PROBLEM SET 3

Problem 1

Show that $(G, K) = (\mathrm{O}(1, n), \mathrm{O}(n))$ and $(G, K) = (\mathrm{O}(n + 1), \mathrm{O}(n))$ are Riemannian symmetric pairs. Describe the corresponding orthogonal involutive Lie algebras and the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Define a metric on $\mathfrak{p}$ and compute its curvature in $p_0 = eK \in G/K$. Show that G/K is a space of constant curvature.

Problem 2

Find various representations of the real and complex Grassmannians of p-planes in $\mathbf{R}^n$ or in $\mathbf{C}^n$ as homogeneous spaces G/H. Discuss which corresponding pairs (G, H) are symmetric pairs, and which are Riemannian symmetric pairs.

Problem 3

Discuss the pair $(G, K) = (\mathrm{GL}(n, \mathbf{R}), \mathrm{O}(n))$ as symmetric pair. Find a dual symmetric pair.

Problem 4

The duality for orthogonal involutive Lie algebras gives rise to a duality for symmetric spaces. Show:

(1) For each symmetric space M, there is a simply connected symmetric space M' which is dual to M, and M' is unique up to isometry.
(2) Let M, N be simply connected dual symmetric spaces and K respectively L denotes the group of isometries fixing a point $p \in M$ respectively $q \in N$. Then there is an orthogonal transformation $A : T_pM \to T_qN$ with $A^*\mathfrak{R}^N_q = -\mathfrak{R}^M_p$ and, for any such transformation, $L = AKA^{-1}$.