On the Fontaine-Mazur Conjecture for CM-Fields

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In [3] Fontaine and Mazur conjecture (as a consequence of a general principle) that a number field k has no infinite unramified Galois extension such that its Galois group is a p-adic analytic pro-p-group. A counter-example to this conjecture would produce an unramified Galois representation with infinite image, that could not "come from geometry". Some evidence for this conjecture is shown in [1] and [4].

Since every p-adic analytic pro-p-group contains an open powerful resp. uni form subgroup one is led to the question whether a given number field possesses an infinite unramified Galois p-extension with powerful resp. uniform Galois group. With regard to this problem, we would like to mention a result of Boston [1]:

> *Let p be a prime number and let $k|k_0$ be a finite cyclic Galois extension of degree prime to p such that p does not divide the class number of k_0. Then, if the Galois group $G(M|k)$ of an unramified Galois p-extension M of k, Galois over k_0, is powerful, it is finite.*

In this paper we will prove a statement which is in some sense weaker as the above and in another sense stronger (and in view of the general conjecture very weak):

> *Let p be odd and let k be a CM-field with maximal totally real subfield k^+ containing the group μ_p of p-th roots of unity. Let $M = L(p)$ be the* maximal *unramified p-extension of k. Assume that the p-rank of the ideal class group $Cl(k^+)$ of k^+ is not equal to 1. Then, if the Galois group $G(L(p)|k)$ is powerful, it is finite.*

If the p-rank of $Cl(k^+)$ is equal to 1, we have two weaker results. First, replacing the word powerful by uniform and assuming that the first step in the p-cyclotomic tower of k is not unramified, then the statement above holds without any condition on $Cl(k^+)$. Secondly, we consider the conjecture in the p-cyclotomic tower of the number field k. Denote the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension k_∞ of k by k_n and let $G(L_n(p)|k_n)$ be the Galois group of the maximal unramified p-extension $L_n(p)$ of k_n. Then the following statement holds.

Let $p \neq 2$ and let k be a CM-field containing μ_p. Assume that the Iwasawa μ-invariant of $k_\infty|k$ is zero. Then there exists a number n_0 such that for all $n \geq n_0$ the following holds: If the Galois group $G(L_n(p)|k_n)$ is powerful, then it is finite.

Similar results hold for the maximal unramified p-extension $L_S(p)$ which is completely decomposed at all primes in S and for the maximal p-extension $k_S(p)$ of k which is unramified outside S, if S contains no prime above p.

Of course, our main interest is the conjecture for general p-adic analytic groups. We will prove the following result.

Let $p \neq 2$ and let k be a CM-field containing μ_p with maximal totally real subfield k^+ and assume that $\mu_p \not\subseteq k^+_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of k^+ above p. Then, if $G(L_k(p)|k)$ is p-adic analytic, $G(L_{k^+}(p)|k^+)$ is finite.

Unfortunately, we do not have Boston's result for general analytic pro-p-groups. Otherwise, in the situation above it would follow that $G(L_k(p)|k)$ is not an infinite p-adic analytic group.

1 A duality theorem

We use the following notation:

p	is a prime number,
k	is a number field,
S_∞	is the set of archimedean primes of k,
S	is a set of primes of k containing S_∞,
$E_S(k)$	is the group of S-units of k,
$Cl_S(k)$	is the S-ideal class group of k,
L_S	is the maximal unramified extension of k which is completely decomposed at S,
$L_S(p)$	is the maximal p-extension of k inside L_S,
L	is the maximal unramified extension of k,
$L(p)$	is the maximal p-extension of k inside L.

We write $E(k)$ for the group $E_{S_\infty}(k)$ of units of k and $Cl(k)$ for the ideal class group $Cl_{S_\infty}(k)$ of k. Obviously,

$$\begin{array}{rcll} L &=& L_{S_\infty}, & \text{if } k \text{ is totally imaginary,} \\ L(p) &=& L_{S_\infty}(p), & \text{if } p \neq 2 \text{ or } k \text{ totally imaginary.} \end{array}$$

If K is an infinite algebraic extension of $\mathbb{Q}$, then $E_S(K) = \varinjlim_k E_S(k)$ where k runs through the finite subextensions of K.

For a profinite group G, a discrete G-module M and any integer i the i-th Tate cohomology is defined by

$$\hat{H}^i(G,M) = H^i(G,M) \text{ for } i \geq 1 \text{ and } \hat{H}^i(G,M) = \varprojlim_{U,def} \hat{H}^i(G/U, M^U) \text{ for } i \leq 0,$$

where U runs through all open normal subgroups of G and the transition maps are given by the deflation, see [7].

Theorem 1.1 *Let S be a set of primes of k containing S_∞. Then the following holds:*

(i) *There are canonical isomorphisms*

$$\hat{H}^i(G(L_S|k), E_S(L_S)) \cong \hat{H}^{2-i}(G(L_S|k), \mathbb{Q}/\mathbb{Z})^\vee$$

for all $i \in \mathbb{Z}$. Here $^\vee$ denotes the Pontryagin dual.

