which define a function $F: R^m \to R^m : x = (x_1, \ldots, x_m) \to (F_1(x), \ldots, F_m(x))$ where $F_j(x) = f_{i_j}(x)$. Also $DF(x(p))$ is invertible so that by the inverse function theorem we can write locally

$$x_i = k_i(f_{i_1}, \ldots, f_{i_m}),$$

where $k_i : R^m \to R$ are C^∞. However, each $f \in F(p)$ equals $G(x_1, \ldots, x_m)$ locally, where G is C^∞, and using the above expression for x's in terms of f_j's we have the results.

To show (b) just note that for $r = m$ we have $df_1, \ldots, df_m$ generate $T_p{}^*(M)$ if and only if they form a basis. Then we can use the above equations expressing $x_i = k_i(f_1, \ldots, f_m)$ and $f_j = h_j(x_1, \ldots, x_m)$ to see $f_1, \ldots, f_m$ are coordinates for some chart at $p \in M$.

Exercise (3) Let U be open in R^m and let $f: U \to R$ be of class C^∞. Compare $Df(p)$ and $df(p)$ for $p \in U$.

5. Tangent Maps (Differentials)

In the preceding section we considered a C^∞-map g from the manifold M into the manifold R and noted that the differential $df(p)$ is a linear map from the tangent space $T(M, p)$ into the vector space $R \cong T(R, f(p))$; this isomorphism uses example (2) of Section 2.4. We shall generalize this situation by showing that a C^∞-map $f: M \to N$ between two manifolds induces a linear map $df(p) : T(M, p) \to T(N, f(p))$. However, by means of coordinate functions this generalized situation reduces to that of the preceding section.

Definition 2.21 Let M and N be C^∞-manifolds and let $f: M \to N$ be a C^∞-mapping. The **differential of** f at $p \in M$ is the map

$$df(p) : T(M, p) \to T(N, f(p))$$

given as follows. For $L \in T(M, p)$ and for $g \in F(f(p))$, we define the action of $df(p)(L)$ on g by

$$[df(p)(L)](g) = L(g \circ f).$$

REMARKS (1) We shall frequently use the less specific notation df for $df(p)$ when there should be no confusion. Also we shall use the notation

$$Tf = Tf(p) = df(p)$$

and also call $Tf(p)$ the **tangent map** of f at p. This notation is very useful in discussing certain functors on categories involving manifolds.

(2) We note that for $g \in F(f(p))$ the function $g \circ f$ is in $F(p)$ so the operation $L(g \circ f)$ is defined. We must next show $Tf(L)$ is actually in $T(N, f(p))$ by showing it is a derivation. Thus for $g, h \in C^\infty(f(p))$,

$$\begin{aligned} Tf(L)(ag + bh) &= L(a(g \circ f) + b(h \circ f)) \\ &= aL(g \circ f) + bL(h \circ f) \\ &= a[Tf(L)](g) + b[Tf(L)](h) \end{aligned}$$

and the product rule is also easy.

The following result shows that $df(p)$ is the correct generalization for $Df(p)$ of Section 1.2, where $f: U \to W$ is a C^∞-map of an open set U in R^m and W is some Euclidean space.

Proposition 2.22 Let $f: M \to N$ be a C^∞-map of C^∞-manifolds and let $p \in M$. Then the map

$$Tf(p) : T(M, p) \to T(N, f(p))$$

is a linear transformation; that is, $Tf(p) \in \mathrm{Hom}(T(M, p), T(N, f(p)))$. Furthermore if (U, x) is a chart at p and (V, y) is a chart at $f(p)$, then $Tf(p)$ has a matrix which is the Jacobian matrix of f represented in these coordinates.

PROOF Let $X, Y \in T(M, p)$. Then for $a, b \in R$ and $g \in F(f(p))$ we have

$$\begin{aligned} [Tf(aX + bY)](g) &= (aX + bY)(g \circ f) \\ &= aX(g \circ f) + bY(g \circ f) \\ &= [a\, Tf(X) + b\, Tf(Y)](g) \end{aligned}$$

so that $Tf(aX + bY) = a\, Tf(X) + b\, Tf(Y)$. Next let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be the given coordinate functions so that we can represent f in terms of coordinates in the neighborhood V by

$$f_k = y_k \circ f = f_k(x_1, \ldots, x_m) \qquad \text{for} \quad k = 1, \ldots, n.$$

Now let $\partial/\partial x_i = \partial_i(p)$ and $\partial/\partial y_i = \partial_i(f(p))$ determine a basis for $T(M, p)$ and $T(N, f(p))$, respectively. Thus to determine a matrix for Tf we compute its action on the basis $\partial/\partial x_i$ in $T(M, p)$. Let

$$Tf(\partial/\partial x_i) = \sum_j b_{ji}\, \partial/\partial y_j$$

be in $T(N, f(p))$. Then we evaluate the matrix (b_{ji}) using the fact that $y_k \in F(f(p))$ as follows

$$\begin{aligned} \partial f_k/\partial x_i(p) &= \partial_i(p)(y_k \circ f) \\ &= [Tf(\partial_i(p))](y_k) \\ &= \sum_j b_{ji}\, \partial(y_k)/\partial y_j = b_{ki} \end{aligned}$$

using $\partial(y_k)/\partial y_j = \delta_{kj}$. Thus $(b_{ji}) = (\partial f_j/\partial x_i(p))$ is the desired Jacobian matrix.

Proposition 2.23 (Chain rule) Let M, N, and P be C^∞-manifolds and let

$$f : M \to N \quad \text{and} \quad g : N \to P$$

be C^∞-maps. Then for $p \in M$,

$$T(g \circ f)(p) = Tg(f(p)) \cdot Tf(p)$$

which is a composition of homomorphisms of tangent spaces.

PROOF Let $X \in T(M, p)$ and $h \in F((g \circ f)(p))$. Then using definitions and $df(p)X \in T(N, f(p))$ we have

$$\begin{aligned}[T(g \circ f)(p)X](h) &= X((h \circ g) \circ f) \\ &= [df(p)X](h \circ g) \\ &= [dg(f(p))\,(df(p))X](h) \\ &= [Tg(f(p)) \cdot Tf(p)X](h).\end{aligned}$$

REMARKS (3) If U is open in a C^∞-manifold M, then U is a C^∞-submanifold such that the inclusion map $i : U \to M : x \to x$ is C^∞. Also for $u \in U$, $Ti(u) : T(U, u) \to T(M, u)$ is an isomorphism and we identify these tangent spaces by this isomorphism.

Many of the preceding results on submanifolds can be easily expressed in terms of tangent maps and are usually taken as definitions. Thus let M and N be C^∞-manifolds of dimension m and n, respectively, and let

$$f : M \to N$$

be a C^∞-map. Then we have the following results.

The **inverse function theorem** can be stated as follows: If $p \in M$ is such that

$$Tf(p) : T(M, p) \to T(N, f(p))$$

is an isomorphism, then $m = n$ and f is a local diffeomorphism. Thus there exists a neighborhood U of p in M such that

(1) $f(U)$ is open in N;
(2) $f : U \to f(U)$ is injective;
(3) the inverse map $f^{-1} : f(U) \to U$ is C^∞.

We now consider separately the injective and surjective parts of the above homomorphism $Tf(p)$; this was discussed in Section 2.3.

