A short course on Siegel modular forms

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"Siegel modular forms", as they are called today, were first introduced by Siegel in a paper of 1935 and nowadays often are given as a first example of holomorphic modular forms in several variables. The theory meanwhile has a long traditional background and is a very important and active area in modern research, combining in many nice ways number theory, complex analysis and algebraic geometry.

The goal of these lectures is twofold: first I would like to introduce the basic concepts of the theory, like the Siegel modular group and its action on the Siegel upper half-space, reduction theory, examples of Siegel modular forms, Hecke operators and L-functions. Secondly, following up, I would like to discuss two rather recent research topics, namely the "Ikeda lifting" and sign changes of Hecke eigenvalues in genus two.

For sections 1-4 we refer to [F] as a basic reference, for the results about L-functions in sect. 4 we refer to [A]. Regarding sect. 5 the reader may consult [E-Z,I,K1,C-K,K-K] for more details, and for sect. 6 we refer to the survey article [K2].

1 Introduction and motivation

First we would like to give some motivation why to study Siegel modular forms. This motivation will stem from two different sources, namely number theory and the theory of compact Riemann surfaces.

a) From number theory

Let $A \in M_m(\mathbb{Z})$ be an integral (m, m)-matrix and suppose that A is even (i.e. all the diagonal entries of A are even), A is symmetric (i.e. $A = A'$, where the prime $'$ denotes the transpose) and $A > 0$ (i.e. A is positive definite). Then

$$Q(x) = \frac{1}{2}x'Ax = \sum_{1\le\mu<\nu\le m} a_{\mu\nu}x_\mu x_\nu + \sum_{\mu=1}^{m} \frac{a_{\mu\mu}}{2}x_\mu^2 \quad (x \in \mathbb{R})$$

is an integral positive definite quadratic form in the variables $x_1, \ldots, x_m$.

For $t \in \mathbb{N}$, let

$$r_Q(t) := \#\{g \in \mathbb{Z}^m | Q(g) = \frac{1}{2}g'Ag = t\}$$

be the number of representations of t by Q.

Lemma 1.

$$r_Q(t) < \infty.$$

Proof: Diagonalize A over $\mathbb{R}$; since $A > 0$, it follows that $M_t := \{x \in \mathbb{R}^m | Q(x) = t\}$ is compact, hence $M_t \cap \mathbb{Z}^m$ (intersection of a compact and a discrete set) is finite.

Problem: Given Q, find finite, closed formulas for $r_Q(t)$, or at least an asymptotic formula for $r_Q(t)$ when $t \to \infty$!

Idea (Jacobi): Study the generating Fourier series (theta series)

$$\theta_Q(z) := 1 + \sum_{t\ge1} r_Q(t)e^{2\pi i t z} = \sum_{x\in\mathbb{Z}^m} e^{2\pi i Q(x) z} \quad (z \in \mathbb{C}).$$

Since $A > 0$, this series converges absolutely locally uniformly in the upper half-plane $\mathcal{H}$, i.e. for $z \in \mathbb{C}$ with $Im\ z > 0$. On $\mathcal{H}$ operates the group $SL_2(\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc = 1\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d}.$$

Fact: For m even, $\theta_Q \in M_{m/2}(N)$= space of modular forms of weight $\frac{m}{2}$ and level N, i.e. essentially

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \pm(cz+d)^{m/2}\theta_Q(z) \;\; (\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), N|c),$$

where N is a certain natural number depending on Q, the so-called level of Q.

Application: $M_{m/2}(N)$ is a *finite-dimensional* $\mathbb{C}$-vector space $\Longrightarrow$ results on $r_Q(t)$.

Example: $m = 4,\ Q(x) = x_1^2 + \cdots + x_4^2$; then $\theta_Q \in M_2(4)$. Known: dim $M_2(4) = 2$, in fact $M_2(4)$ has a basis of two Eisenstein series $E_1(z) := P(z) - 4P(4z), E_2(z) := P(z) - 2P(2z)$, where

$$P(z) = 1 - 24 \sum_{t \geq 1} \sigma_1(t) e^{2\pi i t z} \ \ (\sigma_1(t) := \sum_{d|t} d).$$

Therefore

$$\theta_Q(z) = \alpha E_1(z) + \beta E_2(z)$$

for certain $\alpha, \beta \in \mathbb{C}$. Comparing the first two Fourier coefficients on both sides gives

$$1 = -3\alpha - \beta,$$

$$8 = -24\alpha - 24\beta.$$

This implies

$$\alpha = -\frac{1}{3}, \beta = 0,$$

hence

$$\theta_Q = -\frac{1}{3} E_1.$$

Comparing coefficients for all $t \geq 1$, we obtain therefore

$$r_Q(t) = 8 \left(\sigma_1(t) - 4\sigma_1(\frac{t}{4}) \right) \ \ (Jacobi),$$

in particular

$$r_Q(t) \geq 1, \forall t \geq 1 \ \ (Lagrange).$$

In general, however, due to the presence of cusp forms, one only gets asymptotic formulas.

More general problem: Study the number of representations of a (n, n)-matrix T by Q, i.e. study

$$r_Q(T) := \#\{G \in M_{m,n}(\mathbb{Z}) \mid \frac{1}{2} G' A G = T\}.$$

Note: if T is represented like that, then $T = T'$ and $T \geq 0$ (i.e. T is positive semi-definite). Also T must be half-integral (i.e. $2T$ is even) and $r_Q(T) < \infty$ (since $\frac{1}{2} G' A G = T \Rightarrow \frac{1}{2} g_\nu' A g_\nu = t_{\nu\nu}$ where g_ν is the ν-th column of G).

Put

$$\theta_Q^{(n)}(Z) := \sum_{\substack{T = T' \geq 0 \\ T \ half-integral}} r_Q(T) e^{2\pi i tr(TZ)} \qquad (Z \in M_n(\mathbb{C})).$$

(Note that

$$tr(TZ) = \sum_{1\le\mu<\nu\le n} t_{\mu\nu} z_{\mu\nu} + \sum_{\nu=1}^{n} t_{\nu\nu} z_{\nu\nu}$$

(where $Z = (z_{\mu\nu})$, $T = \begin{pmatrix} t_{11} & & \frac{1}{2}t_{\mu\nu} \\ & \ddots & \\ \frac{1}{2}t_{\mu\nu} & & t_{nn} \end{pmatrix}$) is the most general linear form in the variables $z_{\mu\nu}$).

This series is absolutely convergent on the Siegel upper-space $\mathcal{H}_n := \{Z \in M_n(\mathbb{C}) \mid Z = Z', Im\, Z > 0\}$ of degree n. On $\mathcal{H}_n$ operates the symplectic group

$$\begin{aligned} Sp_n(\mathbb{R}) &= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(\mathbb{R}) \mid A,B,C,D \in M_n(\mathbb{R}), AD' - BC' = E, \\ & \quad AB' = BA', CD' = DC'\} \\ &= \{M \in GL_{2n}(\mathbb{R}) \mid I_n[M] = I_n\} \end{aligned}$$

(where $I_n = \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix}$) by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z = (AZ+B)(CZ+D)^{-1}.$$

(Note that $\mathcal{H}_1 = \mathcal{H}, Sp_1(\mathbb{R}) = SL_2(\mathbb{R})$.)

Fact: For m even, $\theta_Q^{(n)}$ is a Siegel modular form of weight $\frac{m}{2}$, level N and degree n, i. e. essentially

$$\theta_Q^{(u)}((AZ+B)(CZ+d)^{-1}) = \pm det(CZ+D)^{\frac{m}{2}} \theta_Q^{(n)}(Z), (\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}), N \mid c_{\mu\nu}\ \forall \mu,\nu).$$

Aim: To get results on $r_Q(T)$, study Siegel modular form of degree n!

For example, one has the following result due to Siegel: if $A_1, \dots, A_h$ is a full set of $GL_m(\mathbb{Z})$-representatives of even, symmetric, positive definite, unimodular matrices of size m (such matrices exist iff $8|m$, and the number of classes is always finite), then the average value

$$\sum_{\mu=1}^{n} \frac{r_{Q_\mu}(T)}{\epsilon(Q_\mu)} \quad \text{(where } \epsilon(Q_\mu) := \#\{G \in M_n(\mathbb{Z}) | G' A_\mu G = A_\mu\}\text{)}$$

can be expressed by Fourier coefficients of Eisenstein series of degree n. The latter in turn can be described by "elementary" number-theoretic expressions.

b) Compact Riemann surfaces

Let X be a compact Riemann surface, i.e. a compact, connected complex manifold of dimension 1.