(ii) *There are canonical isomorphisms*

$$\hat{H}^i(G(L_S(p)|k), E_S(L_S(p))) \cong \hat{H}^{2-i}(G(L_S(p)|k), \mathbb{Q}_p/\mathbb{Z}_p)^\vee$$

for all $i \in \mathbb{Z}$.

Proof: Let $C_S(L_S)$ be the S-idele class group of L_S. The subgroup $C_S^0(L_S)$ of $C_S(L_S)$ given by the ideles of norm 1 is a level-compact class formation for $G(L_S|k)$ with divisible group of universal norms. From the duality theorem of Nakayama-Tate we obtain the isomorphisms

$$\hat{H}^i(G(L_S|k), C_S(L_S)) \cong \hat{H}^{2-i}(G(L_S|k), \mathbb{Z})^\vee, \quad i \in \mathbb{Z},$$

since $\hat{H}^i(G(L_S|k), C_S(L_S)) \cong \hat{H}^i(G(L_S|k), C_S^0(L_S))$, see [7] proposition 4.

Let $K|k$ be a finite Galois extension inside L_S. From the exact sequence

$$0 \longrightarrow E_S(K) \longrightarrow J_S(K) \longrightarrow C_S(K) \longrightarrow Cl_S(K) \longrightarrow 0,$$

where $J_S(K)$ denotes the group of S-ideles of K, which is a cohomological trivial $G(K|k)$-module ($K|k$ is completely decomposed at S), we obtain isomorphisms

$$\hat{H}^{i+1}(G(K|k), E_S(K)) \cong \hat{H}^i(G(K|k), D(K)),$$

where $D(K)$ denotes the kernel of the surjection $C_S(K) \twoheadrightarrow Cl_S(K)$, and a long exact sequences

$$\longrightarrow \hat{H}^i(G(K|k), D(K)) \longrightarrow \hat{H}^i(G(K|k), C_S(K)) \longrightarrow \hat{H}^i(G(K|k), Cl_S(K)) \longrightarrow .$$

If K' is the maximal abelian extension of K in L_S, then $G(L_S|K')$ is an open subgroup of $G(L_S|K)$ by the finiteness of the class number of K. The commutative diagram

$$\begin{tikzcd}
Cl_S(K') \arrow[r, "norm"] \arrow[d, "rec"', "\sim"] & Cl_S(K) \arrow[d, "rec"', "\sim"] \\
G(L_S|K')^{ab} \arrow[r, "can"] & G(L_S|K)^{ab}
\end{tikzcd}$$

shows, since *can* is the zero map, that

$$Cl_S(K') \stackrel{norm}{\longrightarrow} Cl_S(K)$$

is trivial. It follows that

$$\lim_{\substack{\longleftarrow \\ K}} \hat{H}^i(G(K|k), Cl_S(K)) = 0 \quad \text{for } i \leq 0.$$

Since all groups in the exact sequence above are finite, we can pass to the projective limit and we obtain isomorphisms

$$\lim_{\substack{\longleftarrow \\ K}} \hat{H}^i(G(K|k), D(K)) \cong \hat{H}^i(G(L_S|k), C_S(L_S)) \quad \text{for } i \leq 0,$$

and therefore isomorphisms

$$\hat{H}^{i+1}(G(L_S|k), E_S(L_S)) \cong \hat{H}^i(G(L_S|k), C_S(L_S)) \quad \text{for } i \leq -1.$$

The last assertion also holds for $i = 0$: from the commutative diagram

$$\begin{tikzcd}
\hat{H}^0(G(K'|k), D(K')) \arrow[r, "\delta", "\sim"'] \arrow[d, "def"] & H^1(G(K'|k), E_S(K')) \arrow[d] \\
\hat{H}^0(G(K|k), D(K)) \arrow[r, "\delta", "\sim"'] & H^1(G(K|k), E_S(K)),
\end{tikzcd}$$

where $k \subseteq K \subseteq K'$ are finite Galois extensions inside L_S, it follows that the limit $\lim_{\longleftarrow K} H^1(G(K|k), E_S(K))$ exists. Since

$$H^1(G(K|k), E_S(K)) \subseteq H^1(G(L_S|k), E_S(L_S)) \cong Cl_S(k)$$

and

$$\begin{aligned}
\lim_{\substack{\longleftarrow \\ K}} \hat{H}^0(G(K|k), D(K)) &\cong \hat{H}^0(G(L_S|k), C_S(L_S)) \cong H^2(G(L_S|k), \mathbb{Z})^\vee \\
&\cong H^1(G(L_S|k), \mathbb{Q}/\mathbb{Z})^\vee = G(L_S|k)^{ab} \cong Cl_S(k),
\end{aligned}$$

the projective limit $\lim_{\longleftarrow K} H^1(G(K|k), E_S(K))$ becomes stationary and is equal to $H^1(G(L_S|k), E_S(L_S))$.

