CHAPTER 2

MANIFOLDS

A manifold is a topological space where some neighborhood of a point looks like an open set in a Euclidean space. Thus we are able to translate the calculus of the preceding chapter to this type of a space and we develop the formalism in this chapter. In the first few sections we consider differentiable structures, the definition of a manifold, and real-valued differentiable functions defined on a manifold. Next we consider submanifolds and how they arise from the inverse function theorem; we give many examples of submanifolds which are subgroups of $GL(n, R)$. The derivative is generalized to a tangent at a point p in a manifold M and then the vector space spanned by these tangents generalizes the tangent plane of a surface. As the point p varies over M we obtain a variable tangent vector which is formalized via vector fields. We give many examples concerning $GL(n, R)$ which will be abstracted in later chapters; in particular we consider the invariant vector fields on $GL(n, R)$ and their integral curves.

Most of this chapter is used in the rest of the book and the reader who knows this material need only look at the examples. However, if one is unfamiliar with manifolds it might be best to read through Section 2.3, then read Chapters 3 and 4 for applications before finishing this chapter. The reader should note that we are assuming a neighborhood of a point is an open set in the space.

1. Differentiable Structures

We now extend the basic concepts of Euclidean space to a topological space which locally looks like Euclidean space via suitable choices of "coordinates."

Definition 2.1 (a) Let $V = R^m$ and let $X_1, \ldots, X_m$ be a basis of V so that we can represent any point $p = \sum p_k X_k \in V$ uniquely. Relative to this basis, we define **coordinate functions** u_i for $i = 1, \ldots, m$ on R^m by

$$u_i : R^m \to R : \sum p_k X_k \to p_i .$$

We shall frequently use the usual orthonormal basis $e_1, \ldots, e_m$ to obtain the **usual coordinates** u_i given by $u_i(a_1, \ldots, a_m) = a_i$.

(b) Let M be a topological space and let $p \in M$. An m-**dimensional chart at** $p \in M$ is a pair (U, x), where U is an open neighborhood of p and x is a homeomorphism of U onto an open set in R^m. The **coordinates** of the chart (U, x) are the functions x_i for $i = 1, \ldots, m$ given by

$$x_i = u_i \circ x : U \to R : q \to x_i(q),$$

where $x_i(q) = u_i(x(q))$ and the u_i are coordinates in R^m. We frequently write $x = (x_1, \ldots, x_m)$. The set U is called a **coordinate neighborhood** and (U, x) is called a **coordinate system** at $p \in M$.

Definition 2.2 An m-**dimensional topological manifold** M is a Hausdorff space with a countable basis such that for every $p \in M$ there exists an m-dimensional chart at p. In this case we say that the **dimension of** M is m.

Thus in particular we can find a covering of M by open sets and each open set U in the covering is homeomorphic to the open m-ball $B_m = \{a \in R^m : \|a\| < 1\}$.

Examples (1) Any open subset N of R^m is a manifold of dimension m, since N itself is a coordinate neighborhood of each of its points and the identity map x is such that (N, x) is an m-dimensional chart. Thus for $V = R^n$ we have $GL(V) \subset R^{n^2}$ is a manifold of dimension n^2. Note that for a fixed basis in V, any linear transformation $A \in GL(V)$ has a unique matrix representation (a_{ij}) and coordinate functions x_{ij} can be defined by $x_{ij}(A) = a_{ij}$.

More generally, if N is an open subset of a manifold M, then N becomes a manifold by restricting the topology and charts of M to N, and N is called an **open submanifold** of M.

(2) The unit circle $M = S^1 = \{a \in R^2 : \|a\| = 1\}$ with the topology induced from R^2 is a one-dimensional manifold, and the collection of open sets which covers S^1 can be taken to have two elements. More generally, we shall show later that the n-sphere $S^n = \{a \in R^{n+1} : \|a\| = 1\}$ is an n-dimensional manifold, and the collection of charts can be taken to have two elements.

(3) The closed interval $M = [0, 1]$ is not a one-dimensional manifold since the point 0 is not contained in an open set $U \subset M$ which is homeomorphic to an open set in R. Is the loop M as indicated in Fig. 2.1 a manifold? Thus is the point of intersection contained in an open set $U \subset M$ which is homeomorphic to an open set in R?

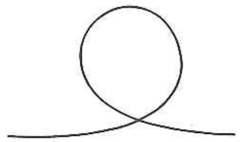


Fig. 2.1.

The coordinate functions given for the manifold $GL(V)$ are differentiable of class C^∞ (actually analytic), and we now define such notions in general.

Definition 2.3 A set $\mathscr{A}$ of (m-dimensional) charts of an m-dimensional manifold M is called a C^∞**-atlas** if $\mathscr{A}$ satisfies the following conditions.

(a) For every $p \in M$, there exists a chart $(U, x) \in \mathscr{A}$ with $p \in U$; that is, $M = \bigcup\{U : (U, x) \in \mathscr{A}\}$.

(b) If $(U(x), x)$ and $(U(y), y)$ are in $\mathscr{A}$, where $U(z)$ is the coordinate neighborhood corresponding to the homeomorphism z, then $U(x) \cap U(y)$ is empty or the maps $x \circ y^{-1}$ and $y \circ x^{-1}$ are of class C^∞.

Note that $x \circ y^{-1}$ (respectively $y \circ x^{-1}$) has domain $y(U(x) \cap U(y))$ [respectively $x(U(x) \cap U(y))$] and transforms these subsets of R^m homeomorphically onto each other (Fig. 2.2). Thus since one of these maps is the inverse

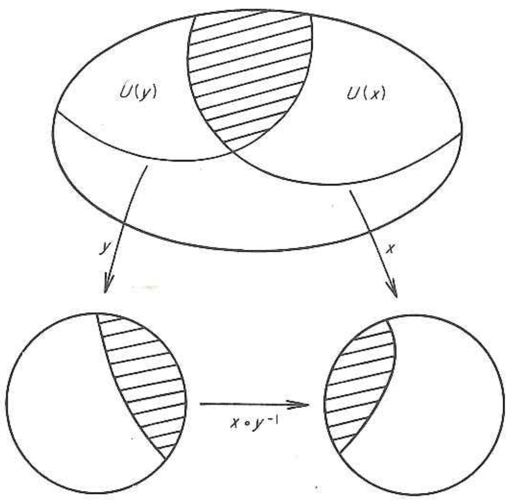


Fig. 2.2.

of the other, their derivatives are invertible linear transformations on R^m, using the chain rule. These maps are called a **change of coordinates** and one says that the corresponding coordinate $(U(x), x)$ and $(U(y), y)$ systems of p are **compatible** when they satisfy condition (b). This will eventually lead to the fact that if a function $f : M \to R$ is differentiable in one coordinate system, then f is differentiable in any compatible coordinate system.

Definition 2.4 Let $\mathscr{A}$ be a C^∞-atlas on an m-dimensional manifold M. Then a chart (U, x) is **admissible to** $\mathscr{A}$ or **compatible with** $\mathscr{A}$ if (U, x) is compatible with every chart in $\mathscr{A}$; that is, for any $(U(y), y) \in \mathscr{A}$, we have (U, x) and $(U(y), y)$ satisfy condition (b) in Definition 2.3.

Now given any atlas $\mathscr{A}$, one can adjoin all charts which are admissible to $\mathscr{A}$ and obtain a collection $\bar{\mathscr{A}}$ which is again an atlas on M. Thus $\bar{\mathscr{A}}$ is **maximal** relative to properties (a) and (b) of Definition 2.3, and any atlas is contained in a unique maximal atlas.