Let g be the genus of X (visualize X as a sphere with g handles). Note that also dim $H_1(X;\mathbb{Z}) = 2g$ and $\chi(X) = 2 - 2g$ is the Euler characteristic of X.

Examples:

i) $X = S^2 = \mathbb{P}^1(\mathbb{C}), g = 0$;

ii) $X = \mathbb{C}/L$, where $L \subset \mathbb{C}$ is a lattice (elliptic curve), $g = 1$.

In the following we assume that $g \geq 1$. It is well-known that g is also the dimension of the space $\mathcal{H}(X)$ of holomorphic differential forms on X. Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a "canonical" $\mathbb{Z}$-basis for $H_1(X;\mathbb{Z})$, i.e. the intersection matrix is I_g (cf. a)). (For example, visualize X as a $4g$-sided polygon with the usual boundary identifications.) Then there is a uniquely determined basis $\{\omega_1, \ldots, \omega_g\}$ of $\mathcal{H}(X)$ such that

$$\int_{a_i} \omega_i = \delta_{ij} \quad (1 \leq i, j \leq g).$$

If one puts $z_{ij} := \int_{b_j} \omega_i (1 \leq i, j \leq g)$, then the period matrix $Z := (z_{ij})$ is in $\mathcal{H}_g$. Moreover, if one chooses a different basis, then the corresponding period matrix $\hat{Z}$ is obtained from Z by

$$\hat{Z} = M \circ Z$$

where $M \in \Gamma_g := Sp_g(\mathbb{Z})$. In this way one obtains a map

$$\phi : \mathcal{M}_g \to \mathcal{A}_g := \Gamma_g \backslash \mathcal{H}_g$$

where $\mathcal{M}_g$ is the set of isomorphism classes of compact Riemann surfaces of genus g.

Theorem 2. *(Torelli): ϕ is injective.*

Known: $g = 1 \Rightarrow \phi$ is bijective.

In general, $\mathcal{M}_g$ and $\mathcal{A}_g$ are "complex spaces" of dimensions $3g - 3$ and $\frac{1}{2}g(g+1)$, respectively. On both spaces one can do complex analysis, and one obtains interesting functions on $\mathcal{M}_g$ by restricting functions on $\mathcal{A}_g$ (i.e. by restricting Siegel modular functions of degree g).

Schottky Problem: Describe the locus of $\mathcal{M}_g$ inside $\mathcal{A}_g$ by algebraic equations! (For $g = 2, 3$ one has $\overline{\phi(\mathcal{M}_g)} = \mathcal{A}_g$, so the problem starts at $g = 4$). One wants a description in terms of equations between modular forms. For example, for $g = 4$ this locus is exactly the zero set of

$$\theta^{(4)}_{E_8 \oplus E_8} - \theta^{(4)}_{D_8^+}$$

where E_8 is the standard E_8-root lattice and in general

$$D_m^+ := D_m \cup (D_m + (\frac{1}{2}, \dots, \frac{1}{2})),$$

with

$$D_m = \{x \in \mathbb{Z}^m \mid x_1 + \cdots + x_m \equiv 0 \pmod 2\}$$

(note $D_8^+ = E_8$).

2 The Siegel modular group

a) Symplectic matrices

Definition: Let R be a commutative ring with 1. Let $E = E_n$ respectively $0 = 0_n$ the unit respectively zero matrix in $M_n(R)$ and put $I := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \in M_{2n}(R)$. Then

$$Sp_n(R) := \{M \in GL_{2n}(R) \mid I[M] = I\}$$

is called the symplectic group of degree n with coefficients in R. (General notation: if $A \in M_m(R), B \in M_n(R)$, we put $A[B] := B'AB$.)

Remark:

i) $Sp_n(R)$ is a subgroup of $GL_{2n}(R)$ (this is clear since $I[M_1M_2] = (I[M_1])[M_2]$, and $I[E_{2n}] = I$), in fact it is the automorphism group of the alternating skew-symmetric form defined by I.

ii) Later on, we will be only interested in the cases $R = \mathbb{Z}, \mathbb{R}$.

Lemma 3. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(R)$ *with* $A, B, C, D \in M_n(R)$. *Then*

i) $$\begin{aligned} M \in Sp_n(R) \quad &\Leftrightarrow \quad A'C = C'A, B'D = D'B, A'D - C'B = E \\ &\Leftrightarrow \quad AB' = BA', CD' = DC', AD' - BC' = E. \end{aligned}$$

ii) $M \in Sp_n(R) \Rightarrow M' \in Sp_n(R)$.

iii) $M \in Sp_n(R) \Rightarrow M^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}$.

Proof: Simple computations, for example

$$\begin{aligned} M'IM = I \quad &\Leftrightarrow \quad \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \\ &\Leftrightarrow \quad \begin{pmatrix} -C'A + A'C & -C'B + A'D \\ -D'A + B'C & -D'B + B'D \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \end{aligned}$$

hence the first part of i) holds.

Remark: From the Lemma, in particular, it follows that $Sp_1(R) = SI_2(R)$.

Examples for symplectic matrices:

i) $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$ with $S = S' \in M_n(R)$;

ii) $\begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}$ with $U \in GL_n(R)$;

iii) $I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$.

Theorem 4. *Let R be a Euclidean ring. Then $Sp_n(R)$ is generated by the special matrices as above under i) - iii).*

Proof: More complicated, use induction on n! Details cannot be given here.

Corollary 5. *$Sp_n(R) \subset SL_{2n}(R)$.*

Proof: The generators given in Theorem 4 obviously have determinant 1.

b) The Siegel upper half-space $\mathcal{H}_n$

Definition: $\mathcal{H}_n := \{Z = X + iY \in M_n(\mathbb{C}) \mid X, Y \text{ real }, Z = Z', Y = Im\ Z > 0)$ is called the Siegel upper half-space of degree n. Obviously $\mathcal{H}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathcal{P}_n \subset \mathbb{R}^{n(n+1)}$, where $\mathcal{P}_n := \{Y \in M_n(\mathbb{R}) \mid Y = Y', Y > 0\}$.

Lemma 6. *$\mathcal{P}_n$ is an open subset of $\mathbb{R}^{\frac{n(n+1)}{2}}$, hence $\mathcal{H}_n$ is an open subset of $\mathbb{R}^{n(n+1)}$.*

Proof: One has $Y > 0$ iff $detY^{(\nu)} > 0\ \forall \nu = 1, ..., n$ where $Y^{(\nu)}$ is the ν-th principal minor of Y, and det is a continuous function.

Theorem 7. *i) The group $Sp_n(\mathbb{R})$ operates on $\mathcal{H}_n$ by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z := (AZ+B)(CZ+D)^{-1}.$$

ii) One has

$$Im\ \ M \circ Z = (CZ+D)'^{-1} Im\ Z\ \ \overline{(CZ+D)}^{-1}.$$

Proof: i) One has to check that

a) $det(CZ+D) \neq 0, M \circ Z \in \mathcal{H}_n\ \ (M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R}), Z \in \mathcal{H}_n),$

b) $E_{2n} \circ Z = Z, (MN) \circ Z = M \circ (N \circ Z).$

This can be done by brute force calculations, using e.g. Theorem 4.

Here, to give a simple example, we only prove:

Claim: $Z \in \mathcal{H}_n \Rightarrow Z$ invertible, $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \circ Z = -Z^{-1} \in \mathcal{H}_n.$

Proof of claim: Choose $V \in GL_n(\mathbb{R})$ with $Y[V] = V'YV' = E$. Then $Z[V] = T + iE$ with $T = X[V]$. Since $(T+iE)(T-iE) = T^2 + E = TT' +\ E$ is positive definite, it is invertible, so $T + iE$ and hence Z is invertible.
Furthermore

$$(T+iE)^{-1} = (T-iE)(T^2+E)^{-1}$$

$$\Rightarrow -Z^{-1} = -(((T+iE)[V^{-1}])^{-1} = -V(T-iE)(T^2+E)^{-1}V' \Rightarrow$$

$$(Im\ Z)^{-1} = V(T^2+E)^{-1}V' > 0.$$

ii) Use that $Im\ Z = \frac{1}{2i}(Z - \bar{Z})$, insert $M \circ Z$ for Z and use the symplectic relations.

c) Reduction theory

Definition: The discrete subgroup $\Gamma_n := Sp_n(\mathbb{Z}) \subset Sp_n(\mathbb{R})$ is called Siegel modular group of degree n.