For $i \geq 1$ the exact sequence

$$0 \longrightarrow E_S(L_S) \longrightarrow J_S(L_S) \longrightarrow C_S(L_S) \longrightarrow 0$$

induces isomorphisms

$$H^i(G(L_S|k), C_S(L_S)) \cong H^{i+1}(G(L_S|k), E_S(L_S)).$$

Putting all together, we obtain canonical isomorphisms

$$\hat{H}^{i+1}(G(L_S|k), E_S(L_S)) \cong \hat{H}^{2-i}(G(L_S|k), \mathbb{Z})^\vee \cong \hat{H}^{1-i}(G(L_S|k), \mathbb{Q}/\mathbb{Z})^\vee$$

for all $i \in \mathbb{Z}$. The proof for the field $L_S(p)$ is analogous. $\square$

Let k be a number field of CM-type with maximal totally real subfield k^+ and let $\Delta = G(k|k^+) = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$. If $p \neq 2$, we put as usual

$$M^\pm = (1 \pm \sigma)M$$

for a $\mathbb{Z}_p[\Delta]$-module M. For a $\mathbb{Z}_p$-module N let ${}_pN = \{x \in N \,|\, px = 0\}$.

Corollary 1.2 *Let p be an odd prime number and let k be a CM-field. Let S be a set of primes of k^+ containing S_∞ and assume that no prime of S splits in the extension $k|k^+$. Then*

$$\dim_{\mathbb{F}_p} {}_pH^2(G(L_S(p)|k), \mathbb{Q}_p/\mathbb{Z}_p)^- \leq \delta,$$

where δ is equal to 1 if k contains the group μ_p of p-th roots of unity and otherwise equal to 0.

Proof: By proposition 1.1, there is a Δ-invariant surjection

$$E_S(k) \twoheadrightarrow \hat{H}^0(G(L_S(p)|k), E_S(L_S(p))) \cong H^2(G(L_S(p)|k), \mathbb{Q}_p/\mathbb{Z}_p)^\vee$$

and so a surjection

$$(E_S(k)/p)^- \twoheadrightarrow ({}_pH^2(G(L_S(p)|k), \mathbb{Q}_p/\mathbb{Z}_p)^-)^\vee.$$

Since no prime of S splits in the extension $k|k^+$, we have $(E_S(k)/p)^- \cong \mu_p(k)$ which gives us the desired result. $\square$

2 Powerful pro-p-groups with involution

Let p be a prime number. For a pro-p-group G the descending p-central series is defined by

$$G_1 = G, \qquad G_{i+1} = (G_i)^p[G_i, G] \quad \text{for } i \geq 1.$$

If a group $\Delta \cong \mathbb{Z}/2\mathbb{Z}$ acts on G and p is odd, then we define

$$d(G)^{\pm} = \dim_{\mathbb{F}_p}(G/G_2)^{\pm} = \dim_{\mathbb{F}_p} H^1(G, \mathbb{Z}/p\mathbb{Z})^{\pm}.$$

The following proposition also follows from Boston result (resp. its proof), but in our situation, where only an involution acts on G, we will give a simple proof.

Proposition 2.1 *Let $p \neq 2$ and let G be a finitely generated powerful pro-p-group with an action by the group $\Delta \cong \mathbb{Z}/2\mathbb{Z}$. Then the following holds:*

If $d(G)^+ = 0$, then G is abelian.

In particular, if $d(G)^+ = 0$ and G^{ab} is finite, then G is finite.

Proof: Since G is powerful, we have

$$[G, G]/H \subseteq G^p H/H \quad \text{where } H = ([G, G])^p[G, G, G].$$

From $G/G_2 = (G/G_2)^-$ it follows that

$$[G, G]/H = ([G, G]/H)^+ \quad \text{and} \quad G^p H/H = (G^p H/H)^-,$$

since $G/[G, G] = (G/[G, G])^-$ and $G^p = \{x^p \mid x \in G\}$, [2] theorem 3.6(iii), and so

$$(x^p)^\sigma \equiv x^{-p} \mod H \quad \text{for } 1 \neq \sigma \in \Delta \text{ and } x \in G.$$

We obtain

$$[G, G] \subseteq ([G, G])^p[G, G, G].$$

This implies $[G, G] = 1$. □

Lemma 2.2 *Let $p \neq 2$ and let G be a finitely generated pro-p-group with an action by the group $\Delta \cong \mathbb{Z}/2\mathbb{Z}$. Then the following inequalities hold:*

$$\begin{aligned} d(G)^+ \cdot d(G)^- &\leq \dim_{\mathbb{F}_p}(G_2/G_3)^- - \operatorname{rank}_{\mathbb{Z}_p}(G^{ab})^- + \dim_{\mathbb{F}_p}{}_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^-, \\ \binom{d(G)^+}{2} + \binom{d(G)^-}{2} &\leq \dim_{\mathbb{F}_p}(G_2/G_3)^+ - \operatorname{rank}_{\mathbb{Z}_p}(G^{ab})^+ + \dim_{\mathbb{F}_p}{}_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^+. \end{aligned}$$

