G_S(p)) \to Б_S(k)$ is equivalent to

$$H^1(G(k(p)|k_S(p)))^{G_S(p)} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\notin S} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)}.$$

Using $T \supseteq S_p$ and (1.3)(iii), we obtain

$$H^1(G(k(p)|k_T(p)))^{G_T(p)} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\notin T} H^1(T_\mathfrak{p}(k))^{G_\mathfrak{p}(k)}.$$

Therefore the commutative and exact diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak{p}\in T\setminus S} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \arrow[r, hook] & \bigoplus_{\mathfrak{p}\notin S} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \arrow[r, two heads] & \bigoplus_{\mathfrak{p}\notin T} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \\
H^1(k_T(p)|k_S(p))^{G_S(p)} \arrow[r, hook] \arrow[u, hook] & H^1(k(p)|k_S(p))^{G_S(p)} \arrow[r] \arrow[u, hook] & H^1(k(p)|k_T(p))^{G_T(p)} \arrow[u, "\cong"']
\end{tikzcd}$$

shows (iii)$\Leftrightarrow$ (iv).

Now let S be finite. By [5] prop.3.2,

$$\begin{aligned}
\chi(Spec(\mathcal{O}_k)\setminus S) &:= \sum_i (-1)^i h^i(Spec(\mathcal{O}_k)\setminus S) \\
&= r_1(k) + r_2(k) - \sum_{\mathfrak{p}\in S\cap S_p} [k_{\mathfrak{p}} : \mathbb{Q}_p].
\end{aligned}$$

Since $cd_p\, G_S(p) \le 2$, we have

$$\begin{aligned}
\chi(G_S(p)) &= \sum_{i=0}^{2} (-1)^i h^i(G_S(p)) \\
&= \chi(Spec(\mathcal{O}_k)\setminus S) + h^2(G_S(p)) - h^2(Spec(\mathcal{O}_k)\setminus S).
\end{aligned}$$

This shows (ii)$\Leftrightarrow$(v).

$\square$

Remarks:

(i) If S contains S_p, then $Spec(\mathcal{O}_k)\setminus S$ is a $K(\pi,1)$ for p. This follows from the equivalence (i)$\Leftrightarrow$(iv) of proposition (2.1) and [3] (8.3.18),(10.4.9), see also [5] prop.2.3

(ii) Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let S be a non-empty finite set of non-archimedean primes of k. Assume that $Spec(\mathcal{O}_k)\setminus S$ is a $K(\pi,1)$ for p. Then the sequence

$$0 \longrightarrow Б_S(k) \longrightarrow H^2(G_S(p)) \longrightarrow \prod_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}}) \xrightarrow{\Sigma} H^0(G_k,\mu_p)^\vee \longrightarrow 0$$

is exact, where Σ is the dual map of the diagonal embedding

$$H^0(G_k,\mu_p) \to \prod_{\mathfrak{p}\in S} H^0(G_{k_{\mathfrak{p}}},\mu_p) \cong \prod_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}})^\vee.$$

This follows from (2.1)(i)⇔(iii) and the commutative and exact diagram

$$\begin{tikzcd}
H^2(G_S(p)) \arrow[r] \arrow[d] & \prod\limits_{\mathfrak{p}\in S} H^2(G_\mathfrak{p}(k)) \arrow[r] \arrow[d, hook] \arrow[dr, two heads] & \mathrm{coker}^2(G_S(p)) \arrow[r] \arrow[d, two heads] & 0 \\
H^2(G_{S\cup S_p}(p)) \arrow[r] & \prod\limits_{\mathfrak{p}\in S\cup S_p} H^2(G_\mathfrak{p}(k)) \arrow[r, "\Sigma"] & H^0(G_k,\mu_p)^\vee \arrow[r] & 0,
\end{tikzcd}$$

where the lower exact sequence is part of the 9-term exact sequence of the theorem of Poitou-Tate.

The following proposition is taken from [5] cor.2.2, and the proof presented here from [1].