Definition 2.5 (a) An m-dimensional topological manifold M has a C^∞**-differentiable structure** or just a C^∞**-structure** if one gives M a maximal C^∞-atlas. Thus to give a C^∞-differentiable structure, one need only exhibit a C^∞-atlas on M, then consider the maximal atlas containing it.

(b) A **differentiable manifold of class** C^∞ or just a C^∞**-manifold** is an m-dimensional topological manifold M to which there is assigned a C^∞-differentiable structure.

Remarks (1) One obtains differentiable manifolds of class C^k, $k \geq 0$, or real analytic manifolds by just demanding that the change of coordinates $y \circ x^{-1}$ and $x \circ y^{-1}$ given in Definition 2.3(b) is of class C^k or analytic.

(2) To define an **m-dimensional complex manifold** just replace R^m in the definition of differentiable manifold of class C^∞ by the m-dimensional complex space C^m. Condition (b) in Definition 2.3 must be modified by demanding that the functions $y \circ x^{-1}$ and $x \circ y^{-1}$ be holomorphic in the respective sets in C^m.

Examples (4) Let $M = R$ and define a coordinate system $(U(x), x)$ by $U(x) = R$ and $x : M \to R : t \to t$. Then $\mathscr{A} = \{(U(x), x)\}$ is a C^∞-atlas which defines a differentiable structure and R is a differentiable manifold of class C^∞ relative to this structure. Now let $M_1 = R$ and define a coordinate system $(U(y), y)$ by $U(y) = R$ and $y : M_1 \to R : t \to t^3$. Then $\mathscr{A}_1 = \{(U(y), y)\}$ is a C^∞-atlas since $U(y)$ covers M_1 and the map $y \circ y^{-1}$, the identity, is of class C^∞. Thus Definition 2.3 is satisfied. The atlas $\mathscr{A}_1$ makes M_1 into a C^∞-manifold. The manifolds are distinct in the sense that the charts $(U(x), x)$ and $(U(y), y)$

on R are not compatible since $x \circ y^{-1} : R \to R : t \to t^{1/3}$ is not differentiable at $t = 0$.

(5) Let $S^n = \{a \in R^{n+1} : \|a\| = 1\}$ be the n-sphere with the topology induced from R^{n+1} and $\|a\|^2 = \sum_{i=1}^{n+1} a_i^2$ for $a = (a_1, \ldots, a_{n+1}) \in R^{n+1}$. We define a differentiable structure on S^n as follows. Let $p = (0, \ldots, 0, 1)$ be the "north pole" and $q = (0, \ldots, 0, -1)$ be the "south pole" of S^n. Then the open sets $U(p) = S^n - \{p\}$ and $U(q) = S^n - \{q\}$ cover S^n, and we define coordinate functions x and y so that $\{(U(p), x), (U(q), y)\}$ is an atlas on S^n. The functions x and y are defined by stereographic projections as follows. For $a \in U(p)$ let λ be the line determined by the points p and a and let π be the plane in R^{n+1} given by $u_{n+1} = 0$. Then the value $x(a)$ is the point in R^{n+1} where λ and π intersect. Thus we have a map $x : U(p) \to R^n$ (see Fig. 2.3).

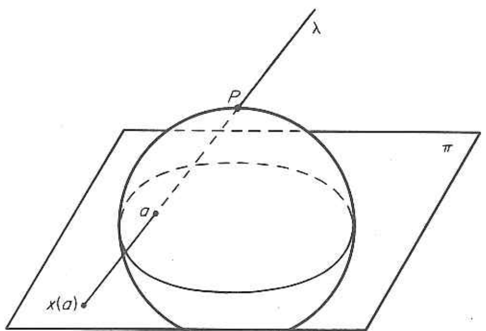


Fig. 2.3.

More specifically if $a = (a_1, \ldots, a_{n+1})$, then $x(a) = (x_1, \ldots, x_n)$, where $x_i = a_i/1 - a_{n+1}$ for $i = 1, \ldots, n$. Similarly y is given by stereographic projection $y : U(q) \to R^n : a \to (y_1, \ldots, y_n)$, where $y_i = a_i/1 + a_{n+1}$ for $i = 1, \ldots, n$. From the formulas, the functions x and y are homeomorphisms onto R^n, and the formulas show that $x \circ y^{-1}$ and $y \circ x^{-1}$ are of class C^∞. Thus we obtain an atlas which makes S^n into a C^∞-manifold.

Note that S^n is a special case of manifolds defined by the implicit function theorem as follows. Let $f : R^{n+1} \to R$ be a C^∞-function and suppose that on the set $M = \{p \in R^{n+1} : f(p) = 0\}$ we have $Df(p) \neq 0$ or more generally $D_{n+1}f(p) \neq 0$. Then one can apply the implicit function theorem to obtain a neighborhood of $p \in M$ which projects in a bijective manner onto the plane $u_{n+1} = 0$ and yields an atlas which makes M into a C^∞ n-dimensional manifold. Thus for S^n, take $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$ and note $Df(p) \neq 0$ for $p \in S^n = f^{-1}(0)$ (just compute $Df(p)$ for $p \in S^n$).

(6) We now consider the product manifold determined by two C^∞-manifolds M and N. Thus let $(U(x), x)$ and $(V(y), y)$ be in the maximal atlases

for M and N with $U(x)$ [respectively $V(y)$] a neighborhood of $p \in M$ (respectively $q \in N$). Then define an atlas on the topological product space $M \times N$ by letting $U(x) \times V(y)$ be the coordinate neighborhood of $(p, q) \in M \times N$ and define the homeomorphism

$$x \times y : U(x) \times V(y) \to R^m \times R^n : (u, v) \to (x(u), y(v)).$$

Thus the set of all these charts $(U(x) \times V(y), x \times y)$ defines a C^∞-atlas on $M \times N$ and the corresponding maximal atlas defines a C^∞-differentiable structure on $M \times N$. The **product manifold of** M **and** N is the Hausdorff space $M \times N$ with the C^∞-structure as given above. Similarly one can define the product of any finite number of differentiable manifolds.

Next let S^1 be the unit circle with the usual C^∞-differentiable structure and let $T^n = S^1 \times \cdots \times S^1$ (n-times) be the product manifold. Then T^n is called an **n-dimensional torus.** Thus, in particular, since $T^2 = \bigcup \{\{x\} \times S^1 : x \in S^1\}$; that is, T^2 is a union of unit circles whose centers are on a unit circle, we obtain Fig. 2.4.

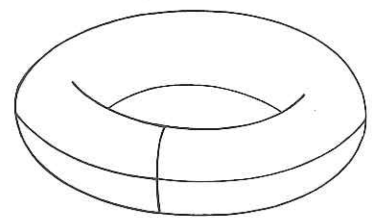


Fig. 2.4.

As shown in Fig. 2.5, T^2 can also be represented as a closed square whose points on the top edge are identified with those directly below on the bottom edge. The points on the right and left edges with the same heights are identified; in particular, the four vertices are identified as the same point. This identification comes from appropriately cutting and bending the above diagram for T^2. Note that since $S^1 = \{\exp 2\pi ix : 0 \leq x \leq 1\}$, we can identify $T^2 = \{(\exp 2\pi ix, \exp 2\pi iy) : 0 \leq x < 1 \text{ and } 0 \leq y < 1\}$ with $[0, 1) \times [0, 1)$ as above.

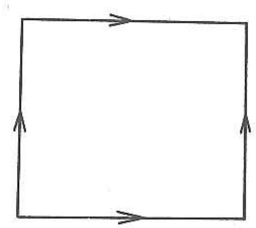


Fig. 2.5.

2. Differentiable Functions

A mapping $f: M \to N$ of two C^∞-manifolds will be seen to be differentiable of class C^∞ if its "coordinate expressions" are differentiable. Thus we shall reduce the differentiability of f to investigating the differentiability of functions $g: R^m \to R$.

