PROBLEM SET 1

Problem 1

Prove the following statements:

(1) Let M be a Riemannian manifold, $f : M \to M$ an affine transformation. Assume that there exist $x \in M$ such that $d_x f : T_x M \to T_x M$ is a linear isometry. Then f is an isometry.

(2) Let M be a Riemannian manifold and $d : M \times M \to \mathbf{R}$ the corresponding distance function. If $f : M \to M$ is a distance preserving map, then f is an isometry.

(3) Let M be a Riemannian manifold. Let $f, g : M \to M$ be isometries. If there exists $x \in M$ such that $f(x) = g(x)$ and $d_x f = d_x g$, then $f = g$.

(4) Let M, N be Riemannian manifolds. Let $f : M \to N$ be a local isometry. If M is complete, then f is a Riemannian covering map.

Problem 2

Let M be a Riemannian manifold, ∇ the Levi-Civita connection and $\mathfrak{R}$ its curvature tensor. Show that $\nabla \mathfrak{R} = 0$ if and only if for every smooth curves $c : I \to M$ and every parallel vector fields X, Y, Z along c the vector field $\mathfrak{R}(X, Y)Z$ is a parallel vector field along c.

Problem 3

(1) Let M be a Riemannian manifold. Show that the sectional curvature determines the curvature tensor.

(2) Let M be a Riemannian manifold with constant sectional curvature. Show that M is locally symmetric.

(3) Let M, N be complete simply connected Riemannian manifolds of the same dimension. Let K_M, K_N be the sectional curvature on M respectively on N. Let $f : M \to N$ be a map such that $f^* K_N = K_M$. Show that M and N are isometric.

Problem 4

Let M, N be locally symmetric spaces and $\mathfrak{R}^M, \mathfrak{R}^N$ their curvature tensors. Let $m \in M, n \in N$ and $L : T_m M \to T_n N$ be a linear isometry with $L^* \mathfrak{R}^N = \mathfrak{R}^M$. Let $r \leq \min\{\mathrm{inj}(M, m), \mathrm{inj}(N, n)\}$. Let $B^M(r, m)$, respectively $B^N(r, n)$, be the ball of radius r around m, respectively n. Then

$$f := \exp_n \circ L \circ \exp_m^{-1} : B^M(r, m) \to B^N(r, n)$$

is an isometry with $f(m) = n$ and $d_m f = L$.

Problem 5

Let M be a symmetric space. Let $G = \mathrm{Isom}(M)$ be the isometry group of M. Assume that ω is a G-invariant differential form on M. Show that ω is closed, i.e. $d\omega = 0$.

What happens if you take $G = \mathrm{Isom}(M)^\circ$ to be the connected component of the isometry?

Problem 6

Let M be a connected locally compact metric space and $\mathrm{Isom}(M)$ the group of distance preserving mappings of M onto itself, topologized by the compact open topology. Let $H \subset \mathrm{Isom}(M)$ be a closed subgroup. Then for each $p \in M$ the orbit $H \cdot p$ is closed.