Proposition 2.2 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let S be a non-empty finite set of non-archimedean primes of k. Let $k'|k$ be a finite extension inside $k_S(p)$. Then the following assertions are equivalent:*

(i) *$Spec(\mathcal{O}_k)\backslash S$ is a $K(\pi,1)$ for p.*

(ii) *$Spec(\mathcal{O}_{k'})\backslash S$ is a $K(\pi,1)$ for p.*

Proof: Let $R(k,S) = r_1(k) + r_2(k) - \sum_{\mathfrak{p}\in S\cap S_p}[k_\mathfrak{p} : \mathbb{Q}_p]$. Since p is odd or k is totally imaginary, we have $R(k',S) = [k' : k]R(k,S)$. Therefore, using the equivalence (i)⇔(v) of proposition (2.1) and $\chi(G_S(k')(p)) = \chi(G_S(k)(p))[k' : k]$, assertion (i) implies (ii). Conversely, let $k''|k$ be a finite extension inside $k_S(p)$ containing k'. Then, using the implication (i)⇒(ii), we obtain

$$\begin{aligned}
\chi_2(G_S(k'')(p)) &= \chi(G_S(k'')(p)) \\
&= \chi(Spec(\mathcal{O}_{k''})\backslash S) \\
&= [k'' : k]\chi(Spec(\mathcal{O}_k)\backslash S) \\
&= [k'' : k]\Big(\chi_2(G_S(k)(p)) + h^2(G_S(k)(p)) - h^2(Spec(\mathcal{O}_k)\backslash S)\Big) \\
&\geq [k'' : k]\chi_2(G_S(k)(p))
\end{aligned}$$

Using [3] (3.3.15) equality follows, and so $h^2(G_S(k)(p)) = h^2(Spec(\mathcal{O}_k)\backslash S)$, and by [3] (3.3.16), $\mathrm{cd}_p\, G_S(k)(p) \leq 2$. □

The following proposition is taken from [5] thm.9.1.

Proposition 2.3 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Assume that $Spec(\mathcal{O}_k)\backslash S$ is a $K(\pi,1)$ for p and $G_S(p) \neq 1$. Then $k_S(p)$ realizes the maximal p-extension $k_{\mathfrak{q}}(p)$ of $k_{\mathfrak{q}}$ where $\mathfrak{q} \in S_{\min}\backslash S_p$.*

Proof: We have only to show that $\mathfrak{q}$ ramifies in $k_S(p)|k$. Suppose not, then $k_S(p) = k_{S'}(p)$, where $S' = S\backslash\{\mathfrak{q}\}$. By proposition (2.1)(i)⇔(v), it follows that $Spec(\mathcal{O}_k)\backslash S'$ is a $K(\pi,1)$ for p, and so $Ш^2(G_{S'}(p)) \xrightarrow{\sim} Б_{S'}(k)$. The commutative and exact diagram

$$\begin{tikzcd} 0 \arrow[r] & Б_{S'}(k) \arrow[r] & H^2(G_{S'}(p)) \arrow[r] \arrow[d, equal] & \prod_{\mathfrak{p}\in S'} H^2(G_{\mathfrak{p}}(k)) \arrow[d, equal] \\ 0 \arrow[r] & Б_S(k) \arrow[r] & H^2(G_S(p)) \arrow[r] & \prod_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}}(k)), \end{tikzcd}$$

shows that $Б_{S'}(k) \xrightarrow{\sim} Б_S(k)$. Using [3] (10.7.12), it follows that $h^1(G_S(p)) = h^1(G_{S'}(p)) + 1$ which is a contradiction. $\square$

Let $T \supseteq S$ be sets of nonarchimedean primes of k. We consider the properties

$$\begin{aligned} &\mathcal{L}_0(k,S,T): && (k_S(p))_{\mathfrak{p}} \neq k_{\mathfrak{p}} && \text{for all } \mathfrak{p} \in (T\backslash S)\cap S_p\,, \\ &\mathcal{L}_1(k,S,T): && (k_S(p))_{\mathfrak{q}} \neq k_{\mathfrak{q}} && \text{for all } \mathfrak{q} \in (T\backslash(S\cup S_p))_{\min}\,, \\ &\mathcal{R}(k,S,T): && \mathop{\ast}_{\mathfrak{p}\in T\backslash S(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_T(p)|k_S(p)). \end{aligned}$$

Using the subgroup theorem for free products, see [3](4.2.1), one has

$$\mathcal{R}(k,S,T) \Rightarrow \big(\mathcal{R}(k,U,T) \text{ and } \mathcal{R}(k,S,U)\big),$$

where $T \supseteq U \supseteq S$.

If $T \cap S_p = \varnothing$, then one part of the following theorem is also proved in [5] prop.8.1 and cor.8.2.