Aim of reduction theory: Study the action of Γ_n on $\mathcal{H}_n$! From each orbit $\{M \circ Z \mid M \in \Gamma_n\}$ choose a "suitable" representative, a so-called "reduced point". The set $\mathcal{F}_n$ of reduced points should be described by "simple" and possibly a finite set of inequalities in the components of Z; $\mathcal{F}_n$ should have "nice" geometric properties (e.g. should be connected, measurable, etc.).

Recall the case $n = 1$, i.e. $\Gamma_1 = SL_2(\mathbb{Z}), \mathcal{H}_1 = \mathcal{H} = \{z \in \mathbb{C} \mid Im\ z > 0\}$.

Idea: $z \in \mathcal{H}, z \mapsto h(z) := Im\ z =$ "height" of z. Then

$$(1) \quad h(M \circ z) = |cz + d|^{-2} h(z) \quad (M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma_1).$$

One shows:

i) Any orbit $\Gamma_1 \circ z$ contains points w of maximal height. These are characterized among the points of $\Gamma_1 \circ z$ by the conditions $|cw + d| \geq 1, \forall c, d \in \mathbb{Z}$ with $(c, d) = 1$;

ii) For every $z \in \mathcal{H}$, $\exists M \in \Gamma_1$ such that $M \circ z \in \mathcal{F}_1' := \{z = x + iy \in \mathcal{H} \mid |x| \leq \frac{1}{2},\ |cz + d| \geq 1, \forall c, d \in \mathbb{Z}$ with $(c, d) = 1\}$;

iii) $\mathcal{F}_1' = \mathcal{F}_1 := \{z = x + iy \in \mathcal{H} \mid |x| \leq \frac{1}{2}, |z| \geq 1\}$;

iv) No two different inner points of $\mathcal{F}_1$ are Γ_1-equivalent.

Proof:

i) Follows easily from (1).

ii) Follows from (i) and the fact that $h(z)$ is invariant under translations.

iii) Follows from the inequalities (valid for $z \in \mathcal{F}_1$),

$$|cz + d|^2 = c^2(x^2 + y^2) + 2cdx + d^2 \geq c^2 - |cd| + d^2 \geq 1,$$

(since the quadratic form $x^2 \pm xy + y^2$ is positive definite).

iv) One uses: $z', z \in \mathcal{F}_1, z' = M \circ z \Rightarrow h(M \circ z) = h(z)$ by construction $\Rightarrow |cz + d| = 1 \ (M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix})$.

For arbitrary $n \geq 1$ try to do something similar: $Z \in \mathcal{H}_n \mapsto h(Z) := det(Im\ Z) =$ "height" of Z. Then by Theorem 7, ii) one has

$$h(M \circ Z) = |det(CZ + D)|^{-2} h(Z).$$

One can show that each orbit contains points W of maximal height and these are characterized by $|det(CW + D)\ | \geq 1\ \forall \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_n$. Additional difficulty for $n \geq 1$: $h(Z)$ is not only invariant under translations $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, but also under $\begin{pmatrix} U' & 0 \\ 0 & U'^{-1} \end{pmatrix}$ with $U \in GL_n(\mathbb{Z})$. Therefore one also has to study the action $Y \mapsto Y[U] = U'YU$ of $GL_n(\mathbb{Z})$ on $\mathcal{P}_n$ (Minkowski reduction theory)!

Definition: We let $\mathcal{F}_n$ be the set of all $Z \in \mathcal{H}_n$ which satisfy

i) $|det(CZ+D)| \geq 1 \ \forall \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_n$;

ii) $Y = Im\ Z$ is Minkowski reduced, i.e. $Y[g] \geq y_k$ (k-th diagonal element of Y) $\forall$ $g = \begin{pmatrix} g_1 \\ \ddots \\ g_n \end{pmatrix} \in M_{n,1}(\mathbb{Z})$ with $(g_k, \cdots, g_n) = 1, \forall k = 1, \cdots, n$, and $y_{k,k+1} \geq 0 \ \forall \ k$ with $1 \leq k < n$;

iii) $|x_{ij}| \leq \frac{1}{2} \ \ \forall \ \ 1 \leq i,j \leq n$.

Points in $\mathcal{F}_n$ are called Siegel reduced.

Theorem 8. *If $Z \in \mathcal{F}_n$, then the following holds:*

i) $y_n \geq y_{n-1} \geq \cdots \geq y_1$, $|y_{\mu\nu}| \leq \frac{1}{2} y_\nu$ *for* $\mu \neq \nu$;

ii) $detY \leq y_1 \cdots y_n \leq C_n detY$ *where* $C_n > 0$ *is a constant depending only on* n;

iii) $y_1 \geq \frac{1}{2}\sqrt{3}$;

iv) $Y \geq \delta_n E_n$ *(i.e.* $Y - \delta_n E_n \geq 0$*) where* $\delta_n > 0$ *is a constant depending only on* n.

Properties i) - ii) hold for any Minkowski reduced matrix $Y > 0$, while iii) - iv) can be proved in addition if Z is Siegel reduced.

Theorem 9. i) $\mathcal{F}_n$ *is a fundamental domain for* Γ_n, *i.e.*

- **a)** *for any* $Z \in \mathcal{F}_n$, $\exists M \in \Gamma_n$ *with* $M \circ Z \in \mathcal{F}_n$;
- **b)** *if* $Z, Z' \in int\mathcal{F}_n$ *and* $Z' = M \circ Z$ *with* $M \in \Gamma_n$, *then* $M = \pm E_{2n}$ *and* $Z = Z'$;
- **c)** $\#\{M \in \Gamma_n \mid M \circ \mathcal{F}_n \cap \mathcal{F}_n \neq \emptyset\} < \infty$.

ii) *The set $\mathcal{F}_n$ can be defined by finitely many inequalities of the type given in the definition of $\mathcal{F}_n$. One has*

$\overline{int\mathcal{F}_n} = \mathcal{F}_n$.

The proofs of Theorems 8 and 9 are difficult and cannot be given here.

3 Siegel modular forms: basic properties

Definition: A function $f : \mathcal{H}_n \to \mathbb{C}$ is called a Siegel modular form of weight $k \in \mathbb{Z}$ and degree n ($w.r.t.$ Γ_n), if

i) f is holomorphic,

ii) $f((AZ+B)(CZ+D^{-1}) = det(CZ+D)^k F(Z) \;\; \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n,$

iii) f is bounded in $Y \geq Y_0$, for any $Y_0 > 0$.

Remarks:

i) We denote $M_k(\Gamma_n)$ the $\mathbb{C}$-vector space of Siegel modular forms of weight k and degree n.

ii)

$$f \in M_n(\Gamma_n) \Rightarrow \quad f(Z+S) = f(Z) \; (\forall S = S' \in M_n(\mathbb{Z})),$$
$$f(Z[U]) = (detU)^k f(Z) \; (\forall U \in GL_n(\mathbb{Z})),$$
$$f(-Z^{-1}) = (detZ)^k f(Z)$$

(apply the transformation formula ii) to the special generators of Theorem 4). The converse is also true, since Γ_n is generated by these special matrices (Theorem 4).

iii) kn odd $\Rightarrow M_k(\Gamma_n) = \{0\}$ (since $-E$ then acts on f by multiplication with $(-1)^{kn}$).

Theorem 10. *Let $f \in M_k(\Gamma_n)$. Then f has a Fourier expansion of the form*

$$f(Z) = \sum_{\substack{T=T' half-integral \\ T \geq 0}} a(T) e^{2\pi i \, tr(TZ)},$$

absolutely convergent on $\mathcal{H}_n$ and uniformly in $Y = Y' \geq Y_0 > 0$. If $n \geq 2$, then condition iii) in the definition of a Siegel modular form follows from conditions i) and ii) ("Koecher principle").