Proof: Let $d^\pm = d(G)^\pm$. From the exact sequences

$$\begin{aligned} 0 \longrightarrow H^1(G/G_2, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^1(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(G_2, \mathbb{Z}/p\mathbb{Z})^G \\ \longrightarrow H^2(G/G_2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(G, \mathbb{Z}/p\mathbb{Z}) \end{aligned}$$

and

$$0 \longrightarrow ({}_pG^{ab})^\vee \longrightarrow H^2(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow {}_pH^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0$$

we obtain the inequalities

$$\begin{aligned} \dim_{\mathbb{F}_p} H^2(G/G_2, \mathbb{Z}/p\mathbb{Z})^\pm \;&\leq\; \dim_{\mathbb{F}_p}(G_2/G_3)^\pm + \dim_{\mathbb{F}_p}({}_pG^{ab})^\pm \\ &\quad + \dim_{\mathbb{F}_p} {}_pH^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^\pm. \end{aligned}$$

Let

$$G/G_2 \cong A_1 \oplus \cdots \oplus A_{d^+} \oplus B_1 \oplus \cdots \oplus B_{d^-}$$

be a Δ-invariant decomposition into cyclic groups of order p such that $A_i = A_i^+$ and $B_j = B_j^-$. For $H^2(G/G_2, \mathbb{Z}/p\mathbb{Z})$ we obtain the Δ-invariant Künneth decomposition:

$$\begin{aligned} H^2(G/G_2, \mathbb{Z}/p\mathbb{Z}) \;\cong\; & \bigoplus_{i=1}^{d^+} H^2(A_i, \mathbb{Z}/p\mathbb{Z}) \\ & \oplus \bigoplus_{i<j} H^1(A_i, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(A_j, \mathbb{Z}/p\mathbb{Z}) \\ & \oplus \bigoplus_{i<j} H^1(B_i, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(B_j, \mathbb{Z}/p\mathbb{Z}) \\ & \oplus \bigoplus_{i=1}^{d^-} H^2(B_i, \mathbb{Z}/p\mathbb{Z}) \\ & \oplus \bigoplus_{i,j} H^1(A_i, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(B_j, \mathbb{Z}/p\mathbb{Z}). \end{aligned}$$

Counting dimensions yields

$$\begin{aligned} \dim_{\mathbb{F}_p} H^2(G/G_2, \mathbb{Z}/p\mathbb{Z})^+ &= d^+ + \binom{d^+}{2} + \binom{d^-}{2}, \\ \dim_{\mathbb{F}_p} H^2(G/G_2, \mathbb{Z}/p\mathbb{Z})^- &= d^- + d^+d^-. \end{aligned}$$

Since

$$d^\pm = \operatorname{rank}_{\mathbb{Z}_p}(G^{ab})^\pm + \dim_{\mathbb{F}_p}({}_pG^{ab})^\pm,$$

we obtain the desired result. $\square$

Proposition 2.3 *Let $p \neq 2$ and let G be a finitely generated powerful pro-p-group with an action by the group $\Delta \cong \mathbb{Z}/2\mathbb{Z}$. Then the following inequalities hold:*

(i) $\quad d(G)^+ \cdot d(G)^- \leq d(G)^- + \dim_{\mathbb{F}_p} {}_pH^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^-,$

(ii) $\quad \binom{d(G)^+}{2} + \binom{d(G)^-}{2} \leq d(G)^+ + \dim_{\mathbb{F}_p} {}_pH^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^+.$

Proof: Since G is powerful, the Δ-invariant homomorphism

$$G/G_2 \stackrel{p}{\twoheadrightarrow} G_2/G_3$$

is surjective, see [2] theorem 3.6, and we obtain

$$\dim_{\mathbb{F}_p}(G_2/G_3)^{\pm} \leq d(G)^{\pm}.$$

Using lemma 2.2, this proves the proposition. $\square$

Now we analyze the case where G is a powerful pro-p-group which is a Poincaré group of dimension 3.

Proposition 2.4 *Let p be odd and let P be a finitely generated powerful pro-p-group with an action of $\Delta \cong \mathbb{Z}/2\mathbb{Z}$.*

(i) *If P is uniform, then*

$$\begin{aligned} \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})^+ &= \tbinom{d(P)^+}{2} + \tbinom{d(P)^-}{2}, \\ \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})^- &= d(P)^+ \cdot d(P)^-. \end{aligned}$$

(ii) *If P is uniform such that P^{ab} is finite and $d(P)^+ = 1$, then*

$$\dim_{\mathbb{F}_p} {}_pH^2(P,\mathbb{Q}_p/\mathbb{Z}_p)^- = 0.$$

(iii) *If P is a Poincaré group of dimension 3 such that P^{ab} is finite, then*

$$\begin{aligned} &d(P)^+ = 1 \quad \textit{and} \quad d(P)^- = 2 \qquad \textit{or} \\ &d(P)^+ = 3 \quad \textit{and} \quad d(P)^- = 0. \end{aligned}$$

Proof: Let P be uniform. By [2] definition 4.1 and theorem 4.26, we have

$$\dim_{\mathbb{F}_p}(H^1(P_2,\mathbb{Z}/p\mathbb{Z})^P)^{\pm} = d(P)^{\pm} \quad \text{and} \quad \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z}) = \tbinom{d(P)}{2}.$$

Counting dimensions shows that

$$\dim_{\mathbb{F}_p} H^2(P/P_2,\mathbb{Z}/p\mathbb{Z}) = \dim_{\mathbb{F}_p} H^1(P_2,\mathbb{Z}/p\mathbb{Z})^P + \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z}),$$

and so the sequence

$$0 \longrightarrow H^1(P_2,\mathbb{Z}/p\mathbb{Z})^P \longrightarrow H^2(P/P_2,\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(P,\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

is exact. Therefore

$$\dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})^{\pm} = \dim_{\mathbb{F}_p} H^2(P/P_2,\mathbb{Z}/p\mathbb{Z})^{\pm} - \dim_{\mathbb{F}_p}(H^1(P_2,\mathbb{Z}/p\mathbb{Z})^P)^{\pm},$$

which proves (i).

