PROBLEM SET 2

Problem 0

Show the following.

Let M^n be a Riemannian manifold of dimension $n \geq 2$. Show that M^n is complete, connected and of constant sectional curvature equal to $\epsilon \in \{-1, 0, 1\}$ if and only if M^n isometric to a quotient

(1) of the sphere $\mathbf{S}^n$ by $\Gamma \subset \mathrm{O}(n+1)$ if $\epsilon = 1$,
(2) of the Euclidean space $\mathbf{R}^n$ by $\Gamma \subset \mathrm{E}(n)$ if $\epsilon = 0$,
(3) of the hyperbolic space $\mathbf{H}^n$ by $\Gamma \subset \mathrm{O}(1, n)$ if $\epsilon = -1$,

where Γ acts freely and properly discontinuously.

Here $\mathrm{E}(n)$ is the group of affine transformations of $\mathbf{R}^n$, i.e. $v \mapsto Av + B$, where $A \in \mathrm{GL}(n, \mathbf{R})$ and $B \in \mathbf{R}^n$.

Problem 1

Let H be a compact real Lie group, let g be a bi-invariant Riemannian metric on H and $\mathfrak{h}$ the Lie algebra of H. Show that

(1) Geodesics through $e \in H$ are given by $\gamma(t) = \exp(tX)$, where $X \in \mathfrak{h}$.
(2) Let X, Y be left invariant vector fiels on H. Then
   - $\nabla_X Y = \frac{1}{2}[X, Y]$
   - $g(R(X, Y)Y, X) = \frac{1}{4}||[X, Y]||^2$
   - $R(X, Y)X = -\frac{1}{2}\mathrm{ad}(X)^2 Y$.

Problem 2

Consider $\mathbf{R}^{n+1}$, $n \geq 2$, with the quadratic form

$$Q(\sum_{i=1}^{n+1} x_i) = -x_1^2 + \sum_{i=2}^{n+1} x_i^2.$$

Define $\Omega = \{[v] \in \mathbf{RP}^n \,|\, Q(v) < 0\}$, then $\partial\Omega = \{[v] \in \mathbf{RP}^n \,|\, Q(v) = 0\}$ Then $\Omega \subset \mathbf{RP}^n$ is a strictly convex domain, i.e. for every $x, y \in \Omega$ the projective line through x and y intersects $\partial\Omega$ in exactly two points $\overline{x}$ and $\overline{y}$.

We define a distance function $d_\Omega : \Omega \times \Omega \to \mathbf{R}$ as follows:

$$d_\Omega(x, y) = |\log |c(\overline{x}, x, y, \overline{y})||,$$

where $c(\overline{x}, x, y, \overline{y})$ is the crossratio of four points on a projective line.

Let $[1 : a], [1 : x], [1 : y], [1 : b]$ be four points in $\mathbf{RP}^1$ written in homogeneous coordinate, then the crossratio $c(a, x, y, b) = \frac{y-a}{x-a}\frac{y-b}{x-b}$.

(1) Determine the group of distance-preserving maps $(\Omega, d_\Omega) \to (\Omega, d_\Omega)$.

(2) Show that at every point $x \in \Omega$ there exists an isometry $S_x : (\Omega, d_\Omega) \to (\Omega, d_\Omega)$, which satisfies $S_x(x) = x$ and has differential $d_x S_x = -\mathrm{id} : T_x\Omega \to T_x\Omega$.
(3) What else can you say about the space (Ω, d_Ω)?

Problem 3

Consider $\mathbf{C}^2$ with a hermitian form h of signature $(1,1)$, i.e. $h(z_1+z_2, w_1+w_2) = -z_1\overline{w_1} + z_2\overline{w_2}$. The complex hyperbolic line $\mathbf{CH}^1$ is defined as

$$\mathbf{CH}^1 = \{v \in \mathbf{C}^2 \mid h(v,v) = -1\}.$$

Show that $\mathbf{CH}^1$ carries a natural Riemannian metric with respect to which it is a symmetric space. Show that $\mathbf{CH}^1$ is isometric to the hyperbolic plane $\mathbf{H}^2$.

Problem 4

Consider $\mathbf{C}^{n+1}$ with a positive definite hermitian form h, i.e. $h(\sum_{i=1}^{n+1} z_i, \sum_{i=1}^{n+1} w_i) = \sum_{i=1}^{n+1} z_i\overline{w_i}$. Consider the complex projective space $\mathbf{CP}^n$. Show that $\mathbf{CP}^n$ carries a natural Riemannian metric with respect to which it is a symmetric space.

Problem 5

Similar as in Problem 3 we can consider $\mathbf{C}^{n+1}$ with a hermitian form h of signature $(1,n)$, i.e. $h(\sum_{i=1}^{n+1} z_i, \sum_{i=1}^{n+1} w_i) = -z_1\overline{w_1} + \sum_{i=2}^{n+1} z_i\overline{w_i}$. The complex hyperbolic n-space $\mathbf{CH}^n$ is defined as

$$\mathbf{CH}^n = \{v \in \mathbf{C}^{n+1} \mid h(v,v) = -1\}.$$

Show that $\mathbf{CH}^n$ carries a natural Riemannian metric with respect to which it is a symmetric space.

Problem 6: Pseudo-Riemannian Spaces of Constant Curvature

Consider $\mathbf{R}^n_s$ to be $\mathbf{R}^n$ with a non-degenerate symmetric bilinear form b^n_s of signature $(s, n-s)$. Define pseudo-Riemannian sphere as

$$\mathbf{S}^n_s = \{v \in \mathbf{R}^{n+1}_s \mid b^{n+1}_s(v) = 1\}$$

and the pseudo-Riemannian hyperbolic space as

$$\mathbf{H}^n_s = \{v \in \mathbf{R}^{n+1}_{s+1} \mid b^{n+1}_{s+1}(v) = -1\}.$$

Show that these spaces carry a natural pseudo-Riemannian metric of signature (n,s) with respect to which they are symmetric spaces.

Note a pseudo-Riemannian symmetric space X is defined as a Riemmanian symmetric space: For every $x \in X$ there exists an isometry $S_x : X \to X$ which fixes x and whose differential at x equals $-\mathrm{id} : T_xX \to T_xX$.

There is an analogue of the statement in Problem 0 classifying all connected complete pseudo-Riemannian manifolds of constant sectional curvature.