Moreover, one has

$$a(T[U]) = (det\; U)^k a(T) \;\; (\forall T \geq 0, \forall U \in GL_n(\mathbb{Z})).$$

Proof: If $n = 1$, the existence of the Fourier expansion of the given shape easily follows from the boundedness of $f(Z)$ for $Y \geq Y_0 > 0$ and Riemann's criterion for removable singularities (consider the map $z \mapsto q := e^{2\pi i z}$ and put $\hat{f}(q) := f(z)(q = e^{2\pi i z}, z \in \mathcal{H})$).

Now suppose $n \geq 2$. Since f is holomorphic and satisfies $f(Z+S) = f(Z)$ $(\forall S = S' \in M_n(\mathbb{Z}))$, f certainly has a Fourier expansion which can be written as

$$f(Z) = \sum_{T=T' \; half-integral} a(T) e^{2\pi i tr(TZ)}$$

(note that

$$tr(TZ) = \sum_{1 \leq \mu < \nu \leq n} t_{\mu\nu} z_{\mu\nu} + \sum_{\nu=1}^{n} t_{\nu\nu} z_{\nu}).$$

Since $f(Z[U]) = (det\ U)^k f(Z)\ \forall U \in GL_n(\mathbb{Z})$, the formula $a(T[U]) = (det\ U)^k a(T) (\forall\ T \geq 0, \forall\ U \in GL_n(\mathbb{Z}))$ easily follows by comparing Fourier coefficients. In particular

$$a(T[U]) = a(T) \;\; (\forall\ U \in SI_n(\mathbb{Z})).$$

Suppose that $a(T) \neq 0$. Then, since the Fourier series is absolutely convergent, it follows that

(2)

$$\sum_S |e^{2\pi i tr(SZ)}| = \sum_S e^{-2\pi tr(SY)}$$

is convergent for any $Y = Y' > 0$, where the summation is over all the different matrices $S = T[U]$ with $U \in SI_n(\mathbb{Z})$.

<u>Claim:</u> If T is not positive semi-definite, the series (2) diverges for $Y = E$.

<u>Proof of Claim:</u> By assumption $\exists g \in \mathbb{R}^n$ with $T[g] < 0 \Rightarrow \exists g \in \mathbb{Q}^n$, hence $\exists g \in \mathbb{Z}^n$ with components relatively prime such that $T[g] < 0$. By Gauss lemma, $\exists U \in SI_n(\mathbb{Z})$ with first column g. Replacing T by $T[U]$ one can assume that $t_{11} = T[e_1] < 0$. Let

$$S_y := T[V_y], V_y := \begin{pmatrix} \begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \text{ with } \; y \in \mathbb{Z}.$$

Choose a sequence $(y_\nu)_{\nu \in \mathbb{N}}$ with $|y_\nu| \to \infty (\nu \to \infty)$ s.t. the corresponding matrices S_{y_ν} are pairwise different. (This is possible, since $S_y[e_2] = t_{11}y^2 + t_{12}y + t_{22}$.)

Then

$$tr(S_{y_\nu}) = t_{11} y_\nu^2 + (\text{ linear terms in } y_\nu) \to -\infty \; (y \to \infty)$$
$$\Rightarrow e^{-2\pi tr(S_{y_\nu})} \to \infty \; (|y_\nu| \; \to \infty),$$

which proves the claim.

This proves the first assertion about the Fourier expansion.

Now suppose that f satisfies conditions i) and ii). From the above arguments it then follows that f has an expansion

$$f(Z) = \sum_{\substack{T=T'half-integral \\ T\geq 0}} a(T)e^{2\pi itr(TZ)}.$$

One then obtains for $Y \geq Y_0 > 0$

$$|f(Z)| \leq \sum_{\substack{T=T'half-integral \\ T\geq 0}} |a(T)|\ e^{-2\pi tr(TY)}$$

$$\leq \sum_{\substack{T=T'half-integral \\ T\geq 0}} |a(T)|\ e^{-2\pi tr(TY_0)} < \infty.$$

(in the last line we have used: $S \geq 0, T \geq 0 \Rightarrow tr(ST) \geq 0$).

This at the same time shows that the Fourier series is uniformly convergent in $Y \geq Y_0 > 0$.

Theorem 11. *("Siegel Φ-operator") For $f \in M_k(\Gamma_n)$ and $Z_1 \in \mathcal{H}_{n-1}$ set*

$$(f|\Phi)(Z_1) := \lim_{t\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & it \end{pmatrix}.$$

Then the limit exists, and the Φ-operator defines a linear map $\Phi : M_k(\Gamma_n) \to M_k(\Gamma_{n-1})$ (with the convention $M_k(\Gamma_0) := \mathbb{C}$). If $f(Z) = \sum_{T=T'\geq 0} a(T)e^{2\pi itr(TZ)}$, then

$$(f|\Phi)(Z_i) = \sum_{\substack{T_1=T_1'\geq 0 \\ T_1\in M_{n-1}(\mathbb{Z})\ half-integral}} a(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix})\ \ e^{2\pi itr(T_1Z_1)}\ \ (Z \in \mathcal{H}_{n-1}).$$

Proof: Because of uniform convergence for $Y = Y' \geq Y_0 > 0$, one can interchange the limit for $t \to \infty$ and the summation over T, and then the existence of the limit and the shape of the Fourier expansion can easily be seen. (Observe that if $T = \begin{pmatrix} T_1 & * \\ * & t_{nn} \end{pmatrix} \geq 0$ and $t_{nn} = 0$, then the last row and last column of T must be zero.)

That $f|\Phi \in M_k(\Gamma_{n-1})$ follows from the fact: $M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma_{n-1}$

$$\Rightarrow M := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & 1 & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_n,\ M \circ \begin{pmatrix} Z_1 & 0 \\ 0 & it \end{pmatrix} = \begin{pmatrix} M_1 \circ Z_1 & 0 \\ 0 & it \end{pmatrix},$$

$$det(CZ+D) = det(C_1Z+D_1) \text{ for } Z = \begin{pmatrix} Z_1 & 0 \\ 0 & it \end{pmatrix}.$$

Definition: We let $S_k(\Gamma_n) := ker\ \Phi$. Elements of $S_k(\Gamma_n)$ are called cup forms. From Theorem 11 one can easily obtain:

Corollary 12. *Let $f \in M_k(\Gamma_n)$. Then $f \in S_k(\Gamma_n) \Leftrightarrow a(T) = 0$ unless $T > 0$.*

Examples of Siegel modular forms:

i) Theta series: We use previous notation. Let $A \in M_m(\mathbb{Z})$, A even, $A = A'$, $A > 0$. Then the series

$$\theta_Q^{(n)}(Z) := \sum_{G \in M_{m,n}(\mathbb{Z})} e^{\pi i tr(A[G]Z)} \quad (Z \in \mathcal{H}_n)$$

is called a theta series. Obviously

$$\theta_Q^{(n)}(Z) = \sum_{\substack{T=T'\ half-integral \\ T \geq 0}} r_Q(T)\, e^{2\pi i tr(TZ)}$$

where $r_Q(T)$ is the number of representations of T by Q (cf. section 1).

One can show: if $detA = 1$, then $\theta_Q^{(n)} \in M_{m/2}(\Gamma_n)$. (One can show: $\exists A \in M_m(\mathbb{Z})$ with A even, $A > 0, detA = 1 \Leftrightarrow 8|m$.) The proof is technically difficult and uses the Poisson summation formula.

ii) Eisenstein series: The formal series

$$E_k^{(n)}(Z) := \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n,0} \backslash \Gamma_n} det(CZ+D)^{-k}$$

(with $\Gamma_{n,0} := \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n\}$) formally behaves like a Siegel modular form of weight k and degree n and is called a Siegel-Eisenstein series. The main difficulty is the proof of convergence (in sufficiently large domains). One can in fact show: Suppose k is even and $k > n+1$. Then $E_k^{(n)} \in M_k(\Gamma_n) \setminus \{0\}$.