If P^{ab} is finite, then $\dim_{\mathbb{F}_p}({}_pP^{ab})^{\pm} = d(P)^{\pm}$, and so by (i)

$$\begin{aligned}\dim_{\mathbb{F}_p} {}_pH^2(P,\mathbb{Q}_p/\mathbb{Z}_p)^- &= \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})^- - \dim_{\mathbb{F}_p}({}_pP^{ab})^- \\ &= d(P)^+ \cdot d(P)^- - d(P)^-.\end{aligned}$$

This gives us the desired result (ii).

Now let P be a powerful Poincaré group of dimension 3; in particular, P is torsionfree and therefore P is uniform, see [2] theorem 4.8. Since

$$\dim_{\mathbb{F}_p} H^1(P,\mathbb{Z}/p\mathbb{Z}) = \dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})$$

and since P^{ab} is finite, the exact sequence

$$0 \longrightarrow ({}_pP^{ab})^{\vee} \longrightarrow H^2(P,\mathbb{Z}/p\mathbb{Z}) \longrightarrow {}_pH^2(P,\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0$$

shows that

$$({}_pP^{ab})^{\vee} \xrightarrow{\sim} H^2(P,\mathbb{Z}/p\mathbb{Z}).$$

It follows that

$$\dim_{\mathbb{F}_p} H^2(P,\mathbb{Z}/p\mathbb{Z})^{\pm} = d(P)^{\pm},$$

and so by (i)

$$d(P)^+ \cdot d(P)^- = d(P)^-.$$

This proves (iii). □

3 On the Fontaine-Mazur Conjecture

We keep the notation of sections 1 and 2. Let

$$d_k^{\pm} = \dim_{\mathbb{F}_p}(Cl(k)/p)^{\pm} = d(G(L(p)|k))^{\pm}.$$

Theorem 3.1 *Let p be an odd prime number and let k be a CM-field such that*

(i) $d_k^- \neq 0$, *if* $\mu_p \nsubseteq k$,

(ii) $d_k^+ \neq 1$.

Then, if the Galois group $G(L(p)|k)$ of the maximal unramified p-extension $L(p)$ of k is powerful, it is finite.

Proof: If $d_k^+ = 0$, then the theorem follows from proposition 2.1. Therefore we assume that $d_k^+ \geq 2$ (assumption (ii)). From assumption (i) and Leopoldt's Spiegelungssatz, see [8] theorem 10.11, it follows that $d_k^- \geq 1$. From proposition 2.3 and corollary 1.2 we obtain the inequality

$$d_k^+ d_k^- \leq d_k^- + \delta.$$

It follows that $d_k^+ = 2$, $d_k^- = 1$ (and $\delta = 1$), and so $d(G(L(p)|k)) = 3$.

If $P = G(L(p)|k)_i$, i large enough, then P is uniform, [2] theorem 4.2, and $d(P) \le 3$, [2] theorem 3.8. Suppose that P is non-trivial. Then P is a Poincaré group of dimension $\dim(P) = d(P) \le 3$, see [5] chap.V theorem (2.2.8) and (2.5.8). But Poincaré groups of dimension $\dim(P) \le 2$ have the group $\mathbb{Z}_p$ as homomorphic image, and so we can assume that $\dim(P) = d(P) = 3$. Since $G(L(p)|k)$ is powerful, we have a surjection

$$G(L(p)|k)/G(L(p)|k)_2 \twoheadrightarrow G(L(p)|k)_i/G(L(p)|k)_{i+1}.$$

Furthermore, by [2] theorem 3.6(ii), $G(L(p)|k)_{i+1} = (G(L(p)|k)_i)_2 = P_2$, and so $G(L(p)|k)_i/G(L(p)|k)_{i+1} = P/P_2$. Therefore $d(P)^+ = 2$ and $d(P)^- = 1$. By proposition 2.4(iii) we get a contradiction. $\square$

If $\mu_p \subseteq k$, then $d_k^+ = 1$ is the only remaining case. Here we only get a weaker result. Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k and denote by k_n the n-th layer of $k_\infty|k$.