We have f is an **immersion** if and only if $Tf(p)$ is injective for all $p \in M$. In case f is injective, f is an **embedding**. Also $f(M)$ is a **submanifold** of N if f is an embedding and if $f(M)$ has a C^∞-structure such that $f : M \to f(M)$ is a diffeomorphism. Thus from preceding results we have that the following are equivalent:

(1) $Tf(p)$ is injective;
(2) there exists a chart (U, x) at p in M and a chart (V, y) at $f(p)$ in N such that $m \leq n$ and $x_i = y_i \circ f$ for $i = 1, \ldots, m$ and $y_j \circ f = 0$ for $j = m + 1, \ldots, n$;
(3) there exists a neighborhood U of p in M and a neighborhood V of $f(p)$ in N and there exists a C^∞-map $g : V \to U$ such that $f(U) \subset V$ and $g \circ f$ is the identity $| U$.

A C^∞-function $f : M \to N$ satisfying (1) at $p \in M$ is called **regular** at p. If f is regular at every $p \in M$, then it is also called a **regular function**.

We have f is a **submersion** if and only if $Tf(p)$ is surjective for all $p \in M$. Also the following are equivalent:

(1) $Tf(p)$ is surjective;
(2) there exists a chart (U, x) at p in M and a chart (V, y) at $f(p)$ in N such that $m \geq n$ and $x_i = y_i \circ f$ for $i = 1, \ldots, n$;
(3) there exists a neighborhood U of p in M and a neighborhood V of $f(p)$ in N and a C^∞-map $g : V \to U$ such that $f(U) \supset V$ and $f \circ g$ is the identity $| V$.

Using the surjective nature of $Tf(p)$ we reformulate Proposition 2.13 and construct submanifolds using the following version of the **implicit function theorem**: Let $f : M \to N$ be a C^∞-map of C^∞-manifolds and let $m = \dim M \geq \dim N = n$. Let $q \in f(M)$ be a fixed element and let

$$f^{-1}(q) = \{p \in M : f(p) = q\}.$$

If for each $p \in f^{-1}(q)$ we have $Tf(p) : T(M, p) \to T(N, f(p))$ is surjective, then $f^{-1}(q)$ has a manifold structure for which the inclusion map $i : f^{-1}(q) \to M$ is C^∞. Thus $f^{-1}(q)$ is a submanifold of M. Furthermore the underlying topology of the submanifold $f^{-1}(q)$ is the relative topology and the dimension of $f^{-1}(q)$ is $m - n$.

Examples (1) We next consider the special case of $f : M \to N$ where $M = R$ or $N = R$. First let $N = R$; that is, $f \in C^\infty(M)$. Then combining the notation of Sections 2.4 and 2.5 we have $Tf(p) = df(p)$ and for $X \in T(M, p)$ we have $Tf(p)(X) \in T(R, f(p)) \cong R$. Thus for $u : R \to R : t \to t$ the coordinate function on the manifold R, we have for some $a \in R$

$$Tf(p)X = a(d/du)$$

and as before $a = a(d(u)/du) = [Tf(p)X](u) = X(u \circ f) = X(f)$, using $u(t) = t$. Consequently the map

$$T(R, f(p)) \to R : a(d/du) \to a$$

is the isomorphism which yields

$$Tf(p) : T(M, p) \to R;$$

that is, which yields the cotangent space.

(2) Next consider $f: R \to M$ formulated in terms of curves. Thus let $I = (a, b)$ and let $\alpha : I \to M$ be a C^∞-curve which admits an extension $\tilde{\alpha} : (a - \varepsilon, b + \varepsilon) \to M$ (Definition 2.14). The **tangent vector** to α at $t \in I$ is denoted by $\dot{\alpha}(t)$ and defined by

$$\dot{\alpha}(t) = [T\alpha(t)](d/du),$$

where $u : R \to R$ is the coordinate function discussed above.

Now let $X \in T(M, p)$. Then there exists a curve $\alpha : I \to M$, where I is an interval containing $0 \in R$ such that $\alpha(0) = p$ and $\dot{\alpha}(0) = X$, for let (U, x) be a coordinate system at p with $x(p) = 0$ and find a curve $\beta : R \to x(U) \subset R^m$ with $\beta(0) = 0$ and $\dot{\beta}(0) = [Tx(p)](X)$; that is, β a straight line. Then $\alpha = x^{-1} \circ \beta$ is the desired curve

$$\begin{aligned} \dot{\alpha}(0) &= [T(x^{-1} \circ \beta)(0)](d/du) \\ &= Tx^{-1}(x(p)) \cdot \dot{\beta}(0), \qquad \text{using the chain rule and} \quad x(p) = 0 \\ &= Tx^{-1}(x(p)) \cdot [Tx(p)](X) = X \end{aligned}$$

and

$$\alpha(0) = x^{-1}(\beta(0)) = p.$$

Also for $f \in F(p)$ we have

$$\begin{aligned} X(f) &= \dot{\alpha}(0)(f) \\ &= [(T\alpha(0))(d/du)](f) = d/du(0)(f \circ \alpha). \end{aligned}$$

Let (U, x) be a chart on M and let $\alpha : (a, b) \to U \subset M$ be a C^∞-curve as above. Then for $t \in (a, b)$ we can represent

$$\dot{\alpha}(t) = [T\alpha(t)](d/du) = \sum a_k \, \partial_k(\alpha(t)) \in T(M, \alpha(t))$$

and evaluate the coefficients $a_i = a_i(t)$ using the dual basis of differentials as follows.

$$\begin{aligned} a_i &= dx_i(\textstyle\sum a_k \, \partial_k) \\ &= dx_i(\dot{\alpha}) \\ &= dx_i[d\alpha(d/du)], \qquad \text{notation} \\ &= d(x_i \circ \alpha)(d/du), \qquad \text{chain rule} \\ &= d/du(x_i \circ \alpha), \end{aligned}$$

where we use the definition of differential of a function applied to a tangent (note paragraph following Definition 2.19). Thus as in calculus the tangent vector to a curve α is obtained by differentiating its coordinate representation.

(3) Consider the special case when $M = G = GL(n, R)$. We shall construct an explicit vector space isomorphism of $g = gl(n, R)$ onto $T_I(G)$. Thus for any fixed $X \in g$ let

$$\alpha : R \to G : t \to \exp tX.$$

Then $\alpha = \tilde{\alpha}$ and define an element $\bar{X} \in T_I(G)$ by

$$\bar{X}(f) = \dot{\alpha}(0)(f)$$

for any $f \in F(I)$. From the preceding example we actually have $\bar{X} \in T_I(G)$ since $\alpha(0) = I$. Next note that

$$\begin{aligned} \bar{X}(f) &= [(T\alpha(0))(d/du)](f) \\ &= d/du(0)(f \circ \alpha) \\ &= \lim_{t \to 0} [f(\exp tX) - f(I)]/t \\ &= [Df(I)](X). \end{aligned} \qquad (*)$$