Definition 2.6 Let M and N be C^∞-manifolds of dimension m and n, respectively, and let

$$f: M \to N$$

be a map defined on a neighborhood of a point $p \in M$. We say that f is **differentiable at** p **of class** C^∞ if there exists a coordinate system (U, x) at p in M and a coordinate system (V, y) at $f(p)$ in N such that

$$y \circ f \circ x^{-1}: x(U) \to y(V)$$

is differentiable at $x(p)$ of class C^∞ (see Fig. 2.6). Note that $x(U) \subset R^m$ and $y(V) \subset R^n$.

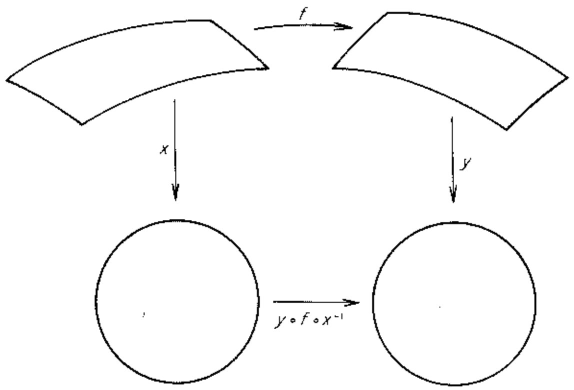


Fig. 2.6.

Since differentiability is given in terms of specific charts, we must show that it is actually independent of the choice of charts. Thus let $(\bar{U}, \bar{x})$ [respectively $(\bar{V}, \bar{y})$] be any other elements of the atlas for M (respectively N) which are neighborhoods of p [respectively $f(p)$]. Then we must show the map

$$\bar{y} \circ f \circ \bar{x}^{-1}: \bar{x}(\bar{U}) \to \bar{y}(\bar{V})$$

is differentiable at $\bar{x}(p)$. However, since differentiability is a local property, it suffices to show this on neighborhoods. Thus for $U \cap \bar{U}$ and $V \cap \bar{V}$ (which

are nonempty), we have on the neighborhoods $\bar{x}(U \cap \bar{U})$ and $\bar{y}(V \cap \bar{V})$ that

$$\bar{y} \circ f \circ \bar{x}^{-1} = \bar{y} \circ y^{-1} \circ (y \circ f \circ x^{-1}) \circ x \circ \bar{x}^{-1}.$$

Thus since $\bar{y} \circ y^{-1}$, $y \circ f \circ x^{-1}$ and $x \circ \bar{x}^{-1}$ are of class C^∞, so is their composition. If $f: M \to N$ is C^∞-differentiable at every point $p \in M$, then f is a **differentiable map of class** C^∞ **from** M **into** N.

Now in terms of coordinates, it suffices to show that the functions

$$f_i = u_i \circ (y \circ f \circ x^{-1}): R^m \to R$$

for $i = 1, \dots, n$ are differentiable on an open subset D of $x(p)$. Thus if (U, x) and (V, y) are the corresponding coordinate systems, then we obtain the coordinate expression

$$y_i = f_i(x_1, \dots, x_m) \quad \text{for} \quad i = 1, \dots, n$$

which must be differentiable at $x(p) = (p_1, \dots, p_m)$. This yields the following; for example, see Bishop and Goldberg [1968, p. 37].

Proposition 2.7 Let $f: M \to N$ be a continuous mapping of two C^∞-manifolds. Then f is of class C^∞ on M if and only if for every real-valued C^∞-function $y: V \to R$ defined on an open submanifold V of N, the function $y \circ f$ is of class C^∞ on the open submanifold $f^{-1}(V)$ of M.

We shall write $C^\infty(M)$ or $F(M)$ for the set of real-valued C^∞-functions on M and $C^\infty(p)$ or $F(p)$ for the set of those real-valued functions which are C^∞-differentiable at $p \in M$. Note that since differentiability of f at $p \in M$ also means f is defined on a neighborhood U of p, the elements of $C^\infty(p)$ are actually pairs (f, U). Consequently one can define an equivalence relation for elements of $C^\infty(p)$ such that $(f_1, U_1) \sim (f_2, U_2)$ if and only if there exists an open set G with $p \in G$ and $f_1(q) = f_2(q)$ for all $q \in G$. The set of equivalence classes are called **germs of** C^∞**-differentiable functions at** p. Note that the coordinate functions x_i on U are in $C^\infty(p)$. We shall usually not use this terminology but just the underlying ideas.

Next note that $F = C^\infty(M)$ is an associative algebra over R with operations given by

$$(af)(p) = af(p) \qquad \text{for} \quad a \in R,$$
$$(f + g)(p) = f(p) + g(p)$$
$$(fg)(p) = f(p)g(p) \qquad \text{for} \quad f, g \in C^\infty(M)$$

and F satisfies the following [Helgason, 1962, p. 5].

(1) If $f_1, \dots, f_r \in F$ and if $g: R^r \to R$ is of class C^∞ on R^r, then $g(f_1, \dots, f_r) \in F$.

(2) If $f: M \to R$ is a function on M such that for each $p \in M$ there is a $g \in F$ and there is a neighborhood U of p such that $f(q) = g(q)$ for all $q \in U$, then $f \in F$.

(3) For each p in the m-dimensional manifold M, there exist m functions $f_1, \ldots, f_m$ in F and an open neighborhood U of p such that the mapping

$$U \to R^m : q \to (f_1(q), \ldots, f_m(q))$$

is a homeomorphism of U onto an open subset of R^m. The functions $f_1, \ldots, f_m$ and the set U can be chosen so that for any $f \in F$, there is $g : R^m \to R$ of class C^∞ and

$$f = g(f_1, \ldots, f_m)$$

on U.

These properties determine a differentiable structure on M as follows (see Helgason [1962, p. 6] for a proof).

Proposition 2.8 Let M be a topological Hausdorff space and let m be an integer greater than 0. Let F be a set of real-valued functions on M satisfying properties (1)–(3). Then there exists a unique collection of charts $\mathscr{A} = \{(U_\alpha; x_\alpha) : \alpha \in A\}$ which form a maximal atlas of M such that the set of real-valued C^∞-functions on the manifold M with atlas $\mathscr{A}$ equal the set F.

Definition 2.9 The C^∞-manifolds M and N are **diffeomorphic** if there exists a homeomorphism $f: M \to N$ such that f and f^{-1} are of class C^∞; f is called a **diffeomorphism.**

Thus a diffeomorphism yields an equivalence relation such that the two manifolds are not only topologically equivalent, but also they have equivalent differentiable structures.

Examples (1) Let R be a manifold with the usual structure $x : R \to R : t \to t$ and $(-1, 1)$ be an open submanifold of R. Then

$$f: (-1, 1) \to R : t \to t/(1 - t^2)$$

is a diffeomorphism.

(2) Let R be the above manifold with the usual structure, and let M_1 be the manifold with space R and coordinate function $y : M_1 \to R : t \to t^3$. Then the map $f: M_1 \to R : s \to s^3$ is a C^∞-homeomorphism and the inverse homeomorphism $f^{-1} : R \to M_1 : u \to u^{1/3}$ is actually differentiable of class C^∞ relative to the above differentiable structures: For $t \in R$ we have the coordinate expression $(y \circ f^{-1} \circ x^{-1})(t) = (y \circ f^{-1})(t) = y(t^{1/3}) = t$. However, note that identity map $g : M_1 \to R : t \to t$ is not C^∞ since $(x \circ g \circ y^{-1})(t) = t^{1/3}$ which is not C^∞, that is, the identity map is not a diffeomorphism.