Theorem 3.2 *Let $p \neq 2$ and let k be a CM-field containing μ_p. Assume that $k_1|k$ is not unramified if $d_k^+ = 1$. Then the Galois group $G(L(p)|k)$ of the maximal unramified p-extension $L(p)$ of k is not uniform.*

Proof: Suppose that $G = G(L(p)|k)$ is uniform. Using theorem 3.1, we may assume that $d(G)^+ = 1$, and so, by proposition 2.4(ii),

$$\dim_{\mathbb{F}_p\, p} H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^- = 0.$$

On the other hand, by theorem 1.1, we have a surjection

$$H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \hat{H}^0(G, E(L(p))) \twoheadrightarrow \hat{H}^0(G(K|k), E(K))$$

where $K|k$ is a finite unramified Galois p-extension of CM-fields (recall that $d(G)^+ \neq 0$), and so a surjection

$$(H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^-)^\vee \twoheadrightarrow \hat{H}^0(G(K|k), E(K))^-.$$

Since K is of CM-type, it follows that

$$\hat{H}^0(G(K|k), E(K))^- \cong \hat{H}^0(G(K|k), \mu(K)(p)).$$

By our assumption, K is disjoint to k_∞, i.e. $\mu(K)(p) = \mu(k)(p)$, and so

$$\dim_{\mathbb{F}_p} \hat{H}^0(G(K|k), \mu(K)(p))/p = 1.$$

It follows that

$$\dim_{\mathbb{F}_p\, p} H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^- = 1.$$

This contradiction proves the theorem. $\square$

Remarks:
(1) The theorems 3.1 and 3.2 also hold in the following situation: Replace $L(p)$ by $L_S(p)$ and Cl by Cl_S where $S \supseteq S_\infty$ is a set of primes which do not split in the extension $k|k^+$. Use corollary 1.2 for S instead of S_∞.
(2) Theorem 3.1 is also true, if we replace $L(p)$ by the maximal p-extension $k_S(p)$ of k which is unramified outside a finite set S which contains S_∞ but no prime above p. Instead of $Cl(k)$ one has to take the ray class group $C(k)/C^{\mathfrak{m}}(k)$ mod $\mathfrak{m} = \prod_{\mathfrak{p}\in S} \mathfrak{p}$ (which is finite). In order to prove an analog of corollary 1.2, use the exact sequence

$$0 \longrightarrow E^S(K) \longrightarrow J_{S_\infty}(K) \times U^1_{S'}(K) \longrightarrow C_S(K) \longrightarrow C(K)/C^{\mathfrak{m}}(K) \longrightarrow 0$$

where $S' = S \backslash S_\infty$ and $U^1_{S'}(K)$ is the product over the principal units at the places of S' and $E^S(K) = \ker(E(K) \to U_{S'}(K)/U^1_{S'}(K))$.

Now we consider the Galois groups $G(L_n(p)|k_n)$ of the maximal unramified p-extension $L_n(p)$ of k_n in the p-cyclotomic tower of k.

Theorem 3.3 *Let $p \neq 2$ and let k be a CM-field containing μ_p. Assume that the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension $k_\infty|k$ is zero.*

Then there exists a number n_0 such that for all $n \geq n_0$ the following holds: If the Galois group $G(L_n(p)|k_n)$ is powerful, then it is finite.

Proof: Let

$$1 \longrightarrow G_\infty \longrightarrow G(L_\infty(p)|k) \longrightarrow \Gamma \longrightarrow 1$$

where $G_\infty = G(L_\infty(p)|k_\infty)$ is the Galois group of the maximal unramified p-extension $L_\infty(p)$ of k_∞ and $\Gamma = G(k_\infty|k) = \langle \gamma \rangle$. Let $\Gamma_n = \langle \gamma^{p^n} \rangle$, $n \geq 0$, be the open subgroups of Γ of index p^n. By our assumption on the Iwasawa μ-invariant G_∞ is a finitely generated pro-p-group.

Let n_1 be large enough such that all primes of k_{n_1} above p are totally ramified in $k_\infty|k_{n_1}$ and let $\langle \gamma_j \rangle \subseteq G(L_\infty(p)|k_{n_1})$, $j = 1, \ldots, s$, be the inertia groups of some extensions of the finitely many primes $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ of k_{n_1} above p.

For $n \geq n_1$ let

$$M_n = (\gamma_j^{p^{n-n_1}}, j = 1, \ldots, s) \subseteq G(L_\infty(p)|k_n)$$

be the normal subgroup generated by all conjugates of the elements $\gamma_j^{p^{n-n_1}}$ and

$$N_n := M_n \cap G_\infty = (\gamma_i^{p^{n-n_1}} \gamma_j^{-p^{n-n_1}}, [\gamma_j^{p^{n-n_1}}, g], i, j = 1, \ldots, s,\ g \in G_\infty).$$

Then the commutative exact diagram

$$\begin{tikzcd}
1 \arrow[r] & N_n \arrow[r] \arrow[d, hook] & M_n \arrow[r] \arrow[d, hook] & \Gamma_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_\infty \arrow[r] & G(L_\infty(p)|k_n) \arrow[r] & \Gamma_n \arrow[r] & 1
\end{tikzcd}$$

shows that

$$G_\infty/N_n \cong G(L_n(p)|k_n)$$

and we have canonical surjections

$$G_\infty \twoheadrightarrow G(L_m(p)|k_m) \twoheadrightarrow G(L_n(p)|k_n)$$

for $m \geq n \geq n_1$.