Now define the mapping

$$\varphi : g \to T_I(G) : X \to \bar{X},$$

where φ is well defined and for $X, Y \in g$ and $a, b \in R$ we use Eq. (*) to obtain, for any $f \in F(I)$,

$$\begin{aligned} \varphi(aX + bY)(f) &= \overline{aX + bY}(f) \\ &= [Df(I)](aX + bY) \\ &= aDf(I)(X) + bDf(I)(Y) \\ &= a\bar{X}(f) + b\bar{Y}(f) \\ &= [a\varphi(X) + b\varphi(Y)](f) \end{aligned}$$

so that φ is a vector space homomorphism. Next suppose $\bar{X} = 0$ and let $u_1, \ldots, u_m$ $(m = n^2)$ be coordinates in g $(\cong R^m)$ corresponding to a basis $X_1, \ldots, X_m$ of g. Let $X = \sum x_i X_i \in g$ with $\varphi(X) = \bar{X} = 0$ and let $f_i = u_i \circ \log \in F(I)$ as previously discussed. Then $f_i(I) = 0$ and since $\bar{X} = 0$,

$$\begin{aligned} 0 &= \bar{X} f_i \\ &= \lim_{t \to 0} [f_i(\exp tX) - f_i(I)]/t, \qquad \text{Eq. } (*) \\ &= x_i \end{aligned}$$

so that $X = 0$ and φ is an isomorphism. We frequently omit this isomorphism and just use the most convenient identification for a given problem.

(4) Let $f: G \to G$ be a C^∞-automorphism of $G = GL(V)$. Then from

Section 1.6 we see that the "tangent map" $Df(I)$ is an automorphism of the Lie algebra $g = gl(V)$. Thus relations on the Lie group are translated to relations on the Lie algebra by the tangent map.

6. Tangent Bundle

In this section we shall show how to make the collection of tangent spaces of a manifold into a manifold. We also discuss mappings of such manifolds and use them to define vector fields in the next section.

Definition 2.24 Let M be a C^∞-manifold of dimension m and let

$$T(M) = \bigcup \{T(M, p) : p \in M\}$$

which is a disjoint union. We call $T(M)$ the **tangent bundle** of M.

We now make $T(M)$ into a manifold (Fig. 2.10). We shall frequently denote the points of $T(M)$ by the pairs (p, Y) where $p \in M$ and $Y \in T(M, p)$; the p is unnecessary in this notation but convenient. First $T(M)$ is a Hausdorff space as follows. Let

$$\pi : T(M) \to M : (p, Y) \to p$$

be the **projection map**. For $(p, Y) \in T(M)$ let (U, x) be a chart at p in the atlas $\mathscr{A}$ of M. Then $\pi^{-1}(U) = \{(q, X) \in T(M) : q \in U\}$. Now if $(q, X) \in \pi^{-1}(U)$, then in terms of coordinates $x(q) = (x_1(q), \ldots, x_m(q))$ and $X = \sum a_j \, \partial/\partial x_j(q)$, where $a_j = a_j(q)$. The map

$$\phi_U : \pi^{-1}(U) \to R^{2m} : (q, X) \to (x_1(q), \ldots, x_m(q), a_1, \ldots, a_m)$$

is injective and there is a unique topology on $T(M)$ such that for all $(U, x) \in \mathscr{A}$, the maps ϕ_U are homeomorphisms (why?). This topology defined by the sets $\pi^{-1}(U)$ can easily be seen to be Hausdorff using the fact that M and R^m are Hausdorff. Also note that since M has a countable basis of neighborhoods, then so does $T(M)$.

Next we define a C^∞-atlas on $T(M)$ so that the projection map $\pi : T(M) \to M$ is a C^∞-map. Thus for each $(p, Y) \in T(M)$ let $\pi^{-1}(U)$ be a neighborhood of (p, Y) where (U, x) is a chart at p and let $\phi(U) \equiv \phi_U : \pi^{-1}(U) \to R^{2m}$ be the above homeomorphism. We claim that $(\pi^{-1}(U), \phi(U))$ is a chart at (p, Y). Thus we must show any two such coordinate neighborhoods are compatible. Therefore, let (U, x), (V, y) be charts at p where the x and y

are C^∞-related by $x = f(y)$ on $U \cap V$; that is, $f = x \circ y^{-1}$. Now in terms of coordinates let $x_k = f_k(y_1, \ldots, y_m)$,

$$\phi(U) = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{2m}), \qquad \phi(V) = (w_1, \ldots, w_m, w_{m+1}, \ldots, w_{2m})$$

where

$$z_i(q, X) = x_i(q) \qquad \text{for } i = 1, \ldots, m,$$
$$z_{j+m}(q, X) = a_j \qquad \text{for } j = 1, \ldots, m,$$

and similarly $w_i(q, X) = y_i(q)$ for $i = 1, \ldots, m$ and $w_{i+m}(q, X) = b_i$ where $X = \sum b_i \, \partial/\partial y_i(q)$.

Now for $(q, X) \in \pi^{-1}(U) \cap \pi^{-1}(V)$ we have first for $i = 1, \ldots, m$

$$z_i(q, X) = x_i(q) = f_i(y_1(q), \ldots, y_m(q)) = f_i(w_1(q, X), \ldots, w_m(q, X))$$

so that the first m coordinate functions are C^∞-related. Next for $j = 1, \ldots, m$ and for $X = \sum a_i \, \partial/\partial x_i(q)$ we note that

$$z_{j+m}(q, X) = a_j(q) = X(x_j) = dx_j(X)$$

and similarly $dy_j(X) = w_{j+m}(q, X)$ for $j = 1, \ldots, m$. Thus by the transformation law for differentials (note remark following Proposition 2.18),

$$\begin{aligned} z_{j+m}(q, X) = dx_j(X) &= \sum_k \partial x_j/\partial y_k(q) \, dy_k(X) \\ &= \sum_k f_{jk}(y_1(q), \ldots, y_m(q)) \, dy_k(X) \\ &= \sum_k f_{jk}(w_1(q, X), \ldots, w_m(q, X)) w_{m+k}(q, X) \end{aligned}$$

which is a C^∞-relationship where $f_{ik} = \partial f_i/\partial y_k$ recalling $x_i = f_i(y_1, \ldots, y_m)$. Thus all the coordinates are compatible. The Hausdorff space $T(M)$ with the maximal atlas determined by the above charts is a C^∞-manifold and we shall always consider the tangent bundle with this C^∞-structure.

Finally we note that the projection map $\pi : T(M) \to M : (p, Y) \to p$ is C^∞. For let (U, x) be a chart at p and let $\phi(U) = (z_1, \ldots, z_{2m})$ be a coordinate system at (p, Y) as above. Then in terms of coordinates, $x_i \circ \pi(q, X) = z_i(q, X)$ for $i = 1, \ldots, m$ which shows that the coordinate expressions $x_i \circ \pi = z_i$ are C^∞ (see Fig. 2.10).

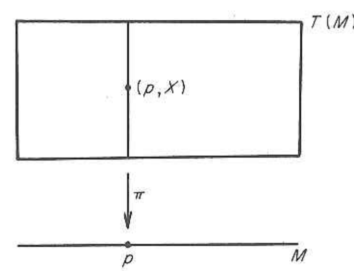


Fig. 2.10.

We now discuss real vector bundles [Lang, 1962; Loos, 1969].

Definition 2.25 A **vector bundle** E over a C^∞-manifold M is given by the following.