Exercise Let M be a C^∞-manifold. Show that the charts (U, x) and (V, y) at $p \in M$ are compatible if and only if x and y are C^∞-related by $y = f(x)$ and $x = g(y)$ for suitable C^∞-functions f and g.

3. Submanifolds

We shall now use the preceding results to study certain substructures of a manifold and return to these topics again after studying the differential of a function.

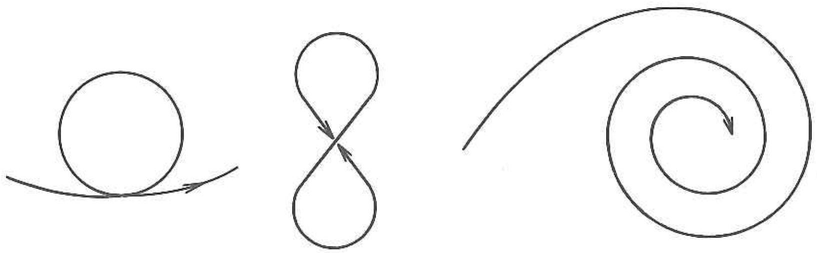
Definition 2.10 Let M and N be C^∞-manifolds of dimensions m and n respectively, and let $f: M \to N$ be a C^∞-mapping.

(a) We call f an **immersion** of M into N if for every $p \in M$, there is a neighborhood U of p in M and a chart (V, y) of $f(p)$ in N such that if we write $y = (y_1, \ldots, y_n)$ in terms of coordinate functions, then $x_i = y_i \circ f \mid U$ for $i = 1, \ldots, m$ are coordinate functions on U in M. That is if $x = (x_1, \ldots, x_m)$, then (U, x) is a chart at p in M. We say that M is **immersed** in N if an immersion $f: M \to N$ exists.

(b) We call f an **embedding** if f is injective and f is an immersion. Also M is said to be **embedded** in N. Thus an immersion is a local embedding.

(c) We call the subset $f(M)$ of N a **submanifold** of N if f is an embedding and $f(M)$ is given a C^∞-differential structure for which the mapping of manifolds $f: M \to f(M)$ is a diffeomorphism. In particular if M is a subset of the C^∞-manifold N and M has its own C^∞-differentiable structure, then M is a submanifold of N if the inclusion map $i: M \to N : x \to x$ is an embedding. Thus a coordinate system on N induces a coordinate system on M.

The subset $f(M)$ is called an **immersed submanifold** if the above mapping f is just an immersion. The topology of a submanifold $M \subset N$ need not be the induced topology of the containing manifold. However, since the inclusion map is C^∞ and consequently continuous, the open sets in the induced topology are open sets in the submanifold topology. Also note that the dimension of a submanifold is less than or equal to the dimension of its containing manifold and in the case of equality we just obtain open submanifolds; this can be easily seen by using the inverse function theorem as stated in Section 2.5.



(a) (b) (c)

Fig. 2.7.

Examples (1) Consider the mappings $f: R \to R^2$ indicated in Fig. 2.7. In (a) f is an immersion (why?) but not an embedding and $f(R)$ is an immersed submanifold but not a submanifold. In (b) the figure "8" is such that the arrow segments approach but do not touch the center p. Then f is an embedding and $f(R)$ is a submanifold when given the obvious C^∞-structure. Note that the submanifold topology is that of a bent open interval and therefore a neighborhood of p in the submanifold topology is just a bent open interval containing p. However, a neighborhood of p in the topology induced from R^2 always contains part of the arrow curves near p. Also the spiral in (c) yields an embedding and a submanifold. What can be said about the submanifold topology and the topology induced from R^2 in (c)?

(2) Consider the torus of Section 2.1

$$T^2 = \{(\exp 2\pi ix, \exp 2\pi iy) : 0 \le x < 1 \text{ and } 0 \le y < 1\}$$

and define

$$f: R \to T^2 : t \to (\exp 2\pi iat, \exp 2\pi ibt)$$

where $a/b = \alpha$ is an irrational number. Then f is injective (by solving the resulting equations and using α is irrational) and f is C^∞. Thus by giving $f(R)$ the obvious C^∞-structure so that $f: R \to f(R)$ is a diffeomorphism, $f(R)$ is a submanifold. Furthermore $f(R)$ wraps around T^2 in a nonintersecting manner and is actually dense in T^2 (exercise or see the text of Auslander and MacKenzie [1963]). Representing T^2 as a square with opposite sides identified as discussed in Section 2.1, we see $f(R)$ can be represented by the line segment $(x, y) = (at, bt)$ and their displacements as in Fig. 2.8. Also we should note

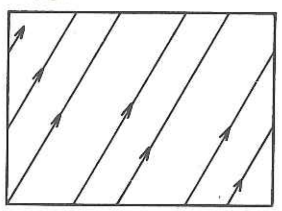


Fig. 2.8.

that points close together in T^2 need not be close in $f(R)$; that is, the topology in $f(R)$ is not the induced topology.

Exercises (1) In general, can one find a one-dimensional submanifold of T^n which is dense in T^n?

(2) Show if $f: M \to N$ defines a submanifold and if M is compact, then $f: M \to f(M)$ is a homeomorphism. (*Hint*: What can be said about a continuous map of a compact space onto a Hausdorff space?)

If $z = f(x, y)$ is a well-behaved function, then it defines a surface $M \subset R^3$ which is a two-dimensional submanifold. For a point $p \in M$ we can define, in a suitable neighborhood V in R^3 of p, coordinates (x, y, u), where $u = z - f(x, y)$. Thus the surface is given locally by the equation $u = 0$. The familiar upper hemisphere given by $z = (1 - x^2 - y^2)^{1/2} > 0$ is an example of such a situation. We have the following generalization of this.

Proposition 2.11 Let M be an m-dimensional C^∞-submanifold of the n-dimensional C^∞-manifold N and let $p \in M$. Then there exists a coordinate system (V, z) of N with $p \in V$ such that:

(a) $z_1(p) = \cdots = z_n(p) = 0$ where the z_i are the coordinate functions;

(b) the set $W = \{r \in V : z_{m+1}(r) = \cdots = z_n(r) = 0\}$ together with the restriction of $z_1, \ldots, z_m$ to W form a chart of M with $p \in W$.

Conversely, if a subset $M \subset N$ has a manifold structure with a coordinate system at each $p \in M$ satisfying the above, then M is a submanifold of N.

PROOF Let $f: Q \to N$ be an embedding which defines $M = f(Q)$ and let $p = f(q)$ for a unique $q \in Q$. Now let (T, y) be a chart for p in N and we can assume $y(p) = 0$ in R^n. Let U be a neighborhood of $q = f^{-1}(p)$ in Q and let $x = y \circ f | U$ be such that (U, x) is a chart for q in Q. Thus $x(q) \in R^m$ and for $i = 1, \ldots, m$ we have $x_i = y_i \circ f | U$ are the corresponding coordinate functions.

Now the composition $y \circ f \circ x^{-1} = F$ defines a C^∞-function $F: x(U) \to y(T)$, where $x(U) \subset R^m$ and $y(T) \subset R^n$, and we can write F in terms of coordinates

$$y_i = f_i(x_1, \ldots, x_m) \qquad \text{for} \quad i = 1, \ldots, n.$$

The hypotheses $M = f(Q)$ is a submanifold, $y \circ f = F \circ x$, and $x = y \circ f | U$ yield $y_i = x_i$ for $i = 1, \ldots, m$ in the above expression for F. Thus the rank of $DF(x(q))$ is m; that is, the $m \times m$ matrix $(\partial f_i / \partial x_j)$, $i, j = 1, \ldots, m$, is the identity. By the inverse function theorem there exists a neighborhood D of $x(q)$ with $D \subset x(U)$ where the first m equations can be locally inverted

$$x_i = g_i(y_1, \ldots, y_m) \qquad \text{for} \quad i = 1, \ldots, m,$$

where $y_1, \ldots, y_m$ are actually the coordinate functions defined on $f \circ x^{-1}(D) \subset T$ but are also used above to denote "coordinates" in $y(T)$. (Note that due to the simple expression of the first m equations $y_i = f_i(x_1, \ldots, x_m)$, the functions g_i can be explicitly computed. What are they?)