Let $n_0 \geq n_1$ be large enough such that

$$G_\infty/(G_\infty)_3 \xrightarrow{\sim} G(L_n(p)|k_n)/(G(L_n(p)|k_n))_3$$

for all $n \geq n_0$, i.e.

$$G(L_\infty(p)|k_n)/(G_\infty)_3 = G_\infty/(G_\infty)_3 \times \Gamma_n \cong G(L_n(p)|k_n)/(G(L_n(p)|k_n))_3 \times \Gamma_n.$$

Then $\langle \gamma_j^{p^{n-n_1}} \rangle$ acts trivially on $G_\infty/(G_\infty)_3$ for all $j \leq s$ and N_n is contained in $(G_\infty)_3$.

Suppose that $G(L_n(p)|k_n)$, $n \geq n_0$, is powerful. Then

$$[G_\infty, G_\infty] \subseteq (G_\infty)^p N_n.$$

By assumption on n_0 the group N_n is contained in $(G_\infty)_3$, and so

$$[G_\infty, G_\infty] \subseteq (G_\infty)^p [G_\infty, [G_\infty, G_\infty]].$$

From this inclusion it follows that

$$[G_\infty, G_\infty] \subseteq (G_\infty)^p,$$

thus G_∞ is powerful.

Using proposition 2.1, we can assume that

$$d^+_{k_n} = \dim_{\mathbb{F}_p}(Cl(k_n)/p)^+ \geq 1.$$

Let $K|k_n$ be an unramified Galois extension of degree p such that $G(K|k_n) = G(K|k_n)^+$ and let $K_\infty = k_\infty K$. Because of our definition of n_1 the field K is not contained in k_∞ and $G(L_\infty(p)|K_\infty)$ is a normal subgroup of $G(L_\infty(p)|k_\infty)$ of index p.

Using results of Iwasawa theory, [6] (11.4.13) and (11.4.8), we obtain

$$d(G(L_\infty(p)|K_\infty))^- = p(d(G(L_\infty(p)|k_\infty))^- - 1) + 1.$$

From [2] theorem 3.8 and the equality above it follows that

$$\begin{aligned}
& d(G(L_\infty(p)|k_\infty))^+ + d(G(L_\infty(p)|k_\infty))^- \\
= \;& d(G(L_\infty(p)|k_\infty)) \\
\geq \;& d(G(L_\infty(p)|K_\infty)) \\
= \;& d(G(L_\infty(p)|K_\infty))^+ + d(G(L_\infty(p)|K_\infty))^- \\
= \;& d(G(L_\infty(p)|K_\infty))^+ + p(d(G(L_\infty(p)|k_\infty))^- - 1) + 1.
\end{aligned}$$

The maximal quotient $G(L_\infty(p)|k_\infty)_\Delta$ of $G(L_\infty(p)|k_\infty)$ with trivial action of Δ is also powerful and we have $d(G(L_\infty(p)|k_\infty)_\Delta) = d(G(L_\infty(p)|k_\infty))^+$. Using again [2] theorem 3.8, we get

$$d(G(L_\infty(p)|k_\infty))^+ \geq d(G(L_\infty(p)|K_\infty))^+.$$

Both inequalities together imply

$$d(G(L_\infty(p)|k_\infty))^- \leq 1.$$

Using [6] (11.4.4), we finally obtain

$$d(G(L_\infty(p)|k_\infty))^+,\ d(G(L_\infty(p)|k_\infty))^- \leq 1.$$

It follows that $G(L_n(p)|k_n)$ is a powerful pro-p-group with $d(G(L_n(p)|k_n)) \leq 2$. If $G(L_n(p)|k_n)$ is not finite, then it contains an open subgroup P which is a Poincaré group (see [5] chap.V theorem (2.2.8) and (2.5.8)) of dimension $\dim P = d(P) \leq 2$ (use again [2] theorem 3.8). But these groups have the group $\mathbb{Z}_p$ as homomorphic image. By the finiteness of the class number it follows that $G(L_n(p)|k_n)$ is finite. □

Remark: Theorem 3.3 also holds if we replace $L(p)$ by $L_\Sigma(p)$ and Cl by Cl_Σ, where $\Sigma = S_\infty \cup S_p$ is the set of archimedean primes and primes above p, and if we assume that no prime of S_p splits in the extension $k|k^+$.

Now we consider the conjecture for general p-adic analytic groups. Let

$$1 \longrightarrow \mathcal{D} \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1$$

be an exact sequence of pro-p-groups. For an open normal subgroup H of G we denote the pre-image of H in $\mathcal{G}$ by $\mathcal{H}$. Thus we get a commutative exact diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{G} \arrow[r] & G \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{D} \arrow[r] \arrow[u, equal] & \mathcal{H} \arrow[r] \arrow[u, hook] & H \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

Proposition 3.4 *With the notation as above assume that*

(i) $\mathcal{G}$ *is finitely generated and* $\mathrm{cd}_p\,\mathcal{G} \le 2$,

(ii) $\mathrm{cd}_p\,G < \infty$,

(iii) *the Euler-Poincaré characteristic of* $\mathcal{G}$ *is zero, i.e.*

$$\chi(\mathcal{G}) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_p} H^i(\mathcal{G}, \mathbb{Z}/p\mathbb{Z}) = 0\,.$$

Then

$d(\mathcal{H})$ *is unbounded for varying open normal subgroups* H *of* G *or* $\mathrm{cd}_p\,G \le 2$.