More generally, one can define "generalized Eisenstein series" (so-called Klingen-Siegel-Eisenstein series) by lifting cup forms of weight k and degree $j < n$ to Γ_n. For n even, $k > 2n$ one can show that

$$M_k(\Gamma_n) = \mathcal{E}_k(\Gamma_n) \oplus S_k(\Gamma_n),$$

where $\mathcal{E}_k(\Gamma_n)$ is the subspace of "generalized Eisenstein series".

Theorem 13. *One has*

i) $M_k(\Gamma_n) = \{0\}$ *for* $k < 0$*;*

ii) $S_0(\Gamma_n) = \{0\}, M_0(\Gamma_n) = \mathbb{C}$*;*

iii) $dim\ S_k(\Gamma_n) = O(k^N), dim\ M_k(\Gamma_n) = O(k^N)\ (n \to \infty)$ *where* $N = \frac{n(n+1)}{2}$.

Proof: One first shows the basic

Lemma 14. *Let* $f \in M_k(\Gamma_n)$ *and* $k \geq 0$. *Then* $g(Z) := (det\ Y)^{k/2}\ |f(Z)|$ *is* Γ_n*-invariant. If* f *is cuspidal, then* g *has a maximum on* $\mathcal{H}_n$.

The first assertion of the Lemma immediately follows from

$$det\ Im\ M\ \circ Z = \ |det\ (CZ + D\ |^{-2} Im\ Z \quad (M \in \Gamma_n).$$

To show the second one, one observes that

$$\mathcal{F}_n(C) := \{Z \in \mathcal{F}_n \mid det\ Y \leq C\} \quad (C > 0)$$

is compact (it is closed by Theorem 8, iv) and bounded by $loc.cit.\ i) - iii)$), hence it suffices to show that $\lim_{\substack{Z \in \mathcal{F}_n \\ det\ Y \to \infty}} g(Z) = 0$. This again follows from the reduction conditions, using the Fourier expansion of f.

We prove e.g. ii): by Lemma 14, if $f \in S_0(\Gamma_n)$, f has a maximum on $\mathcal{H}_n$, hence $f = c$ is constant by the maximum principle. Since f is cuspidal,

$$c = \lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} = 0,$$

so $f = 0$. Also $\mathbb{C} \subset M_0(\Gamma_n)$. We prove the converse by induction on n.

If $n = 0$ there is nothing to prove. Suppose $f \in M_k(\Gamma_n)$, $n \geq 1$. Then $f|\Phi \in M_k(\Gamma_{n-1})$, so $f|\Phi = c$ by induction hypothesis, hence $(f - c)|\Phi = 0$. Therefore $f = c$.

The proof of i) is similar, while the proof of iii) is more difficult: One shows that there exists $\mu_n > 0$ depending only on n such that if $f \in S_k(\Gamma_n)$ and $a(T) = 0$ for all $T > 0$ with $tr(T) < \frac{k}{\mu_n}$, then $f = 0$.

Hence $dim\ S_k(\Gamma_n)$ is less or equal to the number of $T \in M_n(\mathbb{Z})$ such that T is half-integral, $T > 0, tr(T) < \frac{k}{\mu_n}$, and the latter number is easily seen to be $O(k^N)$, by a simple counting argument.

Also

$$dim\ M_k(\Gamma_n) = dim\ ker\Phi + dim\ im\Phi \leq dim\ S_k(\Gamma_n) + dim\ M_k(\Gamma_{n-1}) = O(k^N)$$

by induction.

4 Hecke operators and L-functions

a) The Hecke algebra

Let $\Gamma := \Gamma_n$ and $GSp_n^+(\mathbb{Q}) := \{M \in GL_{2n}(\mathbb{Q} \mid I[M] = \nu_M I, \nu_M \in \mathbb{Q}, \nu_M > 0\}$ be the group of rational symplectic similitudes of size $2n$ with positive scalar factor. Then Γ is a group anf G is a (semi-)group with $\Gamma \subset G$.

Let $L(\Gamma, G)$ be the free $\mathbb{C}$-module generated by the right cosets Γx ($x \in \Gamma\backslash G$).

Then Γ operates on $L(\Gamma, G)$ by right multiplication, and we let

$$H_n := L(\Gamma, G)^\Gamma$$

be the subspace of Γ-invariants. If

$$T_1 = \sum_{x\in\Gamma\backslash G} a_x \Gamma x, \quad T_2 = \sum_{y\in\Gamma\backslash G} b_y \Gamma y \in H_n,$$

one puts

$$T_1 \cdot T_2 := \sum_{x,y\in\Gamma\backslash G} a_x b_y \Gamma xy.$$

Then $T_1 \cdot T_2 \in H_n$. This follow from the fact that H_n is "generated" by double cosets, i.e. by the elements $\sum_i \Gamma x_i$ where $\Gamma\backslash\Gamma x\Gamma = \cup_i \Gamma x_i$ (finite disjoint) and $x \in G$.

The space H_n together with the above multiplication is called the Hecke algebra. It is a commutative associative algebra with 1 (commutativity formally follows from the fact that $\Gamma x \Gamma = \Gamma x' \Gamma$).

The following facts are known:

i) $H_n = \otimes_{p\, prim} H_{n,p}$
where $H_{n,p}$ is defined in the same way as H_n, however with G replaced by $G^{(p)} := G \cap GL_{2n}(\mathbb{Z}[p^{-1}])$.

ii) The local component $H_{n,p}$ is generated by the $n+1$ double cosets

$$T(p) = \Gamma \begin{pmatrix} 1_n & 0 \\ 0 & p1_n \end{pmatrix} \Gamma$$

and

$$T_{i,j}(p^2) = \Gamma \begin{pmatrix} \begin{matrix} 1_i & 0 \\ 0 & p1_j \end{matrix} & 0 \\ 0 & \begin{matrix} p^2 1_i & 0 \\ 0 & p1_j \end{matrix} \end{pmatrix} \Gamma \qquad (0 \le i < n, i+j = n)$$

where 1_n denotes the unit matrix of size n (the elements $T(p), T_{i,j}(p^2)$ are algebraically independent). Moreover, one has

$$H_{n,p} \cong \mathbb{C}\left[X_0^{\pm 1}, \ldots, X_n^{\pm 1}\right]^W,$$

where W is the Weyl group generated by the permuatations of the $X_i (i = 1, \ldots, n)$ and the maps $X_0 \mapsto X_0 X_j,\ X_j \mapsto X_j^{-1},\ X_i \mapsto X_i \quad (1 \le i \le n, i \neq j)$, for $j \in \{1, \ldots, n\}$. In particular, one has

$$Hom_{\mathbb{C}}(H_{n,p}, \mathbb{C}) = (\mathbb{C}^*)^{n+1}/W$$

(with the obvious operation of W on $(\mathbb{C}^*)^{n+1}$).

iii) There exist special Hecke operators $T(m)(m \in \mathbb{N}),\ T(m) := \sum_{X \in \Gamma \setminus O_{m,n}} \Gamma x (O_{m,n} := \{x \in GL_{2n}(\mathbb{Z}) \mid I[x] = mI\})$.

iv) The Hecke algebra operates on $M_k(\Gamma_n)$ resp. $S_k(\Gamma_n)$ by

$$F|_k \left(\sum a_x \Gamma x\right) = \sum a_x F\ |_k x$$

where

$$(f|_k x)(z) = r_x^{nk-n(n+1)/2} \cdot det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1})$$

$(x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, r_x =$ scalar factor of $x)$.

v) The Hecke operators are hermitian with respect to the scalar product

$$< F, G >= \int_{F_n} F(Z)\overline{G(Z)}(detY)^k \frac{dY\ dY}{(detY)^{n+1}}.$$

Hence the space $S_k(\Gamma_n)$ has a basis of common eigenfunctions of all $T \in H_n$. If F is such an eigenfunction and $F|T = \lambda(T)F$ $(T \in H_n)$, then $T \mapsto \lambda(T)$ is a homomorphism $H_{n,p} \to \mathbb{C}$ for each p, hence by ii) is determined by an element $(\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}) \in (\mathbb{C}^*)^{n+1}/W$ ("Satake p-paramenters").

b) L-functions

If $F \in S_k(\Gamma_n)$ is a common eigenform, for $Re(s) \gg 0$ define

$$L_{St}(F, s) := \zeta(s) \prod_p L_{St,p}\ (F; p^{-s})^{-1} \quad (\text{"standard zeta function"})$$

and

$$L_{spin}(F, s) := \prod_p L_{spin,p}\ (F; p^{-s})^{-1} \quad (\text{"spinor zeta function"})$$

where

$$L_{St,p}(F;X) := \prod_{i=1}^{n}(1-\alpha_{i,p}X)(1-\alpha_{i,p}^{-1}X),$$

$$L_{spin,p}(F;X) := (1-\alpha_{0,p}X)\prod_{\nu=1}^{n}\prod_{1\le i_1<\cdots<i_\nu\le n}(1-\alpha_{0,p}\,\alpha_{i_1,p}\dots\,\alpha_{i_\nu,p}X).$$

One knows that $L_{St}(F,s)$ has a meromorphic continuation $\mathbb{C}$ with finitely many poles and has a functional equation under $s \mapsto 1-s$, for all n (Böcherer a.o.).