We now change from the y-coordinates to the $z = (z_1, \ldots, z_n)$ coordinates given by

$$z_j = y_j \qquad \text{for} \quad j = 1, \ldots, m$$
$$z_i = y_i - f_i(g_1(y_1, \ldots, y_m), \ldots, g_m(y_1, \ldots, y_m))$$

for $i = m+1, \ldots, n$. These equations for y's are defined on $f \circ x^{-1}(D) \subset T$ and are C^∞. They form a change of coordinates because z^{-1} exists locally [show $\det(\partial z_i/\partial y_j(p)) \neq 0$] and $y \circ z^{-1}$ and $z \circ y^{-1}$ are C^∞ on their domains.

Let $V \supset f \circ x^{-1}(D)$ be the subset of the domain of y in T where z is defined. Then V is a neighborhood of p in N and by unscrambling the definitions of f_i and g_i and using $y(p) = 0$ we have $z(p) = 0$. Now the set W in (b) given by

$$W = \{r \in V : z_{m+1}(r) = \cdots = z_n(r) = 0\}$$

contains p and is in M since in terms of the defining equations for $z_{m+1}, \ldots, z_n$ we see $W \subset f(x^{-1}(D)) \subset f(U) \subset f(Q) = M$ and also W is open in M. The restriction of $z_i = y_i$ for $i = 1, \ldots, m$ to W equals x_i (second paragraph) and so are coordinates on W.

The converse follows from various definitions.

Remark The above proof contains some machinery which is not necessary in view of our definition of a submanifold and for a more direct proof see the book by Bishop and Goldberg [1968, p. 42]. However, it can be modified to obtain the following result which is frequently used as the definition of a submanifold [Helgason, 1962; Singer and Thorpe, 1967].

Corollary 2.12 Let P be an m-dimensional C^∞-manifold, let N be an n-dimensional C^∞-manifold with $n \geq m$, and let $f : P \to N$ be an injective C^∞-function. If for every $q \in P$, there exists a chart (U, x) of q in P and there exists a chart (T, y) of $f(q) = p$ in N such that the linear transformation

$$D(y \circ f \circ x^{-1})(x(q)) : R^m \to R^n$$

is injective, then $M = f(P)$ is a submanifold of N provided $f(P)$ is given a C^∞-structure so that $f : P \to f(P)$ is a diffeomorphism.

PROOF We shall use the converse of Proposition 2.11 by showing (a) and (b) hold. By a simple translation argument we can assume that $x(q) = y(p) = 0$. Now near $x(q)$ we can represent the composition $y \circ f \circ x^{-1} = F$ in terms of coordinates

$$y_i = f_i(x_1, \ldots, x_m) \qquad \text{for} \quad i = 1, \ldots, n.$$

Since $D(y \circ f \circ x^{-1})(x(q))$ has rank m we have that some subsystem of m equations

$$y_{i_j} = f_{i_j}(x_1, \ldots, x_m) \qquad \text{for} \quad j = 1, \ldots, m$$

is such that $m \times m$ matrix $(\partial f_{i_j}/\partial x_k)$ is invertible. We can assume this subsystem consists of the first m equations and consequently can define a function

$$\bar{F} : x(U) \to R^m : (x_1, \ldots, x_m) \to (y_1, \ldots, y_m)$$

where we use the coordinate function to also denote the corresponding point. Thus by the inverse function theorem there exists a neighborhood D of $x(q)$ and $D \subset x(U)$ on which $\bar{F}$ has a local inverse G. Thus the first m equations can be locally inverted

$$x_i = g_i(y_1, \ldots, y_m) \qquad \text{for} \quad i = 1, \ldots, m,$$

and we proceed as in the above proof. Note that from the defining equations of z we see that $x(q) = y(p) = 0$ implies $z(p) = 0$.

Remark Let M be an m-dimensional C^∞-manifold. Then it can be proved that M is diffeomorphic to a submanifold of R^n with $n \leq 2m + 1$. This theorem of Whitney can be found in the work of Auslander and MacKenzie [1963].

Proposition 2.13 Let M and N be C^∞-manifolds of dimension m and n, respectively, with $m \geq n$. Let $f : M \to N$ be a C^∞-map and for some fixed $p \in N$ let $f^{-1}(p) = \{q \in M : f(q) = p\}$. Let every $q \in f^{-1}(p)$ have a chart (U, x) in M and let p have a chart (T, y) in N such that $D(y \circ f \circ x^{-1})(x(q)) : R^m \to R^n$ is surjective. Then $f^{-1}(p)$ is a closed $(m - n)$-dimensional submanifold of M or $f^{-1}(p)$ is empty.

Proof This follows from the variation of the inverse function theorem given in Proposition 1.17 using the inverse image of the set $\{p\}$ is closed (or see the book by Spivak [1965, p. 111]).

A C^∞-map $f : M \to N$ such that for every $q \in M$ there is a chart (U, x) at q and a chart (T, y) at $f(q)$ with $D(y \circ f \circ x^{-1})(x(q))$ surjective is called a **submersion**. Thus the injective or surjective nature of $D(y \circ f \circ x^{-1})(x(q))$ determines submanifolds.

Exercise (3) If $f : R \to R^m$ is C^∞, then show the graph of f given by $G(f) = \{(t, f(t)) : t \in R\}$ is a submanifold of $R^{m+1} = R^1 \times R^m$ with the induced topology. Does $f : R \to R^2 : t \to (t^2, t^3)$ define a submanifold? The above can be generalized to C^∞-functions $f : M \to N$.

Definition 2.14 Let M be a C^∞-manifold. A C^∞-**curve** in M is a C^∞-map f from some interval I contained in R into M such that f has an extension $\tilde{f}$ which is a C^∞-map of an open interval $J \supset I$ into M. Thus if $I = [a, b]$, then there exists an $\varepsilon > 0$ such that $J = (a - \varepsilon, b + \varepsilon)$ and there exists a C^∞-function $\tilde{f}: J \to M$ such that $f(t) = \tilde{f}(t)$ for all $t \in I$. Then f is frequently called **a curve segment** in case $I = [a, b]$. A **broken** C^∞-**curve** in M is a continuous map $f: [a, b] \to M$ together with a partition of $[a, b]$ such that on the corresponding closed subintervals f is a C^∞-curve.

Examples (3) The map $f: R \to R^2 : t \to (t^2, t^3)$ is a C^∞-curve in R^2 with a cusp at (0, 0).

(4) The "wrap around" map on the torus T^2 given in Section 2.3 with "irrational slope" is actually a C^∞-curve which is dense in T^2.

(5) The map $f: [0, 1] \to R^2$ given by

$$f(t) = \begin{cases} (t, \sin 1/t) & \text{if } t \neq 0, \\ (0, 0) & \text{if } t = 0 \end{cases}$$

is not a C^∞-curve in R^2 since it does not have a C^∞-extension to an open interval containing 0.