(a) E is a C^∞-manifold.
(b) There is a C^∞-surjection $\pi : E \to M$ called the **projection map.**
(c) Each **fiber** $E_p = \pi^{-1}(p)$ has the structure of a vector space over R.
(d) E is locally trivial; that is, there is a fixed integer n so that for each $p \in M$ there exists an open neighborhood U of p such that $U \times R^n$ is diffeomorphic to $\pi^{-1}(U)$ by a diffeomorphism ϕ so that the accompanying diagram is commutative, where pr_1 is the projection onto the first factor; specifically,

$$\begin{array}{ccc} U \times R^n & \xrightarrow{\ \phi\ } & \pi^{-1}(U) \\ & {\scriptstyle pr_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U & \end{array}$$

$\pi \circ \phi(q, X) = q$. Furthermore we require for each $q \in U$ that $\phi(q, \ \ )$ is a vector space isomorphism of R^n onto $E_q = \pi^{-1}(q)$.

Examples (1) $E = T(M)$ the tangent bundle where $n = m$ and $R^m \cong T(M, p) = E_p$.

(2) Let M be a C^∞-manifold and let

$$T^*(M) = \bigcup \{T^*(M, p)\} : p \in M\},$$

which is called the **cotangent bundle.** Then analogous to the construction of $T(M)$ we make $T^*(M)$ into a C^∞-manifold. Thus for $E = T^*(M)$ we see that $T^*(M)$ is a vector bundle and $E_p = \pi^{-1}(p) = T^*(M, p)$.

Definition 2.26 Let M and N be C^∞-manifolds and let E and E' be vector bundles over M and N, respectively. A **bundle homomorphism** is a pair of (surjective) maps (F, f) such that:

(a) $F : E \to E'$ and $f : M \to N$ are C^∞-maps;
(b) the accompanying diagram is commutative; that is, $\pi' \circ F = f \circ \pi$.

$$\begin{array}{ccc} E & \xrightarrow{\ F\ } & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[\ f\ ]{} & N \end{array}$$

Thus for each $p \in M$, $F(\pi^{-1}(p)) \subset (\pi')^{-1}(f(p))$.

(c) For each $p \in M$, the restriction $F : E_p \to E'_{f(p)}$ of the fibers is a linear transformation of the corresponding vector space structures. Also (F, f) is a **bundle isomorphism** of E onto E' if it is a bundle homomorphism such that the maps F and f are surjective diffeomorphisms. It is easy to see that in this case the pair (F^{-1}, f^{-1}) is a bundle isomorphism of E' onto E.

Examples (3) Let $g : M \to N$ be a C^∞-map. Then we define the map

$$T(g) : T(M) \to T(N) : (p, X) \to (g(p), [T(g)(p)](X))$$

where $X \in T(M, p)$ and therefore $[T(g)(p)](X) \in T(N, g(p))$. Then (Tg, g) is a bundle homomorphism of $T(M)$ into $T(N)$ because the diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{\ Tg\ } & T(N) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\ g\ ]{} & N \end{array}$$

is commutative and $T(g)(p) : T(M, p) \to T(N, g(p))$ is a vector space homomorphism.

Next note if we also have another C^∞-map $h : L \to M$ of manifolds, then $g \circ h : L \to N$ is a C^∞-map and

$$T(g \circ h) = T(g) \circ T(h) : T(L) \to T(N),$$

so that $(T(g \circ h), \ g \circ h)$ is a bundle homomorphism. Thus T can be regarded as a covariant functor from the category whose objects are manifolds and morphisms are C^∞-maps into the category whose objects are vector bundles and morphisms are bundle homomorphisms [Loos, 1969].

It will be easy to see later that if G is a Lie group, then the tangent bundle $T(G)$ is a Lie group and is isomorphic as a vector bundle and as a Lie group to the Lie group $g \times G$ (semi-direct product) where g is the Lie algebra of G. Thus the tangent bundles which we want to consider are of a relatively simple nature.

Exercise (1) Let M and N be C^∞-manifolds and let $M \times N$ be the corresponding product manifold. Show the tangent bundle $T(M \times N)$ is bundle isomorphic to $T(M) \times T(N)$.

7. Vector Fields

We have previously discussed the coordinate vector fields $\partial_i = \partial/\partial x_i$ and saw that they were functions defined on a neighborhood U of $p \in M$ which assigns to each $q \in U$ a tangent vector $\partial_i(q) \in T(M, q)$.

Definition 2.27 Let M be a C^∞-manifold and let $T(M)$ be the corresponding tangent bundle. A **vector field on a subset** $A \subset M$ is a map $X : A \to T(M)$ such that $\pi \circ X = \text{idy} | A$. Thus X assigns to each $p \in A$ a tangent vector $X(q)$, where $X(q) \in T(M, q)$, but such that $p = \text{idy}(p) = (\pi \circ X)(p) = \pi(X(q)) = q$. That is, the tangent vector assigned to p by X is actually in $T(M, p)$. Also X is a C^∞**-vector field on** A if A is open and if for each $f \in C^\infty(A)$ the function Xf is in $C^\infty(A)$ where we define Xf by the action of the corresponding tangent vector: $(Xf)(p) = [X(p)](f)$. Thus X is C^∞ on M if and only if $X : M \to T(M)$ is a C^∞-mapping of manifolds.

Example (1) Let $M = R^2$ and let $A = B(0, r)$ the open ball of radius r and center 0. Then with coordinates u_1, u_2 on A a C^∞-vector field X on A can be written

$$X = a_1(u_1, u_2)\, \partial/\partial u_1 + a_2(u_1, u_2)\, \partial/\partial u_2$$

where the a_1 and a_2 are C^∞-functions on A as shown below. Thus a C^∞-vector field is a well-behaved variable tangent vector.

REMARKS We now consider a vector field on M locally in terms of coordinates. Thus let (U, x) be a chart on M with U open in M, then we have the following.

(1) The coordinate vector fields $\partial_i = \partial/\partial x_i$ are C^∞-vector fields on U. This follows from the previous discussion: $\partial_i(p) \in T(M, p)$ so that $(\pi \circ \partial_i)(p) = p$. Next if $f \in C^\infty(U)$, then $g = f \circ x^{-1} : x(U) \to R$ is C^∞ on the open set $x(U) \subset R^m$. Also $\partial_i(f) = \partial g/\partial u_i \circ x$ is C^∞ on U where $u_1, \ldots, u_m$ are coordinates on R^m.

(2) If X is a C^∞-vector field on U, then there exist functions $a_i \in C^\infty(U)$ such that $X = \sum a_i \partial_i$ on U. Furthermore the $a_i = X(x_i)$. Thus the functions $a_i : U \to R$ exist because for each $q \in U$, the tangents $\partial_i(q)$, $i = 1, \ldots, m$, form a basis of $T(M, q)$ and $X(q) = \sum a_i(q)\, \partial_i(q)$ for some $a_i(q) \in R$. The a_i are C^∞ since $\partial_k(x_i) = \delta_{ki}$ and $a_i = \sum a_k\, \partial_k(x_i) = X(x_i)$ which is in $C^\infty(U)$. Also note that if X is a C^∞-vector field on M, then the restriction $X | U$ is a C^∞-vector field on U and has the above expression in coordinates. Thus summarizing

we see that a vector field X is C^∞ on M if and only if for every chart (U, x) the corresponding component functions $a_i = X(x_i)$ are in $C^\infty(U)$.