Conjecture (Andrianov, Langlands): The function $L_{spin}(F,s)$ when completed with appropriate Γ-factors, has meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto nk - \frac{n(n+1)}{2} + 1 - s$.

This conjecture is known only for $n \le 2$ (for $n = 1$ this is classical by Hecke), for $n = 3$ one knows meromorphic continuation (R. Schmidt, 2002). For $n = 2$ one has more precisely

Theorem 15. *(Andrianov, 1974) Suppose $n = 2$ and let us write $Z_F(s) = L_{spin}(F,s)$.*

i) *Suppose that $f|T(m) = \lambda(m)f$ for all $m \ge 1$. Then*

$$Z_{F,p}(X) = 1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})X^2 - \lambda(p)p^{2k-3}X^3 + p^{4k-6}X^4;$$

ii) *The function*

$$Z_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

has meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$Z_F^*(2k-2-s) = (-1)^k Z_F^*(s).$$

Proof: One uses Andrianov's formulas: if $D < 0$ is a fundamental discriminant, $\{T_1,\dots,T_{h(D)}\}$ is a set of Γ_1-representatives of primitive positive definite integral binary quadratic forms of discriminant D and χ is a character of the ideal class group $CL(\mathbb{Q}(\sqrt{D}))$,

then

$$L(s-k+2,\chi)\sum_{i=1}^{h(D)}\left(\sum_{n\ge 1}\frac{a(nT_i)}{n^s}\right) = \left(\sum_{i=1}^{h(D)} a(T_i)\right) Z_F(s).$$

(If $D < 0$ is an arbitrary discriminant, then there exist similar but more complicated identities.)

The left hand side can be written as the integral over a portion of the 3-dimensional hyperbolic space (suitably embedded in $\mathcal{H}_2$) of the product of the restriction of F to this portion and a non-holomorphic Eisenstein series for the hyperbolic 3-space. The analytic properties of $Z_F(s)$ then follow from the analytic properties of the Eisenstein series.

5 Liftings

a) The Saito-Kurokawa lift

Theorem 16. *(Maass, Andrianov, Eichler, Zagier, 1981) Let k be even. Then for each normalized ($a(1) = 1$) Hecke eigenform $f \in S_{2k-2}(\Gamma_1)$, there is a Hecke eigenform $F \in S_k(\Gamma_2)$ (uniquely determined up to a non-zero scalar) such that*

$$Z_F(s) = \zeta(s-k+1)\,\zeta(s-k+2)L(f,s),$$

where $L(f,s)$ is the Hecke L-function of f.

The space generated by the above F's is called the Maass space $S_k^*(\Gamma_2) \subset S_k(\Gamma_2)$. For the proof one uses

i) $S_{2k-2}(\Gamma_1) \cong S^+_{k-\frac{1}{2}} = \{g \in S_{k-\frac{1}{2}}(\Gamma_0(4)) \mid g = \sum_{n\geq 1, n\equiv 0,3(4)} c(n)\, e^{2\pi i n z}\}$ as Hecke modules (Shimura isomorphism, trace formula);

ii) (Eichler, Zagier) The map

$$g = \sum_{\substack{n\geq 1\\ n\equiv 0,3(4)}} c(n)\, e^{2\pi i n z} \mapsto \sum_{T>0} a(T) e^{2\pi i tr(TZ)}$$

where

$$a(T) := \sum_{d\,\mid\,(n,r,m)} d^{k-1} c\left(\frac{4mn-r^2}{d^2}\right) \quad \left(T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right)$$

is a linear isomorphism from $S^+_{k-\frac{1}{2}}$ onto $S_k^*(\Gamma_2)$ commuting with all Hecke operators.

From ii) one can also obtain the following linear description of the space $S_k^*(\Gamma_2)$.

Theorem 17. *(Maass, Andrianov, Eichler, Zagier, 1981)*
One has

$$S_k^*(\Gamma_2) = \left\{ F \in S_k(\Gamma_2) \mid a\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = \sum_{d\,\mid\,(n,r,m)} d^{k-1} a\begin{pmatrix} \frac{nm}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix} \forall \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0 \right\}$$

("Maass relations")

b) Ikeda's lifting theorem
Note (easy computation): if $F \in S_k^*(\Gamma_2)$ is a Hecke eigenform, then

$$L_{St}(F,s) = \zeta(s)(f, s+k-1)L(f, s+k-2).$$

Conjecture (Duke-Imamoglu, 1996). Let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform. Let $n \in \mathbb{N}$ with $n \equiv k \pmod 2$. Then there is a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ such that

$$L_{St}(F,s) = \zeta(s) \prod_{j=1}^{2n} L(f, s+k+n-j).$$

Using the functional equation of $L(f,s)$ under $s \mapsto 2k-s$, one easily checks a functional equation of the right hand side above under $s \mapsto 1-s$.

Theorem 18. *(Ikeda, 1999) The conjecture is true. Moreover the Fourier coefficients of F are given by*

$$a(T) = c(|D_{T,0}|) f_T^{k-\frac{1}{2}} \prod_{p|D_T} \tilde{F}_p(T;\alpha_p)$$

In the above the notation is as follows:

i) $D_T = (-1)^n det(2T) (\equiv 0, 1 \pmod 4)$ is the discriminant of $T, D_T = D_{T,0} f_T^2$ with $D_{T,0}$ a fundamental discriminant and $f_T \in \mathbb{N}$.

ii) $c(|D_{T,0}|) = |D_{T,0}|$-th Fourier coefficient of a Hecke eigenform

$$g = \sum_{m \geq 1, (-1)^k m \equiv 0,(4)} c(m) e^{2\pi i m z} \in S^+_{k+\frac{1}{2}} \subset S_{k+\frac{1}{2}}(\Gamma_0(4))$$

corresponding to f under the Shimura correspondence.

iii) $\tilde{F}_p(T;X)$is a "non-trivial" part of a modified local singular series (Laurent-) polynomial attached to T. More precisely, let

$$b_p(T;s) = \sum_R v_p(R)^{-s} e_p(tr(TR)) \quad (s \in \mathbb{C})$$

where R runs over all symmetric matrices in $M_{2n}(\mathbb{Q}_p/\mathbb{Z}_p), v_p(R) =$ power of p equal to the product of denominators of the elementary divisors of $R, e_p(x) = e^{2\pi i x'} (x \in \mathbb{Q}_p)$ with x'= fractional part of x. One knows

$$b_p(T;s) = \gamma_p(T;p^{-s}) F_p(T;p^{-s})$$

where

$$\gamma_p(T;X) = (1-X)(1 - \left(\frac{D_{T,0}}{p}\right) p^n X)^{-1} \prod_{j=1}^{n} (1 - p^{2j} X^2)$$

and $F_p(T;X)$ is a polynomial. Put

$$\tilde{F}_p(T;X) = X^{-ord_p\, f_T} F_p(T; p^{-n-\frac{1}{2}} X) \ (= \textit{Laurent polynomial}).$$

Katsurada(1999) has proved: $\tilde{F}_p(T;X)$ is symmetric, i.e. $\tilde{F}_p(T;X^{-1}) = \tilde{F}_p(T;X)$.

iv) $\alpha_p = p$-th Satake parameter of f, i.a.w. if $a(p) = p$-th Fourier coefficient of f, then

$$1 - a(p)X + p^{2k-1}X^2 = (1 - p^{k-\frac{1}{2}}\alpha_p X)(1 - p^{k-\frac{1}{2}}\alpha_p^{-1}X).$$

The proof (difficult) uses representation theory and techniques from the theory of Fourier-Jacobi expansions.