We recall that a topological space M is **connected** if it satisfies any of the following equivalent conditions:

(1) M is not the union of two nonempty disjoint closed subsets;
(2) M is not the union of two nonempty disjoint open subsets;
(3) the only subsets of M which are both open and closed are M and the empty set;
(4) if $M = \bigcup_a E_a$, where E_a are open and $E_a \cap E_b$ is empty if $a \neq b$, then only one of the E_a is nonempty;
(5) if $f: M \to N$ is a continuous map into a discrete set, then $f(M)$ is a single point.

A topological space M is **path connected** if for every $p, q \in M$, there exists a continuous curve $f: [a, b] \to M$ with $p = f(a)$ and $q = f(b)$. We have the fact that a path connected space must be connected [Singer and Thorpe, 1967].

Proposition 2.15 Let M be a C^∞-manifold.

(a) If M is connected, then every pair of points can be joined by a broken C^∞-curve.
(b) M is connected if and only if M is path connected.

PROOF Part (b) follows from the preceding remarks and part (a). Thus let $p \in M$ and for $q \in M$, define $q \sim p$ if and only if q can be joined to p by a broken C^∞-curve. Then since $\sim$ is an equivalence relation, M is the union of the disjoint equivalence classes

$$E_p = \{q \in M : q \sim p\}.$$

Now each E_p is open in M, for if $q \in E_p$, let (U, x) be a chart of M such that $q \in U$ with $x(q) = 0$ and $x(U) = B_m$, an open m-ball. Now for each $u \in U$ the point $x(u) \in B_m$ can be joined to $x(q)$ by a C^∞-curve λ in B_m; that is, λ a straight line segment. Therefore $u \in U$ can be joined to q by the C^∞-curve $x^{-1} \circ \lambda$ and consequently $u \in U$ can be joined to p by a broken C^∞-curve. Thus $u \sim p$, so that $U \subset E_p$ and E_p is open. However, since $M = \bigcup E_q$ (disjoint), we have by condition (4) that all the E_q are empty except one. Thus every point in M can be joined to p by a broken C^∞-curve.

Example (6) Let $V = R^n$ and let $G = GL(V)$. Then G is an open n^2-dimensional submanifold of R^{n^2}. Now let

$$SL(V) = \{A \in GL(V) : \det(A) = 1\}.$$

Then $SL(V)$ is clearly a subgroup of $GL(V)$ and is called the **special linear group** and is sometimes denoted by $SL(n, R)$. Now $SL(V)$ is a closed submanifold because if we let

$$f: GL(V) \to R - \{0\} : A \to \det(A),$$

then using exercise (5), Section 1.4 for the derivative of det we see that for all $A \in GL(V)$, $D(f)(A)$ is surjective; that is, of rank 1. Thus by Proposition 2.13, $SL(V) = f^{-1}(1)$ is closed and of dimension $n^2 - 1$.

We shall now use exp to obtain a coordinate system at the point $I \in SL(V)$ and then for any point $A \in SL(V)$. For $g = gl(V)$ we let

$$sl(V) = \{X \in g : \operatorname{tr} X = 0\}.$$

Thus since tr is linear and $\operatorname{tr}[X, Y] = \operatorname{tr} XY - \operatorname{tr} YX = 0$ we see that $sl(V)$ is a Lie subalgebra of g; that is, $sl(V)$ is a vector subspace of g so that for all $X, Y \in sl(V)$ we have $[X, Y] = XY - YX \in sl(V)$. Also for any $X \in g$,

$$X = \frac{1}{n}(\operatorname{tr} X)I + \left[X - \frac{1}{n}(\operatorname{tr} X)I\right] = \frac{1}{n}(\operatorname{tr} X)I + Y,$$

where $\operatorname{tr} Y = 0$. Consequently $\dim sl(V) = n^2 - 1$. Next note that exp restricted to $sl(V)$ is actually in $SL(V)$, since we have from exercise (3), Section 1.1 that

$$\det(\exp X) = e^{\operatorname{tr} X} = 1$$

if $X \in sl(V)$. Thus if we let $F = \exp | sl(V)$, we have from the proof of Proposition 1.19 that $DF(0)$ is the identity. Therefore by the inverse function theorem there exists a neighborhood U_0 of 0 in $sl(V)$ and a neighborhood U_I

of I in $SL(V)$ such that $F: U_0 \to U_I : X \to \exp X$ is a diffeomorphism of U_0 onto U_I. Thus for t in a sufficiently small interval $(-\delta, \delta)$ of $0 \in R$ and for X fixed in $sl(V)$ we see that the map

$$\exp : sl(V) \to SL(V) : tX \to \exp tX$$

maps the line segment tX into a C^∞-curve segment in $SL(V)$.

To coordinatize $SL(V)$ by exp we proceed as follows. First as in the remarks following Proposition 1.19 we have the local C^∞-diffeomorphism

$$\log : U_I \to U_0$$

and since U_0 is open in $sl(V)$ we find that $(U_I, \log)$ is a chart at I in $SL(V)$ [noting that $\operatorname{tr}(\log \exp X) = 0$]. Now for any other point $A \in SL(V)$ we have that

$$L(A) : SL(V) \to SL(V) : B \to AB$$

is a diffeomorphism of $SL(V)$ and therefore the set

$$L(A)U_I = AU_I = \{Au : u \in U_I\}$$

is an open neighborhood of A, using $A = AI$. Let $V = AU_I$ and let $y = \log \circ L(A)^{-1}$. Then (V, y) is a chart at A in $SL(V)$ as shown in Fig. 2.9. Finally we remark that $G = GL(V)$ is not connected; for if it were, then since det is continuous, $\det(G) = R - \{0\}$ is connected, a contradiction. However, $SL(V)$ is connected and this follows from Proposition 2.15 and the following result.

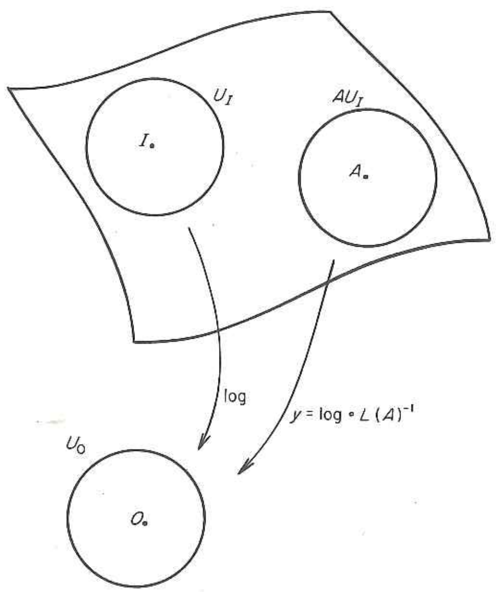


Fig. 2.9.

Exercise (4) Let $P(V) = \{A \in GL(V) : \det A > 0\}$. Then $P(V)$ is path connected.

Now to show $SL(V)$ is connected we note that the map

$$\theta : P(V) \to SL(V) : A \to (\det A)^{-1/n} A$$

is a continuous surjection so that $SL(V)$ is connected.

Example (7) Again let $V = R^n$ and let

$$B : V \times V \to R : (X, Y) \to B(X, Y)$$

be a nondegenerate bilinear form (symmetric or skew-symmetric). Then the adjoint A^* relative to B is uniquely given by

$$B(AX, Y) = B(X, A^*Y)$$

for $A \in \operatorname{End}(V)$. We have the usual rules

$$(aA + bB)^* + aA^* + bB^* \qquad \text{and} \qquad (AB)^* = B^*A^*.$$

Let

$$K = \{B \in \operatorname{End}(V) : B^* = B\}.$$

Then K is a vector space and a manifold and the manifold dimension equals the vector space dimension. Also for any $A \in G = GL(V)$ let

$$f : G \to \operatorname{End}(V) : A \to AA^* - I.$$

Then $f(A) \in K$ and let

$$\begin{aligned} H &= \{A \in G : B(AX, AY) = B(X, Y) \text{ all } X, Y \in V\} \\ &= \{A \in G : AA^* - I = 0\} \\ &= f^{-1}(0). \end{aligned}$$

Then H is clearly a subgroup of G and H is a closed submanifold of G of dimension $n^2 - \dim K$ as follows.