The following result is frequently taken as a definition [Helgason, 1962].

Proposition 2.28 We have that X is a C^∞-vector field on M if and only if X is a derivation of the algebra $C^\infty(M)$ into $C^\infty(M)$.

PROOF For each $f, g \in C^\infty(M)$ and $a, b \in R$ the properties

$$X(af + bg) = aX(f) + bX(g) \qquad \text{and} \qquad X(fg) = (Xf)g + f(Xg)$$

follow from the corresponding properties for tangents (Definition 2.16). Also by definition X is C^∞ if and only if $Xf \in C^\infty(M)$.

We have seen that a C^∞-vector field X on M restricts to a tangent $X(p)$ and we now consider the converse of extending a tangent to a vector field.

Proposition 2.29 Let M be a C^∞-manifold and let $X \in T(M, p)$. Then there exists a vector field $\tilde{X}$ which is C^∞ on M such that $\tilde{X}(p) = X$.

PROOF We can choose a chart (U, x) at p such that $X = \sum b_i\, \partial_i(p)$. Thus defining the constant functions $a_i : U \to R : q \to b_i$ we see that $Y = \sum a_i\, \partial_i$ is a C^∞-vector field on U such that $X = Y(p)$. Now let $\phi : M \to R$ be a C^∞ "bump function" at p; that is, from exercise (6), Section 1.4 we have $p \in D \subset U$, where D is an open neighborhood of p and the C^∞-function ϕ satisfies $0 \le \phi(x) \le 1$ for all $x \in M$ and $\phi(q) = 1$ if $q \in D$ and $\phi(x) = 0$ if $x \in M - U$. Then we define

$$\tilde{X} = \begin{cases} \phi Y & \text{on } U, \\ 0 & \text{on } M - U. \end{cases}$$

Thus $\tilde{X}(p) = \phi(p) Y(p) = X$ and by construction $\tilde{X}$ is a C^∞-vector field on M.

Example (2) For the manifold $G = GL(V)$ we identified in Section 2.5, $T_I(G)$ with $g = gl(V)$ and for $X \in g$ we define a C^∞-vector field $\tilde{X}$ on G by its action on $f \in C^\infty(G)$ at $p \in G$

$$(\tilde{X}f)(p) = X(f \circ L(p)) = [TL(p)(I) \cdot X](f)$$

where $L(p) : G \to G : q \to pq$. Then $\tilde{X}$ is C^∞ since the right side of the equality consists of C^∞-operations and note $(\tilde{X}f)(I) = X(f)$ so that $\tilde{X}(I) = X$. Also

$\tilde{X}(p) = [TL(p)(I)]X \in T(G, p)$ so that $[\pi \circ \tilde{X}](p) = p$ which shows $\tilde{X}$ is a vector field. Using (*) of example (3), Section 2.5 we have

$$\begin{aligned}(\tilde{X}f)(p) &= X(f \circ L(p)) \\ &= \lim_{t \to 0} \frac{[f \circ L(p)](\exp tX) - [f \circ L(p)](I)}{t} \\ &= \lim_{t \to 0} \frac{f(p \exp tX) - f(p)}{t} \\ &= \frac{d}{dt} f(p \exp tX)\big|_{t=0}.\end{aligned}$$

We shall let $D(M)$ denote the set of all C^∞-vector fields on M. Then we have the following algebraic results concerning these derivations

Proposition 2.30 (a) $D(M)$ is a Lie algebra over R relative to the **bracket multiplication** $[X, Y] = XY - YX$.

(b) $D(M)$ is a left F-module over the ring $F = C^\infty(M)$.

Proof (a) Clearly if $X, Y \in D(M)$ and $a, b \in R$, then $aX + bY \in D(M)$ by just checking the properties of a derivation. Next we shall show $[X, Y] = XY - YX$ is a derivation

$$\begin{aligned}[X, Y](fg) &= X[(Yf)g + f(Yg)] - Y[(Xf)g + f(Xg)] \\ &= (XYf)g + (Yf)(Xg) + (Xf)(Yg) + f(XYg) \\ &\quad - (YXf)g - (Xf)(Yg) - (Yf)(Xg) - f(YXg) \\ &= ([X, Y]f)g + f([X, Y]g).\end{aligned}$$

The multiplication $[X, Y]$ is bilinear and satisfies $[X, Y] = -[Y, X]$. Also the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

is a straightforward computation which is always satisfied for the bracket of endomorphisms.

(b) It is easy to see that the various defining properties of a left module are satisfied; for example, $(f + g)X = fX + gX$ or $(fg)X = f(gX)$ for $f, g \in F$ and $X \in D(M)$. However, note that $D(M)$ is not a "Lie algebra" over F since for "scalars" $f, g \in F$ we do not obtain the correct action relative to the product

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \neq fg[X, Y].$$

Remark If for a chart (U, x) on M we let $X = \sum a_i \partial_i$ and $Y = \sum b_i \partial_i$ be in $D(U)$, then for any $f \in C^\infty(U)$, $(XY)(f) = X(Yf) \in C^\infty(U)$ and

$$\begin{aligned}(XY)(f) &= \sum_k X(b_k \partial_k(f)) \\ &= \sum_{k,i} \left(a_i \frac{\partial b_k}{\partial x_i} \frac{\partial}{\partial x_k} + b_k a_i \frac{\partial^2}{\partial x_i \partial x_k}\right)(f)\end{aligned}$$

which shows XY is not a tangent vector. However because the order of differentiation can be interchanged, the second-order derivatives vanish in $[X, Y]$.

Example (3) Let u_1, u_2, u_3 be coordinate functions on $M = R^3$ and let

$$X = u_2 \partial_3 - u_3 \partial_2, \qquad Y = u_3 \partial_1 - u_1 \partial_3, \qquad Z = u_1 \partial_2 - u_2 \partial_1.$$

Then X, Y, Z are linearly independent (over R) C^∞-vector fields and the vector space L spanned by X, Y, Z is a Lie algebra because the products

$$[X, Y] = -Z, \qquad [Y, Z] = -X, \qquad [Z, X] = -Y$$

are all in L.

Next we consider the action of a C^∞-map $f: M \to N$ on vector fields. First we note that f induces a map $Tf(p): T(M, p) \to T(N, f(p))$ which maps tangent vectors into tangent vectors. However, in general, it is not possible to map vector fields on M into vector fields on N by Tf. Thus for any $X \in D(M)$ define the map

$$Tf(X): M \to T(N): p \to [Tf(p)]X(p)$$

noting that $[Tf(p)]X(p) \in T(N, f(p))$.

One would like to use $(Tf)X$ to define a vector field over N or even over $f(M)$ by taking a point $r = f(p) \in N$ and defining $[(Tf)X](r)$ to be $[Tf(p)]X(p)$. However, this is not always possible as shown by the following. Suppose $p \neq q$ but $f(p) = f(q)$. Let $X \in D(M)$ be a C^∞-vector field such that $Tf(p)X(p) \neq Tf(q)X(q)$ both of which are in $T(N, f(p))$. Then we can not assign a unique value to $(Tf)X$ at $r \in N$ by the desired process.

Exercise (1) If $f: M \to N$ is a diffeomorphism, then show that a vector field can be defined on N by the formula $[(Tf)X] \circ f^{-1}: N \to T(N)$.