Theorem 2.4 (Reducing and enlarging the set of primes) *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let $T \supseteq S$*

be finite sets of nonarchimedean primes of k. Assume that $\mathcal{L}_0(k,S,T)$ holds. Then we have the following assertions are equivalent:

(i) *$Spec(\mathcal{O}_k)\backslash S$ is a $K(\pi,1)$ for p and $(k_S(p))_{\mathfrak{q}} \neq k_{\mathfrak{q}}$ for all $\mathfrak{q} \in (T\backslash(S \cup S_p))_{\min}$.*

(ii) *$Spec(\mathcal{O}_k)\backslash T$ is a $K(\pi,1)$ for p and*

$$\mathop{\ast}_{\mathfrak{p}\in T\backslash S(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_T(p)|k_S(p)).$$

The implication (i)$\Rightarrow$(ii) *also holds when S or T is infinite.*

Proof: Assume that $\mathcal{L}_1(k,S,T)$ and $\mathcal{K}(\mathcal{O}_k,S)$ holds. We may further assume that $(T\backslash S)_{\min} \neq \varnothing$; in particular, $G_S(p) \neq 1$. By proposition (2.2), it follows that $\mathcal{K}(\mathcal{O}_{k'},S)$ for all finite extensions $k'|k$ inside $k_S(p)$. Thus, using proposition (2.1) (i)$\Leftrightarrow$(iii),

$$\varinjlim_{k'\subseteq k_S(p)} Б_S(k') = \varinjlim_{k'\subseteq k_S(p)} Ш^2(G_S(k')(p)) \subseteq \varinjlim_{k'\subseteq k_S(p)} H^2(G_S(k')(p)) = 0.$$

Using proposition (2.3) and $\mathcal{L}_i(k,S,T)$, $i=0,1$, we see that $(k_S(p))_{\mathfrak{q}} \neq k_{\mathfrak{q}}$ for all $\mathfrak{q} \in T_{\min}$. By proposition (1.4), it follows that $\mathcal{R}(k,S,T)$ holds. The spectral sequence

$$H^i(G_S(p), H^j(G(k_T(p)|k_S(p))) \Rightarrow H^{i+j}(G_T((p)))$$

now shows that $cd_{\mathfrak{p}}\, G_T(p) \leq 2$. Consider the commutative and exact diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak{p}\in T\backslash S} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \arrow[r, hook] & \bigoplus_{\mathfrak{p}\notin S} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \arrow[r, two heads] & \bigoplus_{\mathfrak{p}\notin T} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} \\
H^1(k_T(p)|k_S(p))^{G_S(p)} \arrow[r, hook] \arrow[u, "\cong"] & H^1(k(p)|k_S(p))^{G_S(p)} \arrow[r] \arrow[u, hook, "\mathrm{res}"] & H^1(k(p)|k_T(p))^{G_T(p)}, \arrow[u, hook]
\end{tikzcd}$$

Since $\mathcal{K}(\mathcal{O}_k,S)$ holds, the map res is an isomorphism, and we obtain

$$H^1(k(p)|k_T(p))^{G_T(p)} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\notin T} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)}.$$

Using proposition (2.1)(i)$\Leftrightarrow$(iv), it follows that $\mathcal{K}(\mathcal{O}_k,T)$ holds.

Conversely, assume that $\mathcal{K}(\mathcal{O}_k,T)$ and $\mathcal{R}(k,S,T)$ hold. Then, by lemma (1.1), we obtain $cd_p\, G_S(p) \leq 3$. It follows exactly in the same way as in the proof of corollary (1.2), using the remark (ii), that $cd_p\, G_S(p) \leq 2$. Furthermore, since

$cd_p\, G_S(p) \leq 2$, $cd_p\, G_T(p) \leq 2$ and $\mathcal{R}(k, S, T)$ holds, we can apply lemma (1.1) and obtain

$$\begin{aligned}\chi(G_T(p)) - \chi(G_S(p)) &= \sum_{\mathfrak{p}\in(T\setminus S)_{\min}} \big(\dim_{\mathbb{F}_p} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} - \dim_{\mathbb{F}_p} H^2(G_{\mathfrak{p}}(k))\big)\\ &= \sum_{\mathfrak{p}\in(T\setminus S)\cap S_p} \big(\dim_{\mathbb{F}_p} H^1(T_{\mathfrak{p}}(k))^{G_{\mathfrak{p}}(k)} - \dim_{\mathbb{F}_p} H^2(G_{\mathfrak{p}}(k))\big)\\ &= \sum_{\mathfrak{p}\in(T\setminus S)\cap S_p} [k_{\mathfrak{p}} : \mathbb{Q}_p]\end{aligned}$$

Using proposition (2.1)(i)$\Leftrightarrow$(v), we see that $\mathcal{K}(\mathcal{O}_k, S)$ holds.