We now want to give a linear version of Ikeda's lifting theorem.

Notation:

i) Let $T \in M_{2n}(\mathbb{Q})$ be half-integral, $T = T'$, p a prime. Let $V/\mathbb{F}_p$ be the quadratic space over $\mathbb{F}_p$ obtained from reducing the quadratic form attached to T modulo p, $R(V) =$ radical of V, $s_p = dim\ R(V), V = R(V)\widehat{\oplus}W$,

$$\lambda_p(T) := \begin{cases} 1 \text{ if } W \text{ hyperbolic} \\ -1 \text{ otherwise }. \end{cases}$$

Put

$$H_{n,p}(T,X) := \begin{cases} 1 & if\ s_p = 0 \\ \prod_{j=1}^{[\frac{S_p-1}{2}]}(1 - p^{2j-1}X^2) & if\ s_p > 0, \text{odd} \\ (1 + \lambda p(T)p^{\frac{S_p-1}{2}}X)\prod_{j=1}^{[\frac{S_p-1}{2}]}(1 - p^{2j-1}X^2) & if\ s_p > 0, \text{even}. \end{cases}$$

ii) Define $\rho_T(p^\mu)$ (p prime, $\mu \geq 0$) by

$$\sum_{\mu \geq 0} \rho_T(p^\mu)X^\mu := \begin{cases} (1 - X^2)H_{n,p}(T,X) & \text{if } p|f_T \\ 1 & \text{if not} \end{cases}$$

and $\rho_T(a)(a \geq 1)$ by

$$\sum_{n \geq 1} \rho_T(a)a^{-s} := \prod_{p|f_T}(1 - p^{-2s})H_{n,p}(T, p^{-s}).$$

iii) For $a \in \mathbb{N}$ with $a \mid f_T$ put

$$\varphi(a;T) := \sqrt{a} \sum_{d^2|a} \sum_{G \in \mathbb{D}(T),\ |\ detG\ |=d} \rho_{T[G^{-1}]}(\frac{a}{d^2})$$

where $\mathbb{D}(T) = GL_{2n}(\mathbb{Z} \setminus \{G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) \mid T[G^{-1}] := G'^{-1}TG^{-1}$ half-integral $\}$ (finite set); note that $\varphi(a;T) \in \mathbb{Z}$.

Theorem 19. *(Kohnen, 2001) The Fourier coefficients of the Ikeda lift F of f are given by*

$$a(T) = \sum_{a \mid f_T} a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right).$$

Corollary 20. *The map*

$$\sum_{\substack{m\geq 1 \\ (-1^k m \equiv 0,(4)}} c(m)e^{2\pi imz} \mapsto \sum_{T>0} \ (\sum_{a \mid f_T} a^{k-1}\phi(a;T)c(|D_{T_0}|\frac{f_T^2}{a^2}))e^{2\pi itr(TZ)}$$

is a linear map $I_{k,n}$ from $S^+_{k+\frac{1}{2}}$ to $S_{k+n}(\Gamma_{2n})$ which on Hecke eigenforms coincides with the Ikeda lift.

Proof: One combines

i) formulas of Böcherer-Kitaoka for the local singular series polynomials which expresses $\Pi_{p \mid f_T}\tilde{F}_p(T;X)$ as a finite sum over $G \in \mathbb{D}(T)$ of "simple" polynomials (without any obvious functional equations);

ii) the functional equation of $\tilde{F}_p(T;X)$ and a symmetrization trick;

iii) the multiplicative structure of the Fourier coefficients $c(|D_{T,0}|m^2)$ $(m \in \mathbb{N})$.

Remark: If $n = 1$, one can show that

$$\phi(a;T) = \begin{cases} a & if\ a|(n,r,m) \\ 0 & \text{otherwise} \end{cases} \quad (T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix})$$

and so one recovers the formula of Eichler-Zagier (see a)).

The same technique as above can also be applied in the context of Eisenstein series. One obtains

Theorem 21. *(Kohnen, 2001) Let $k \equiv 0 \pmod 2, k > 2n+1$. Then the T-th Fourier coefficient $(T > 0)$ of the Siegel-Eisenstein series $E_k^{(2n)}$ of weight k and degree $2n$ is given by*

$$a_{k,2n}(T) = \frac{2^n}{\zeta(1-k)\prod_{j=1}^n \zeta(1-2k+2j)} \sum_{a|f_T} a^{k-n-1}\ \varphi(a;T)H(k-n,\ |D_{T,0}|\ (\frac{f_T}{a})^2),$$

where $H(k,m)$ is the generalized Cohen class number function.

Problem: The function $\varphi(a;T)$ looks nasty and seems to be difficult to compute for $n>1$. Can one give a simpler interpretation for $\varphi(a;T)$?

Let V_ℓ be the $(\ell+1)$-dimensional $\mathbb{C}$-vector space of symmetric Laurent polynomials

$$\sum_{\nu=0}^{\ell} c_\nu (X^\nu + X^{-\nu}) \quad (c_\nu \in \mathbb{C})$$

of degree $\le \ell$. For $j \in \{0,1,\ldots,\ell+1\}$ set

$$\psi_j(X) := \frac{X^j - X^{-j}}{X - X^{-1}}.$$

Then the ψ_j ($j=1,\ldots,\ell+1$) form a basis of V_ℓ. Note that $\tilde{F}_p(T;X) \in V_\ell$ where $e = e_p := ord_p f_T$, by Katsurada's results.

Theorem 22. *(Choie, Kohnen, 2006) Let p be a prime with $p \mid f_T$ and $e = e_p := ord_p f_T$. Then*

$$\tilde{F}_p(T;X) = \sum_{j=1}^{\ell+1} \phi(p^{\ell-j+1};T) \quad p^{-\frac{\ell-j+1}{2}} \quad [\psi_j(X) - (\frac{D_{T,0}}{p})p^{-\frac{1}{2}}\ \psi_{j-1}(X)].$$

c) A Maass relation in higher genus

Problem: Is it possible to characterize the image of the Ikeda lift by linear relations, similarly as in the case $n=1$ (Saito-Kurokawa lift, Maass relations)?

Let $I_{k,n}$ be the linear map $S^+_{k+\frac{1}{2}} \to S_{k+n}(\Gamma_{2n})$ defined in Corollary 20 and put $S^*_{k+n}(\Gamma_{2n}) := im\ I_{k,n}$.

Theorem 23. *(Kojima, Kohnen, 2005) Suppose that $n \equiv 0,1 \pmod 4$ and let $k \equiv n \pmod 2$. Let $F \in S_{k+n}(\Gamma_{2n})$. Then the following assertions are equivalent:*

i) $F \in S^*_{k+n}(\Gamma_{2n})$;

ii) $\exists\ c(m) \in \mathbb{C}$ $(m \ge 1, (-1)^k m \equiv 0,1 \pmod 4)$ *such that*

$$a(T) = \sum_{a \mid f_T} a^{k-1} \ \phi(a;T) c\left(\frac{|D_T|}{a^2}\right) \quad \forall\, T > 0.$$

Remark: For $g \in \mathbb{N}$ there exists a unique genus of even integral symmetric matrices of size g with determinant equal to 2. A matrix in this genus is positive definite if and only if $g \equiv \pm 1 \pmod 8$, and then as representative one can take

$$(3) \quad S_0 := \begin{cases} E_8^{\frac{g-1}{8}} \oplus 2 \text{ if } g \equiv 1 \pmod 8 \\ E_8^{\frac{g-7}{8}} \oplus E_7 \text{ if } g \equiv 7 \pmod 8. \end{cases}$$

Let $g := 2n-1$ and define S_0 by the right-hand-side of (3). For $m \in \mathbb{N}, (-1)^n m \equiv 0, 1 \pmod 4$ put

$$\mathcal{T}_m := \begin{cases} \begin{pmatrix} \frac{1}{2}S_0 & 0 \\ 0 & \frac{m}{4} \end{pmatrix} \text{ if } m \equiv 0 \pmod 4 \\ \begin{pmatrix} \frac{1}{2}S_0 & \frac{1}{2}\, e_{2n-1} \\ \frac{1}{2}e'_{2n-1} & \frac{m+2+(-1)^n}{4} \end{pmatrix} \text{ if } (-1)^n m \equiv 1 \pmod 4). \end{cases}$$

One checks: $\mathcal{T}_m > 0, det\ (2\mathcal{T}_m) = m$.