Definition 2.31 Let $f: M \to N$ be a C^∞-map and let $X \in D(M)$, $Y \in D(N)$ be vector fields. Then X and Y are **f-related** if $(Tf)X = Y \circ f$; that is, for all $p \in M$, $Tf(p) \cdot X(p) = Y(f(p))$.

Thus if X and Y are f-related, then for every $g \in C^\infty(N)$

$$(Yg) \circ f = X(g \circ f),$$

for if $p \in M$, then

$$\begin{aligned}[(Yg) \circ f](p) &= (Yg)(f(p)) = Y(f(p))g \\ &= [Tf(p) \cdot X(p)]g = X(p)(g \circ f) = [X(g \circ f)](p).\end{aligned}$$

Definition 2.32 Let $f: M \to M$ be a C^∞-map and let $X \in D(M)$ be a vector field on M. Then X is f-**invariant** if X is f-related to X. Thus X "commutes" with the action of f by means of the formula $TfX = X \circ f$; that is $Tf(p)X(p) = X(f(p))$.

Another way of viewing the f-invariance of X is by noting that $Tf(p)X(p)$ and $X(f(p))$ are both in $T(M, f(p))$ so that the f-invariance of X means they are equal.

Example (4) For $G = GL(V)$ and $X \in gl(V)$ we defined the vector field $\tilde{X}$ on G by $(\tilde{X}g)(p) = X(g \circ L(p))$ where $g \in C^\infty(G)$. Now for any $a \in G$, $\tilde{X}$ is $L(a)$-invariant. Thus let $f = L(a)$, then for any $g \in C^\infty(G)$

$$\begin{aligned}[Tf(p)\tilde{X}(p)](g) &= \tilde{X}(p)(g \circ f) \\ &= [\tilde{X}(g \circ f)](p) \\ &= X((g \circ f) \circ L(p)), \qquad \text{definition of } \tilde{X} \\ &= X(g \circ L(a) \circ L(p)), \qquad \text{using } f = L(a) \\ &= X(g \circ L(ap)) = (\tilde{X}g)(ap) = [\tilde{X}(ap)](g).\end{aligned}$$

Thus $Tf(p)\tilde{X}(p) = \tilde{X}(f(p))$; that is, $TL(a)\tilde{X} = \tilde{X} \circ L(a)$. The vector field $\tilde{X}$ is called **left invariant** or G-**invariant** and will be used in yet another definition of the Lie algebra of G.

Proposition 2.33 Let X_1 and Y_1, X_2 and Y_2 be f-related. Then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

PROOF Since the X's and Y's are f-related we have using the paragraph following Definition 2.31 for any $g \in C^\infty(N)$ that

$$\begin{aligned}Y_2(Y_1 g)f(p) &= X_2(Y_1 g \circ f)(p) \\ &= X_2(X_1(g \circ f))(p) = [X_2 X_1(g \circ f)](p).\end{aligned}$$

Thus since a similar formula holds for $Y_1 Y_2$ we have

$$([Y_1, Y_2]g)f(p) = ([X_1, X_2](g \circ f))(p)$$

so that $[Y_1, Y_2]$ is f-related to $[X_1, X_2]$.

Proposition 2.34 Let $f: M \to N$ be a C^∞-map.

(a) If f is an immersion, then for every $Y \in D(N)$ there is at most one $X \in D(M)$ such that X and Y are f-related. In this case the $X \in D(M)$ exists if and only if for every $p \in M$ we have $Y(f(p)) \in Tf(p)T(M, p)$.

(b) If f is a surjection, then for every $X \in D(M)$ there is at most one $Y \in D(N)$ such that X and Y are f-related.

PROOF (a) Let $Y \in D(N)$ and let $X, Z \in D(M)$ with $Tf(p)X(p) = Y(f(p)) = Tf(p)Z(p)$. Then since $Tf(p)$ is injective $X(p) = Z(p)$; that is, $X = Z$. Now if X exists, then by definition $Y(f(p)) = Tf(p)X(p) \in Tf(p)T(M, p)$ and conversely one can define X by $X(p) = Tf(p)^{-1}Y(f(p))$ and this defines a vector field on M.

All that remains to show is that X is C^∞. Now since $Tf(p)$ is injective we have from Section 2.5 that for $p \in M$ there is a chart (V, y) at $f(p)$ in N so that (U, x) is a chart at p where $x_i = y_i \circ f$ for $i = 1, \ldots, m$. Now with these coordinates we let $X = \sum a_i \partial_i$ on U, then for $q \in U$ we have

$$\begin{aligned}a_i(q) = X(x_i)(q) &= [X(y_i \circ f)](q) \\ &= [(TfX)(y_i)](q) = [(TfX)(q)](y_i) \\ &= [Y(f(q))](y_i) = [(Yy_i) \circ f](q)\end{aligned}$$

which shows a_i is C^∞ because Yy_i and f are C^∞.

The proof of (b) is a straightforward exercise.

Exercises (2) Let $f: M \to M$ be a C^∞-map. Show that the set of f-invariant vector fields in $D(M)$ is a Lie subalgebra of $D(M)$.

(3) Let $G = GL(V)$ and let μ be an analytic multiplication on G; that is,

$$\mu : G \times G \to G : (x, y) \to \mu(x, y)$$

is an analytic mapping of manifolds. Now form the differential

$$T\mu : T(G, x) \times T(G, y) \to T(G, \mu(x, y));$$

that is, for $X \in T(G, x)$ and $Y \in T(G, y)$ we have

$$[(T\mu)(x, y)](X, Y) \in T(G, \mu(x, y)).$$

(i) For $X \in T_I(G) = gl(V)$ show the map

$$l(\mu, X) : G \to T(G) : x \to [(T\mu)(x, I)](0, X)$$

is an analytic vector field on G if and only if $\mu(x, I) = x$ for all $x \in G$;

(ii) Similarly discuss the function

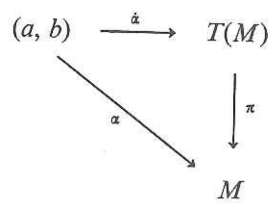
$$r(\mu, X) : G \to T(G) : x \to [(T\mu)(I, x)](X, 0);$$

(iii) In case μ is the usual associative multiplication on G, compare the vector fields $l(\mu, X)$, $r(\mu, X)$, and $\tilde{X}$ of example (2).

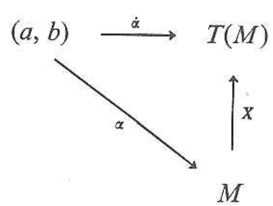
(4) What can be said about a function $f \in C^\infty(R^m)$ such that $[fX, Y] = f[X, Y]$ for all C^∞-vector fields X, $Y \in D(R^m)$ (note Proposition 2.30)?

8. Integral Curves

Let α be a C^∞-curve defined on (a, b) into M as discussed in Section 2.5. Then the tangent vector $\dot{\alpha}(t)$ is given by $\dot{\alpha}(t) = [T\alpha(d/du)](t) \in T(M, \alpha(t))$. Thus $\dot{\alpha} : (a, b) \to T(M)$ is a C^∞-map such that the accompanying diagram is commutative.