Let $\mathfrak{q} \in (T\setminus(S \cup S_p))_{\min}$. By proposition (2.3), $G_{\mathfrak{q}}(k)$ is a subgroup of $G(k_T(p)|k)$. Since $\mathrm{cd}_p\, G_{\mathfrak{q}}(k) = 2$, it can not be a subgroup of the free pro-p group $G(k_T(p)|k_S(p))$. Therefore $G_{\mathfrak{q}}(k_S(p)|k)$ is non-trivial, and so $\mathcal{L}_1(k, S, T)$ holds. $\square$

Using remark (i), theorem (2.4) in the case $T = S \cup S_p$, and

$$\mathcal{R}(k, S, S \cup S_p) \Rightarrow \mathcal{R}(k, S \cup W, S \cup S_p)$$

for $W \subseteq S_p$, we obtain

Corollary 2.5 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let S be a finite set of nonarchimedean primes of k with $S \cap S_p = \varnothing$ and $W \subseteq S_p$. Assume that $(k_S(p))_{\mathfrak{p}} \neq k_{\mathfrak{p}}$ for $\mathfrak{p} \in S_p$. Then*

(i)
$$\mathcal{K}(\mathcal{O}_k, S) \Leftrightarrow \mathcal{R}(k, S, S \cup S_p).$$

(ii) *Assume that $\mathcal{K}(\mathcal{O}_k, S)$ holds. Then also $\mathcal{K}(\mathcal{O}_k, S \cup W)$ holds, and in particular,*
$$\mathrm{cd}_p\, G(k_{W\cup S}(p)|k) = 2.$$

Corollary 2.6 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Let S be a finite set of nonarchimedean primes of k with $Spec(\mathcal{O}_k)\setminus S$ is a $K(\pi, 1)$ for p. Then there exists a set T of nonarchimedean primes with $T \cap S = \varnothing$ and $\delta(T) = 1$, such that there are free product decompositions*

(i)
$$\mathop{\ast}_{\mathfrak{p}\in T(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_{T\cup S}(p)|k_S(p)),$$

(ii)
$$\mathop{\ast}_{\mathfrak{p}\notin(T\cup S)(k_{T\cup S}(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k(p)|k_{T\cup S}(p)).$$

Proof: Since $k_S(p)|k$ is infinite, it follows from Čebotarev density theorem that the set

$$V = \{\mathfrak{q} \text{ a prime of } k \mid \mathfrak{q} \text{ is completely decomposed in } k_S(p)|k\}$$

has density zero. Let T be the complement of the set $S_\infty \cup S \cup V$, hence $\delta(T) = 1$. By theorem (2.4), we obtain that $Spec(\mathcal{O}_k)\backslash(T \cup S)$ is a $K(\pi, 1)$ for p and that there is an isomorphism

$$\mathop{\ast}_{\mathfrak{p}\in(T\cup S)(k_S(p))} T_{\mathfrak{p}}(k) \xrightarrow{\sim} G(k_{T\cup S}(p)|k_S(p)).$$

Since $\delta(T \cup S) = 1$, it follows from [3] (10.5.9) that we have the desired decomposition (ii). $\square$

Remarks: (1) If S contains S_p, then the corollary above is well-known, see [3] (10.5.1): one can take for T all primes not in S.
(2) It is easy to see, that the corollary above implies that the pro-p-group $G(k(p)|k_S(p))$ is minimal generated by a system of minimal generators of the inertia groups $T_{\mathfrak{p}}(k)$, $\mathfrak{p} \notin S$, with defining relations given by the local relations of the groups $G_{\mathfrak{p}}(k)$, $\mathfrak{p} \in V$.

Using a result of A.Schmidt we will give another application of theorem (2.4). We start with a lemma and introduce the following notation: For a prime number q with $q \equiv 1 \mod p$ let $L_{q,p}$ be the maximal p-extension of $\mathbb{Q}$ inside $\mathbb{Q}(\zeta_q)$, where ζ_q is a primitive q-th root of unity.

Lemma 2.7 *Let p be a prime number and k a number field.*

(i) *Let $r \in \mathbb{N}$. Then the set $M_r(k)$ of prime numbers q which are completely decomposed in k and for which the congruences*

$$q \equiv 1 \mod p^{2r} \quad \textit{and} \quad p^{\frac{q-1}{p^r}} \not\equiv 1 \mod q$$

hold has density $1/[k(\zeta_{p^{2r}}) : \mathbb{Q}] - 1/[k(\zeta_{p^{2r}}, \sqrt[p^r]{p}) : \mathbb{Q}]$.