Theorem 24. *(Kojima, Kohnen, 2005) Under the assumption as above, $F \in S^*_{k+n}(\Gamma_{2n})$ iff*

$$a(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)a(\mathcal{T}_{|D_T|/a^2})$$

$\forall T > 0$.

Idea of proof: Let $T_0 := \frac{1}{2}S_0$ and study the Fourier-Jacobi coefficients $\phi_{T_0}(\tau, z)(\tau \in \mathcal{H}, z \in \mathbb{C}^{2n-1,1})$ of index T_0 of F. Then φ_{T_0} is a Jacobi cusp form of even weight $k+n$ on the semi-direct product of Γ_1 and $\mathbb{Z}^{2n-1,1} \times \mathbb{Z}^{2n-1,1}$. Write

$$\phi_{T_0}(\tau, z) = \sum_{\lambda \in \Lambda} h_\lambda(\tau)\vartheta_\lambda(\tau, z)$$

where $\Lambda = S_0^{-1}\mathbb{Z}^{2n-1,1}/\mathbb{Z}^{2n-1,1}$ and $(h_\lambda)_{\lambda\in\Lambda}$ is a vector-valued cuspform of weight $k + n - \frac{2n-1}{2} = k + \frac{1}{2}$ on the metaplectic cover of Γ_1. Now $|\Lambda| = 2$, and then $h(\tau) := h_{\lambda_0}(4\tau) + h_{\lambda_1}(4\tau) \in S^+_{k+\frac{1}{2}}$. Let $\tilde{c}(m)\,(m \in \mathbb{N}, (-1)^k m \equiv 0, 1 \pmod 4)$ be the Fourier coefficients of h. One shows $\tilde{c}(m) = c(m)$ for all m.

6 Sign changes of eigenvalues

Fourier coefficients of cup forms are mysterious objects and in general no simple arithmetic formulas are known for them. If one checks tables, for example one finds that quite often sign changes of these coefficients occur and it is a natural question to try to understand them. For example, one may ask if there are infinitely many sign changes or when the first sign change occurs, hoping for a bound depending only on the weight and the level.

This might be particularly interesting when the cusp form is a Hecke eigenform and so the Fourier coefficients are proportional to the eigenvalues in degree 1 or in degree 2, where the relations between Fourier coefficients and eigenvalues are quite well understood.

a) Elliptic modular forms

The result in the following Theorem seems to be well-known. As usual, we define

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \,|\, c \equiv 0 \pmod{N}\}.$$

Theorem 25. *Let f be a non-zero cusp form of even integral weight k on $\Gamma_0(N)$ and suppose that its Fourier coefficients $a(n)$ are real for all $n \geq 1$. Then the sequence $(a(n))_{n\in\mathbf{N}}$ has infinitely many sign changes, i. e. there are infinitely many n such that $a(n) > 0$ and there are infinitely many n such that $a(n) < 0$.*

According to the above Theorem, a reasonable question to ask is if it is possible to obtain a bound on the first sign change, say in terms of k and N.

In the following, we look at a normalized Hecke eigenform f that is a newform of level N. Recall that "normalized" means that $a(1) = 1$.

Theorem 26. *(Kohnen, Sengupta, 2006) Suppose that f is a normalized Hecke eigenform of even integral weight k and squarefree level N that is a newform. Then one has $a(n) < 0$ for some n with*

$$n \ll kN\,\exp\Big(c\sqrt{\log N/\log\log 3N}\Big)(\log k)^{27}, \quad (n, N) = 1.$$

Here $c > 2$ and the constant implied in $\ll$ is absolute.

Note that it is reasonable to assume that $(n, N) = 1$, since the eigenvalues $a(p)$ with $p|N$ are explicitly known by Atkin-Lehner theory. The proof of the above result uses techniques from analytic number theory. Recently, the above result was improved as follows.

Theorem 27. *(Iwaniec, Kohnen, Sengupta, 2006) Suppose that f is a normalized Hecke eigenform of even integral weight k and level N (not necessarily squarefree) that is a newform. Then one has $a(n) < 0$ for some n with*

$$n << k\sqrt{N} \cdot \log^{8+\epsilon}(kN), \quad (n, N) = 1 \qquad (\epsilon > 0).$$

The proof is "elementary" in the sense that it avoids the use of the symmetric square L-function. Instead, the Hecke relations for the eigenvalues are exploited. Using similar ideas one can obtain

Theorem 28. *(Iwaniec, Kohnen, Sengupta, 2006) Suppose that f is a normalized Hecke eigenform of level N (not necessarily squarefree) and even integral weight k. Then $a(n) < 0$ for some n with*

$$n << (k^2N)^{\frac{29}{60}}, \quad (n, N) = 1.$$

b) Siegel modular forms of genus two.

Using the analytic properties of the Koecher-Maass Dirichlet series attached to F and of the Rankin-Selberg Dirichlet zeta function attached to F, it should be easy to generalize Theorem 25 to the situation here. Now suppose that F is an eigenfunction of all Hecke operators and $g = 2$. As a first surprise, it is not generally true that the eigenvalues of a Hecke eigenform in $S_k(\Gamma_2)$ change signs infinitely often.

Theorem 29. *(Breulmann, 1999) Suppose that k is even and let F be a Hecke eigenform in $S_k^*(\Gamma_2)$, with eigenvalues $\lambda(n)\,(n \in \mathbf{N})$. Then $\lambda(n) > 0$ for all n.*

On the contrary to the above, one has the following

Theorem 30. *(Kohnen, 2005) Let F be a Hecke eigenform in $S_k(\Gamma_2)$ with Hecke eigenvalues $\lambda_n\,(n \in \mathbf{N})$. Suppose that F lies in the orthogonal complement of $S_k^*(\Gamma_2)$ if k is even. Then the sequence $(\lambda_n)_{n\in\mathbf{N}}$ has infinitely many sign changes.*

The proof is based on Landau's theorem coupled with the analytic properties of the spinor zeta function of F and a theorem of Weissauer according to which the generalized Ramanujan-Petersson conjecture (saying that $|\alpha_{1,p}| = |\alpha_{2,p}| = 1$ for all p) is true for forms as in Theorem 30. Of course, after Theorem 30 the question arises when the first sign change occurs.

Theorem 31. *(Kohnen, Sengupta, 2006) Let F be a Siegel-Hecke eigenform in $S_k(\Gamma_2)$ and suppose either that k is odd or that k is even and F is in the orthogonal complement of $S_k^*(\Gamma_2)$. Denote by $\lambda(n)\,(n \in \mathbf{N})$ the eigenvalues of F. Then there exists $n \in \mathbf{N}$ with*

$$n \ll k^2 \log^{20} k$$

such that $\lambda(n) < 0$. Here the constant implied in $\ll$ is absolute.

The proof follows a similar pattern as that of Theorem 28, with the Hecke L-function $L_f(s)$ replaced by the spinor zeta function.

References

[A] A.N. Andrianov: Euler products corresponding to Siegel modular forms of genus 2. Russ. Math. Surv. 29 (1974), 45-116

[C-K] Y.J. Choie and W. Kohnen: A remark on the Ikeda lift and local singular series polynomials. To appear in Proc. AMS

[E-Z] M. Eichler and D. Zagier: The theory of Jacobi forms. Birkhäuser 1985

[F] E. Freitag: Siegelsche Modulformen. Springer 1983

[I] T. Ikeda: On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$. Ann of Maths. 154 (2001), 641-681

[K-K] H. Kojima and W. Kohnen: A Maass space in higher genus. Compos. Math. 141 (2005), 313-322

[K1] W. Kohnen: Lifting modular forms of half-integral weight to Siegel modular forms of even genus. Math. Ann. 322 (2002), 787-809

[K2] W. Kohnen: Sign changes of Fourier coefficients and Hecke eigenvalues of cusp forms: a survey. To appear in Proc. of a conference on Number Theory, Shandong Univ., Weihai, China 2006

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