Definition 2.35 Let M be an m-dimensional C^∞-manifold and let X be a C^∞-vector field on M. An **integral curve of** X is a C^∞-curve $\alpha : (a, b) \to M$ such that the tangent vector to α at each $t \in (a, b)$ equals the value of X at $\alpha(t)$; that is, $\dot{\alpha}(t) = X(\alpha(t))$ all $t \in (a, b)$. Thus the accompanying diagram is commutative.



In terms of a chart (U, x) of M we have from Section 2.5,

$$\dot{\alpha} = \sum d(x_i \circ \alpha)/dt \; \partial_i$$

and writing X in coordinates on U

$$X = \sum a_i \, \partial_i$$

we obtain α as an integral curve of X if and only if

$$d(x_i \circ \alpha)/dt = a_i(x_1 \circ \alpha, \ldots, x_m \circ \alpha) \qquad \text{for} \quad i = 1, \ldots, m.$$

We now summarize the facts we shall need concerning the solutions of such differential equations and the proofs can be found in the work of Dieudonné [1960] and Lang [1968].

Proposition 2.36 Let U be an open subset of R^m, let $p \in U$, and let $a_i \in C^\infty(U)$ for $i = 1, \ldots, m$. Then

(a) there exists an open neighborhood D of p with $D \subset U$;
(b) there exists an open interval $(-\varepsilon, \varepsilon) \subset R$;
(c) there exists a C^∞-map $f : (-\varepsilon, \varepsilon) \times D \to U : (t, w) \to f(t, w)$ such that for each $w \in D$ the function $\alpha_w : (-\varepsilon, \varepsilon) \to U : t \to f(t, w)$ with $\alpha_i = u_i \circ \alpha_w$ for $i = 1, \ldots, m$ satisfies

(i) $d\alpha_i/du(t) = a_i(\alpha_1(t), \ldots, \alpha_m(t))$ all $t \in (-\varepsilon, \varepsilon)$, and
(ii) $\alpha_i(0) = w_i$ where $w_i = u_i(w)$.

Moreover if $\bar{\alpha}_w : (-\bar{\varepsilon}, \bar{\varepsilon}) \to U$ with $(-\bar{\varepsilon}, \bar{\varepsilon}) \subset (-\varepsilon, \varepsilon)$ satisfies (i) and (ii), then $\bar{\alpha}_w = \alpha_w | (-\bar{\varepsilon}, \bar{\varepsilon})$.

Thus the unique solutions to the above differential equations depend in a C^∞-manner on the initial conditions. We now translate these facts to manifolds [Bishop and Goldberg, 1968; Singer and Thorpe, 1967].

Theorem 2.37 (a) Let M be a C^∞-manifold and $X \in D(M)$ a C^∞-vector field on M and let $p \in M$. Then there exists an open neighborhood D of p in M and an open interval $(-\varepsilon, \varepsilon) \subset R$ and a C^∞-map $f : (-\varepsilon, \varepsilon) \times D \to M$ such that for each $w \in D$ the curve

$$\alpha_w : (-\varepsilon, \varepsilon) \to M : t \to f(t, w)$$

is the unique local integral curve of X defined on $(-\varepsilon, \varepsilon)$ with $\alpha_w(0) = w$. In particular α_p is a local integral curve through $p \in M$.

(b) For each $t \in (-\varepsilon, \varepsilon)$ the C^∞-map $\phi(t)$ given by $\phi(t) : D \to M : w \to f(t, w)$ satisfies:

(i) if s, t and $s + t$ are in $(-\varepsilon, \varepsilon)$, then $\phi(s + t) = \phi(s) \circ \phi(t)$ on $\phi(t)^{-1}(D) \cap D$;
(ii) if $t \in (-\varepsilon, \varepsilon)$, then $\phi(t)^{-1}$ exists on $D \cap \phi(t)(D)$ and $\phi(t)^{-1} = \phi(-t)$.

A map $\phi : (-\varepsilon, \varepsilon) \times D \to M$ such that $\phi(t)$ satisfies (i) and (ii) above is called a **local one-parameter group on** M.

PROOF (a) Let (U, x) be a chart at p in M and let $U' = x(U) \subset R^m$. Then on U' we have $X = \sum a_i \partial_i$ where $a_i \in C^\infty(U')$. By Proposition 2.36, there exist $D' \subset U'$ and $(-\varepsilon, \varepsilon) \subset R$ and $f' : (-\varepsilon, \varepsilon) \times D' \to U'$ with the desired properties which can be translated back to M by x^{-1}.

(b) We use the uniqueness part of Proposition 2.36 as follows. For fixed $t \in (-\varepsilon, \varepsilon)$ the curves $u(s) = f(s + t, w)$ and $v(s) = f(s, \phi(t)(w))$ are integral curves of X defined on a subinterval of $(-\varepsilon, \varepsilon)$ which contains 0 and by the initial conditions we have $u(0) = v(0) = f(t, w)$. Thus by the uniqueness $u = v$; that is, $\phi(s + t) = \phi(s) \circ \phi(t)$. Also $\phi(t)^{-1} = \phi(-t)$.

The preceding results on differential equations are also true when C^∞ is replaced by "analytic." Furthermore, if the vector field depends analytically upon a parameter, then the integral curve does also as follows.

Definition 2.38 Let M be an analytic manifold and let V be a Euclidean vector space over R. Let A denote an element in V and let $X(A)$ be an analytic vector field which is a function of $A \in V$. Then $X(A)$ **depends analytically on the parameter** $A \in V$ if for any $p \in M$ and any function f analytic at p, the mapping $\text{Dom}(f) \times V \to R : (q, A) \to [X(A)(f)](q)$ is analytic.

Using the results of Dieudonné [1960] and Lang [1968] on this dependence we have the following.

Theorem 2.39 Let M be an analytic manifold, V a Euclidean vector space over R, and $X(A)$ an analytic vector field which depends analytically upon the parameter $A \in V$. Then for any $p \in M$, there exist an open interval $(-\varepsilon, \varepsilon) \subset R$ and an open convex neighborhood U of 0 in V and an analytic map $u : (-\varepsilon, \varepsilon) \times U \to M : (t, A) \to u(t, A)$ which is the unique local integral curve of $X(A)$ through $p \in M$.

PROOF Since this is a local result, we can assume M is an open set in R^m so that the vector field $X(A)$ can be represented by analytic functions $a_i(x, A)$ on $M \times V$ for $i = 1, \ldots, m$; that is, $X(A) = \sum a_i \partial$ where $a_i : M \times V \to R$ are analytic. Thus we now have as before a system of (parameterized) differential equations for the integral curve, and the results follow from Dieudonné [1960, Theorem 10.7.5], for example.

Exercise (1) Show that the vector fields $\tilde{X}$, $l(\mu, X)$, and $r(\mu, X)$ in exercise (3), Section 2.7 depend analytically on the parameter $X \in gl(V)$.