(ii) *The set of prime numbers $q \equiv 1 \mod p$ which are completely decomposed in k and which have the property that $(L_{q,p}\, k)_{\mathfrak{p}} \neq k_{\mathfrak{p}}$ for all $\mathfrak{p} \in S_p$ has positive density.*

Proof: (i) Let q be a prime number which is completely decomposed in $k(\zeta_{p^{2r}})$; in particular, we have $q \equiv 1 \mod p^{2r}$. Let $\mathfrak{q}$ be a prime of $k(\zeta_{p^{2r}})$ above q. Then

$$p^{\frac{N(\mathfrak{q})-1}{p^r}} \equiv 1 \mod \mathfrak{q}\,, \quad \text{i.e.} \quad (\sqrt[p^r]{p})^{N(\mathfrak{q})} \equiv (\sqrt[p^r]{p}) \mod \mathfrak{q}\,,$$

if and only if $\mathfrak{q}$ is completely decomposed in $k(\zeta_{p^{2r}}, \sqrt[p^r]{p})$. Therefore the density of the set

$$\{q \text{ is completely decomposed in } k, q \equiv 1 \bmod p^{2r}, p^{\frac{q-1}{p^r}} \equiv 1 \bmod q\}$$

is equal to $1/[k(\zeta_{p^{2r}}):\mathbb{Q}] \cdot 1/[k(\zeta_{p^{2r}}, \sqrt[p^r]{p}) : k(\zeta_{p^{2r}})]$, and the set $M_r(k)$ has density $1/[k(\zeta_{p^{2r}}):\mathbb{Q}] \cdot (1 - 1/[k(\zeta_{p^{2r}}, \sqrt[p^r]{p}) : k(\zeta_{p^{2r}})])$.

(ii) Let $r \in \mathbb{N}$ be big enough such that $\sqrt[p^r]{p} \notin k(\zeta_{p^{2r}})$ and $p^r > [k:\mathbb{Q}]$. Then, by (i), the set $M_r(k)$ has positive density. Obviously, if $q \equiv 1 \bmod p^{2r}$ and $p^{\frac{q-1}{p^r}} \not\equiv 1 \bmod q$, then the local unramified extension $(L_{q,p})_p|\mathbb{Q}_p$ has degree at least p^r. Therefore $(L_{q,p}\,k)_\mathfrak{p}$ is a non-trivial unramified extension of $k_\mathfrak{p}$ for $\mathfrak{p}|p$. $\square$

Proposition 2.8 *Let p be a prime number and k a number field where p is odd or k is totally imaginary. Assume that $\mu_p \not\subseteq k$ and $\mathrm{Cl}_k(p) = 0$. Let S be a finite set of nonarchimedean primes of k with $S \cap S_p = \varnothing$ and $W \subseteq S_p$. Let, in addition, T be a set of primes of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subseteq T$ such that $\mathcal{K}(\mathcal{O}_k, W \cup S \cup T_1)$ holds and*

$$\mathop{\ast}_{\mathfrak{p} \in (S_p \setminus W)(k_{W\cup S\cup T_1}(p))} T_\mathfrak{p}(k) \xrightarrow{\sim} G(k_{S_p \cup S \cup T_1}(p)|k_{W\cup S\cup T_1}(p)).$$

In particular,

$$\mathrm{cd}_p\, G(k_{W\cup S\cup T_1}(p)|k) = 2.$$

Proof: Obviously we may assume that $T \cap (S_p \cup S_\infty) = \varnothing$ and that the underlying prime numbers of the primes of T are completely decomposed in k. We have to show that there exists a finite subset $T_1 \subseteq T$ such that $(k_{S\cup T_1}(p))_\mathfrak{p} \neq k_\mathfrak{p}$ for $\mathfrak{p} \in S_p$ and that $\mathcal{K}(\mathcal{O}_k, S \cup T_1)$ holds.

Using lemma (2.7)(ii), there is a prime number q such that $S_q \subseteq T$ and $(k_{S\cup S_q}(p))_\mathfrak{p} \neq k_\mathfrak{p}$ for $\mathfrak{p} \in S_p$. By a result of A. Schmidt, [5] thm.6.2, we obtain a finite subset $T_1 \subseteq T$ containing S_q with the desired properties. $\square$

Remark: The proposition above shows that besides the tame case ($W = \varnothing$) also in the "mixed case" ($W \neq \varnothing$ and $W \neq S_p$) we have examples of Galois groups with cohomological dimension equal to 2.

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