Proof: Suppose that $\dim_{\mathbb{F}_p} H^1(\mathcal{H}, \mathbb{Z}/p\mathbb{Z})$ is bounded for varying H. Since $\chi(\mathcal{G}) = 0$, the same is true for $\dim_{\mathbb{F}_p} H^2(\mathcal{H}, \mathbb{Z}/p\mathbb{Z})$. It follows that $H^i(\mathcal{D}, \mathbb{Z}/p\mathbb{Z})$ is finite for $i = 1, 2$. By [6] proposition (3.3.7), we obtain

$$cd_p\,\mathcal{G} = cd_p\,G + cd_p\,\mathcal{D} \ge cd_p\,G.$$

This proves the proposition. $\square$

As an application to our problem we get the following result for the maximal unramified p-extension $L_k(p)$ of a number field k.

Theorem 3.5 *Let* $p \neq 2$ *and let* k *be a CM-field containing* μ_p *with maximal totally real subfield* k^+. *Assume that* $\mu_p \nsubseteq k^+_{\mathfrak{p}}$ *for all primes* $\mathfrak{p}$ *of* k^+ *above* p. *Then the following holds:*

either	(i)	$G(L_{k^+}(p)\vert k^+)$	*is finite,*
or	(ii)	$G(L_k(p)\vert k)$	*is not* p*-adic analytic,*

with other words, if $G(L_k(p)|k)$ *is* p*-adic analytic, then* $G(L_{k^+}(p)|k^+)$ *is finite.*

Proof: Suppose that (i) and (ii) do not hold. Then the maximal quotient $G(L_{k^+}(p)|k^+)$ of the p-adic analytic group $G(L_k(p)|k)$ with trivial action by $\Delta = G(k|k^+)$ is an infinite analytic group. Passing to a finite extension of k^+, we may assume that $G(L_{k^+}(p)|k^+)$ is uniform (our assumptions on k are still valid). The dimension of $G(L_{k^+}(p)|k^+)$ is greater or equal to 3, since otherwise it would have the group $\mathbb{Z}_p$ as quotient which is impossible by the finiteness of the class number.

If $k^+_{S_p}(p)$ is the maximal p-extension of k^+ which is unramified outside p, then $cd_p\,G(k^+_{S_p}(p)|k^+) \le 2$ and $\chi(G(k^+_{S_p}(p)|k^+)) = 0$, see [6] (8.3.17), (8.6.16) and (10.4.8). Applying proposition 3.4, we obtain that

$$\dim_{\mathbb{F}_p} H^1(G(k^+_{S_p}(p)|K^+), \mathbb{Z}/p\mathbb{Z}) = \dim_{\mathbb{F}_p} H^1(G(k_{S_p}(p)|K^+(\mu_p)), \mathbb{Z}/p\mathbb{Z})^+$$

is unbounded, if K^+ varies over the finite Galois extension of k^+ inside $L_{k^+}(p)$. By [6] theorem (8.7.3) and the assumption that $\mu_p \nsubseteq k^+_{\mathfrak{p}}$ for all primes $\mathfrak{p}|p$, it follows that

$$\begin{aligned} d(G(L_k(p)|K^+(\mu_p)) &= \dim_{\mathbb{F}_p} Cl(K^+(\mu_p))/p \\ &\geq \dim_{\mathbb{F}_p}(Cl_{S_p}(K^+(\mu_p))/p)^- \\ &= \dim_{\mathbb{F}_p} H^1(G(k_{S_p}(p)|K^+(\mu_p)), \mathbb{Z}/p\mathbb{Z})^+ - 1 \end{aligned}$$

is unbounded for varying K^+ inside $L_{k^+}(p)$ and therefore $G(L_k(p)|k)$ is not p-adic analytic. This contradiction proves the theorem. □

References

[1] Boston, N. *Some cases of the Fontaine-Mazur conjecture II.* J. Number Theory **75** (1999), 161-169

[2] Dixon, J.D., du Sautoy, M.P.F., Mann, A., Segal, D. *Analytic pro-p Groups* LMS Lectures Notes **157**, Cambridge University Press 1991

[3] Fontaine, J-M., Mazur B. *Geometric Galois representations.* In: Elliptic Curves, Modular Forms, & Fermat's Last Theorem, edited by J. Coates, and S.T. Yau. International Press, Boston 1995

[4] Hajir F. *On the groth of p-class groups in p-class towers* J. Algebra **188** (1997), 256-271

[5] Lazard, M. *Groupes analytiques p-adiques.* Inst. Hautes Études Scientifiques, Publ. Math. **26** 1965

[6] Neukirch, J., Schmidt, A., Wingberg, K. *Cohomology of Number Fields.* Springer 2000

[7] Schmidt, A. *On Poitou's duality theorem.* J. reine angew. Math. **517** (1999), 145-160

[8] Washington, L.C. *Introduction to Cyclotomic Fields.* Springer 1982 (Second ed. 1997)

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