To see this we shall use Proposition 2.13. Thus we must show for every $A \in G$ such that $f(A) = 0$ that $Df(A) : \operatorname{End}(V) \to K$ is surjective. For any $A \in G$ and any $Y \in K$, let $X = \frac{1}{2} Y A^{*-1} \in \operatorname{End}(V)$. Then we shall show

$$Y = [Df(A)](X)$$

so that $Df(A)$ is surjective. Thus

$$\begin{aligned} [Df(A)](X) &= \lim_{t \to 0} \frac{1}{t} [f(A + tX) - f(A)] \\ &= \lim_{t \to 0} \frac{1}{t} \{[(A + tX)(A + tX)^* - I] - (AA^* - I)]\} \\ &= XA^* + (XA^*)^* = Y. \end{aligned}$$

To coordinatize H we proceed as follows. Let $g = gl(V)$ and let

$$h = \{P \in g : B(PX, Y) = -B(X, PY) \text{ all } X, Y \in V\}$$
$$= \{P \in g : P^* = -P\}.$$

Then h is a Lie subalgebra of g; that is, h is a vector subspace of g and for $P, Q \in h$ we have $[P, Q] = PQ - QP \in h$. Thus for $P, Q \in h$ and $a, b \in R$ we have

$$(aP + bQ)^* = aP^* + bQ^* = -(aP + bQ)$$

so that h is a subspace and

$$[P, Q]^* = (PQ)^* - (QP)^* = -(PQ - QP)$$

so that $[P, Q] \in h$ as desired.

Now for any $P \in h$ we have for all $X, Y \in V$ that

$$B((\exp P)X, (\exp P)Y) = B(X, (\exp P)^*(\exp P)Y)$$
$$= B(X, (\exp P^*)(\exp P)Y)$$
$$= B(X, \exp(-P)(\exp P)Y) = B(X, Y).$$

Thus $\exp : h \to H$ so that as in the preceding example there exist a neighborhood U_0 of 0 in h and a neighborhood U_I of I in H such that $\exp : U_0 \to U_I$ is an analytic diffeomorphism and $(U_I, \log)$ is a chart at I in H which induces the chart $(AU_I, \log \circ L(A)^{-1})$ at A in H. Also if for any $P \in g$ we demand that the C^∞-curve $R \to G : t \to \exp tP$ actually be in H for t in an interval about 0 in R, then by differentiating the formula $B((\exp tP)X, (\exp tP)Y) = B(X, Y)$ we obtain $P \in h$ using example (1), Section 1.2. Note that the manifold dimension of H equals the vector space dimension of h.

There are various subcases depending on B.

B Symmetric (1) (i) Let B be positive definite; that is, $B(X, X) = 0$ implies $X = 0$. Thus there exists a basis $e_1, \ldots, e_n$ of V such that if $X = \sum x_i e_i$, $Y = \sum y_i e_i$, then $B(X, Y) = \sum x_i y_i$. In this case H is called the **orthogonal group** and denoted by $O(n)$. We also note that the vector space $K = \{B \in \text{End}(V) : B = B^*\}$ is just the set of symmetric matrices and has dimension $n(n+1)/2$. Thus the manifold dimension of $H = O(n)$ is $n^2 - n(n+1)/2 = n(n-1)/2$.

Now for $A \in O(n)$, $AA^* = I$ yields $(\det A)^2 = 1$ so that $\det A = \pm 1$. Thus noting

$$A = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix} \in O(n),$$

we have, since $\det : O(n) \to R - \{0\}$ is continuous, that $O(n)$ is not connected.

Let

$$SO(n) = \{A \in O(n) : \det A = 1\}$$
$$= O(n) \cap SL(n, R)$$

which is called the **special orthogonal group**. We know that $SO(n)$ is also a manifold of dimension $n(n-1)/2$ and we shall show later that $SO(n)$ is connected. In this case the Lie algebra $h \cap sl(V)$ associated with $SO(n)$ is denoted by $so(n)$.

(ii) Now assume the general form for the nondegenerate form B; that is, there exists a basis $f_1, \ldots, f_n$ of V such that for $X = \sum x_i f_i$, $Y = \sum y_i f_i$, then

$$B(X, Y) = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{n} x_i y_i$$

[Jacobson, 1953, Vol. II]. In this case the group $H \cap SL(n, R)$ is frequently denoted by $SO(p, q)$, where $p + q = n$ and the Lie algebra $h \cap sl(n)$ is denoted by $so(p, q)$.

Next we shall consider $V = R^n$ as column vectors with

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

relative to the basis $f_1, \ldots, f_n$ so that we can write $B(X, Y)$ in block form

$$B(X, Y) = X^t \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} Y \equiv X^t B Y,$$

where t denotes transpose and I_p, I_q are the appropriate identity matrices. Then for

$$P \in so(p, q) = \{P \in sl(n) : B(PX, Y) = -B(X, PY)\}$$

we have

$$0 = (PX)^t B Y + X^t B(PY) = X^t(P^t B + BP)Y.$$

Thus $P^t B + BP = 0$ and if we partition P into appropriate blocks

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

then $P_{11}^t = -P_{11}$, $P_{22}^t = -P_{22}$, $P_{12} = P_{21}^t$, and P_{12} arbitrary. Thus we obtain the form of the Lie algebra $so(p, q)$ and that it is of dimension

$$p(p-1)/2 + q(q-1)/2 + pq = n(n-1)/2$$

using $p + q = n$.

As before we can use $(U_I, \log)$ to coordinatize $SO(p, q)$ where log: $U_I \to U_0 \subset so(p, q)$. Thus we see that $SO(p, q)$ is an $n(n-1)/2$-dimensional manifold.

Exercise (5) Show $SO(p, q)$ is not connected (*Hint*: Investigate matrices of the form

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

where A_{ii} are orthogonal matrices of the appropriate size satisfying $\det A_{11} \det A_{22} = 1$).

B Skew-symmetric (2) Thus $B(X, Y) = -B(Y, X)$ and there exists a basis $f_1, \ldots, f_n$ of V such that $n = 2p$ and using the preceding notation $B(X, Y)$ has the block form [Jacobson, 1953, Vol. II]

$$B(X, Y) = X^t \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} Y = \sum_{k=1}^{p} (x_k y_{k+p} - x_{k+p} y_k).$$

In this case we shall consider $H \cap GL(2p, R)$ which is freqently denoted by $Sp(p, R)$, where $n = 2p$, or $Sp(p)$, or $Sp(n, R)$ and is called the **symplectic group**. The Lie algebra associated with $Sp(p, R)$ equals $h \cap gl(2p)$ and is denoted by $sp(p, R)$. Next for $P \in sp(p, R)$ we put it into block form and find

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where $P_{22} = -P_{11}^t$, $P_{12}^t = P_{12}$, $P_{21}^t = P_{21}$, and P_{11} arbitrary. Thus $p^2 + p(p+1)/2 + p(p+1)/2 = 2p^2 + p = \dim sp(p, R)$ which equals the manifold dimension of $Sp(p, R)$.