Example (1) Let $p = (0, 0) \in R^2 = M$ with coordinates (u_1, u_2) and let $X = \partial_1 + \exp(-u_2)\,\partial_2$ be a vector field on all of M. Then the equation for the integral curve α is

$$d\alpha_1/dt = 1 \qquad \text{and} \qquad d\alpha_2/dt = \exp(-\alpha_2).$$

Let $D = \{(x, y) \in R^2 : -1 < y < 1\}$ be an open neighborhood of p and let $\varepsilon = e^{-1}$. Then for $t \in (-\varepsilon, \varepsilon)$ and for $w = (w_1, w_2) \in D$, the C^∞-map

$$f : (-\varepsilon, \varepsilon) \times D \to M : (t, w) \to (t + w_1, \log(t + \exp w_2))$$

is such that

$$\alpha_w(t) = (\alpha_1(t), \alpha_2(t))$$

with

$$\alpha_1(t) = t + w_1 \qquad \text{and} \qquad \alpha_2(t) = \log(t + \exp w_2)$$

is a solution to the above equation with $\alpha_w(0) = w$.

REMARK Theorem 2.37 gives only local existence and uniqueness of integral curves and it is not always possible to find global curves; that is, it is not always possible to extend the domain $(-\varepsilon, \varepsilon)$ to all of R. Thus, for example, let $M = R^2 - \{(0, 0)\}$ with coordinates (u_1, u_2) and let $X = \partial_1$. Then the integral curve $\alpha(t)$ of X through $(1, 0)$ is $\alpha(t) = (t + 1, 0)$ which cannot be extended to a curve in M defined on all of R because $(0, 0)$ is not in M.

Let $M = R^2$ and let $X = -u_2\,\partial_1 + u_1\,\partial_2$ be a vector field on M. Then the general form for the integral curve $\alpha_w(t)$ is

$$\alpha_w(t) = (w_1 \cos t - w_2 \sin t,\ w_2 \cos t + w_1 \sin t)$$

and $\alpha_w(0) = w$. Note that $\alpha_w(t)$ is defined for all $t \in R$.

Definition 2.40 A vector field is **complete** if all its integral curves have domains all of R.

Exercise (2) Show $X = -u_2\,\partial_1 + u_1\,\partial_2$ is complete on R^2. Is $X = \exp(-u_1)\,\partial_1 + \partial_2$ complete on R^2?

Examples (2) Let $G = GL(V)$ and let $g = gl(V)$ be identified with $T_I(G)$. For $X \in g$ we have defined the G-invariant vector field $\tilde{X}$ by $(\tilde{X}f)(a) = X(f \circ L(a))$ for all $a \in G$ and $f \in C^\infty(G)$. Let E_{ij} be the usual matrix basis of $\text{End}(V)$ which gives coordinate functions u_{ij} on G; that is, $u_{ij}(a) = (a_{ij})$. We write $\tilde{X} = \sum X_{ij}\,\partial/\partial u_{ij}$ so that $X_{ij} = \tilde{X}(u_{ij})$ are in $C^\infty(G)$ and we now compute the coordinate functions X_{ij}.

For $a, x \in G$ we have

$$(u_{ij} \circ L(a))(x) = u_{ij}(ax) = \sum_{k=1}^{m} u_{ik}(a)u_{kj}(x),$$

using matrix multiplication. Thus applying X to this formula for $u_{ij} \circ L(a)$ we have

$$X(u_{ij} \circ L(a)) = X(\sum_k u_{ik}(a)u_{kj}) = \sum_k u_{ik}(a)X(u_{kj}) = \sum_k u_{ik}(a)x_{kj},$$

where we write $X = \sum x_{ij}\, \partial/\partial u_{ij}(I) \in g$ and have $x_{pq} = X(u_{pq}) = X_{pq}(I)$. Thus letting $f = u_{ij}$ in the definition of $\tilde{X}$ we obtain

$$X_{ij}(a) = (\tilde{X}u_{ij})(a) = X(u_{ij} \circ L(a)) = \sum_k u_{ik}(a)x_{kj}, \qquad (*)$$

so that the equation for an integral curve α of $\tilde{X}$ is

$$d(u_{ij} \circ \alpha)/ds = \sum_k (u_{ik} \circ \alpha)x_{kj} \qquad \text{for} \quad i = 1, \ldots, m.$$

From example (2) in Section 2.7 on $\tilde{X}$ we have

$$(\tilde{X}f)(p) = df(p \exp tX)/dt\,|_{t=0}$$

so that for $p = q \exp sX$ we have

$$(\tilde{X}f)(q \exp sX) = df[q \cdot \exp(s + t)X]/dt\,|_{t=0} = df(q \cdot \exp uX)/du\,|_{u=s} = df(q \cdot \exp sX)/ds,$$

where associativity is used in the first equality. Thus for $q = I$ and $f = u_{ij}$ we have

$$du_{ij}(\exp sX)/ds = (\tilde{X}u_{ij})(\exp sX) = \sum_k u_{ik}(\exp sX)x_{kj}$$

using $(*)$ above for the last equality. This shows that $\alpha(s) = \exp sX$ is the solution of the equation for the integral curve of $\tilde{X}$

$$d\alpha_{ij}/ds = \sum_k \alpha_{ik}x_{kj} \qquad \text{and} \qquad \alpha(0) = I,$$

where $\alpha_{ij} = u_{ij} \circ \alpha$. In terms of the given matrix $X = (X_{ij})$ this equation can be written: $d\alpha/ds = \alpha X$ which yields $\alpha(s) = \exp sX$ which is a one-parameter group defined on all of R. If the initial condition is changed to $\alpha(0) = A$, then for $\tilde{X}$ we obtain the integral curve $\alpha_A(s) = A \cdot \exp sX$ and $\alpha_A(0) = A$. From this we see $\tilde{X}$ is a complete vector field on G.

(3) We now consider a Taylor's series expansion for a real-valued analytic function f on the analytic manifold $G = GL(V)$. Thus let $X \in g$ and $\tilde{X}$ be as in the preceding example and let f be analytic at $p \in G$. Then from this example we have

$$df(p \exp sX)/ds = (\tilde{X}f)(p \exp sX) = [\tilde{X}(p \exp sX)](f)$$

and by induction

$$d^n f(p \exp sX)/ds^n = [\tilde{X}(p \exp sX)](\tilde{X}^{n-1}(f)) = (\tilde{X}^n f)(p \exp sX).$$

Thus if we write $g(s) = f(p \exp sX)$, we have, since g is the composition of analytic functions, the power series in a suitable interval containing $0 \in R$

$$g(s) = \sum \frac{a_n}{n!} s^n,$$

where the $a_n \in R$ are computed by differentiation as usual

$$a_n = d^n f(p \exp sX)/ds^n\,|_{s=0} = (\tilde{X}^n f)(p).$$

Thus if we define the operator formula

$$[(\exp s\tilde{X})(f)](p) = \sum_{n=0}^{\infty} \frac{s^n \tilde{X}^n f}{n!}(p),$$

we obtain the following version of Taylor's formula for $GL(V)$

$$f(p \exp sX) = [(\exp s\tilde{X})(f)](p).$$

Exercise (3) (i) Consider the C^∞-vector field on R^3 defined by

$$X(p) = p_2\,(\partial/\partial x_1)(p) + p_3\,(\partial/\partial x_2)(p) + p_1\,(\partial/\partial x_3)(p)$$

where $p = (p_1, p_2, p_3)$. Find the integral curve $\alpha(t)$ of X so that $\alpha(0) = (-1, 1, 1)$.

(ii) Let the C^∞-vector field on R^3 be given by $Y(p) = p_1 p_2\,(\partial/\partial x_3)(p)$. Compute $[X, Y](p)$.