For future reference we present a short list of important Lie groups and Lie algebras in Tables 2.1 and 2.2. We will describe the groups and algebras entirely in terms of matrices. For convenience we include the groups and algebras that have been previously discussed. First define $I_{p,q} \in GL(p+q, R)$ and $J_n \in GL(2n, R)$ by

$$I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}, \qquad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

In the definition of the **unitary group** $U(n)$ the matrix $\bar{X} = (\bar{a}_{ij})$ is the complex conjugate matrix of $X = (a_{ij})$.

TABLE 2.1

LIE GROUPS

$GL(n, C)$	nonsingular $n \times n$ complex matrices,
$GL(n, R)$	nonsingular $n \times n$ real matrices,
$SL(n, C)$	$\{X \in GL(n, C) : \det(X) = 1\}$,
$SL(n, R)$	$\{X \in GL(n, R) : \det(X) = 1\}$,
$O(n, R)$	$\{X \in GL(n, R) : X^t = X^{-1}\}$,
$SO(n, R)$	$O(n, R) \cap SL(n, R)$,
$O(p, q)$	$\{X \in GL(p+q, R) : I_{p,q} X^t I_{p,q}^{-1} = X^{-1}\}$,
$SO(p, q)$	$O(p, q) \cap SL(p+q, R)$,
$Sp(n, C)$	$\{X \in GL(2n, C) : J_n X^t J_n^{-1} = X^{-1}\}$,
$Sp(n, R)$	$Sp(n, C) \cap GL(n, R)$,
$U(n)$	$\{X \in GL(n, C) : \bar{X}^t = X^{-1}\}$,
$SU(n)$	$U(n) \cap SL(n, C)$.

TABLE 2.2

LIE ALGEBRAS

$gl(n, C)$	$n \times n$ complex matrices,
$gl(n, R)$	$n \times n$ real matrices,
$sl(n, C)$	$\{X \in gl(n, C) : \mathrm{tr}(X) = 0\}$,
$sl(n, R)$	$sl(n, C) \cap gl(n, R)$,
$so(n, R)$	$\{X \in gl(n, R) : X^t = -X\}$,
$so(p, q)$	$\{X \in gl(p+q, R) : I_{p,q} X^t I_{p,q}^{-1} = -X\}$,
$sp(n, C)$	$\{X \in gl(2n, C) : J_n X^t J_n^{-1} = -X\}$,
$sp(n, R)$	$sp(n, C) \cap gl(2n, R)$,
$u(n)$	$\{X \in gl(n, C) : \bar{X}^t = -X\}$,
$su(n)$	$u(n) \cap sl(n, C)$.

For more details on matrix groups the reader should consider the work of Chevalley [1946, Chapter 1] and Helgason [1962, p. 339].

4. Tangents and Cotangents

Let $M \subset R^3$ be a well-behaved surface given by the differentiable function $z = f(x, y)$ and going through the point $p = (0, 0, 0)$. Then from calculus the tangent plane to M at p is given by the equation

$$z = x\, \partial f(0, 0)/\partial x + y\, \partial f(0, 0)/\partial y \qquad \text{for} \quad x, y \in R.$$

If this plane is cut by the plane $x = 0$, then the equation of the line of intersection is $z = y\, \partial f(0, 0)/\partial y$ and we obtain the vector $(0, 1, \partial f(0, 0)/\partial y)$ in the tangent plane. Similarly $(1, 0, \partial f(0, 0)/\partial x)$ is in the tangent plane. These two vectors which give the tangent plane are determined by the partial differentiation of f. Thus we are led to study operators on real-valued functions which have the properties of differentiation and we now abstract this situation to manifolds.

First recall that for a C^∞-manifold M and for $p \in M$ the set $F(p) = C^\infty(p)$ of C^∞-functions at $p \in M$ is an associative algebra using the pointwise operations: Let U, V be open sets of M containing p and let $f : U \to R$, $g : V \to R$ be in $F(p)$. Then define for $a, b \in R$

$$af + bg : U \cap V \to R : q \to af(q) + bg(q) \quad \text{and} \quad fg : U \cap V \to R : q \to f(q)g(q).$$

Definition 2.16 A **tangent at** $p \in M$ is a mapping $L : F(p) \to R$ such that for all $f, g \in F(p)$ and $a, b \in R$,

(a) $L(af + bg) = aL(f) + bL(g)$;
(b) $L(fg) = L(f)g(p) + f(p)L(g)$.

That is, L is a **derivation** of $F(p)$ into R. Let $T_p(M)$, or $T(M, p)$, or M_p denote the set of tangents at $p \in M$.

Example (1) For $p \in M = R^m$ and for fixed $X \in R^m$ the map

$$L_X : F(p) \to R : f \to [Df(p)](X)$$

is a tangent at p using Proposition 1.3 concerning the product rule.

Proposition 2.17 Let M be an m-dimensional C^∞-manifold and let $p \in M$.

(a) If $f, g \in F(p)$ and $f(q) = g(q)$ for all q in a neighborhood U of p, then $L(f) = L(g)$ for all $L \in T(M, p)$.
(b) $T(M, p)$ is a vector space over R.

PROOF (a) The function k defined on U by $k(q) = 1$ for all $q \in U$ is in $F(p)$ and we have for any $L \in T(M, p)$ that

$$\begin{aligned} L(k) &= L(k^2), \quad \text{using } 1 = 1^2 \\ &= L(k)k(p) + k(p)L(k) \\ &= 2L(k). \end{aligned}$$

Thus $L(k) = 0$. Now we have $f = kf = kg$ on U and therefore

$$\begin{aligned} L(f) &= L(k)f(p) + k(p)L(f) \\ &= L(kf) = L(kg) \\ &= L(k)g(p) + k(p)L(g) = L(g). \end{aligned}$$

(Can the "bump function" of exercise 6, Section 1.4 be used above?)

(b) For L_1 and L_2 in $T(M, p)$ and for $a, b \in R$ we see that $aL_1 + bL_2$ is a linear operator; that is, it satisfies (a) of the definition. Also for $f, g \in F(p)$ we have

$$\begin{aligned} (aL_1 + bL_2)(fg) &= aL_1(fg) + bL_2(fg) \\ &= a(L_1(f)g(p) + f(p)L_1(g)) + b(L_2(f)g(p) + f(p)L_2(g)) \\ &= (aL_1 + bL_2)(f)g(p) + f(p)(aL_1 + bL_2)(g). \end{aligned}$$

Thus $aL_1 + bL_2 \in T(M, p)$.

We shall now show that the vector space dimension of $T(M, p)$ is m; that is, equal to the manifold dimension of M. We shall do this by taking a chart (U, x) of M at p such that for $u_i : x(U) \to R$ where the u_i are coordinate functions of R^m, the partial derivative operators $D_i(x(p)) = \partial/\partial u_i(x(p))$ in R^m for $i = 1, \ldots, m$ eventually yield a basis $\partial_i(p)$ for $i = 1, \ldots, m$ of $T(M, p)$.

Thus let (U, x) be a fixed chart at p in M and let $f \in F(p)$, where f is defined on an open neighborhood V of p with $f : V \to R$ of class C^∞. Now f is of class C^∞ on the neighborhood $U \cap V \subset U$ so that we can write f in terms of the fixed coordinates $x = (x_1, \ldots, x_m)$ where $x_i = u_i \circ x$. Therefore for $D = x(U)$ an open set in R^m the function $g = f \circ x^{-1} : D \to R$ is C^∞ on D. Thus $f = g \circ x = g(x_1, \ldots, x_m)$ where we write $g = g(u_1, \ldots, u_m)$ on D. We now define the maps

$$\partial_i : F(p) \to F(p) : f \to \partial(f \circ x^{-1})/\partial u_i \circ x$$

which are called **coordinate vector fields** relative to (U, x); that is, we form the real-valued C^∞-function $h = \partial(f \circ x^{-1})/\partial u_i = \partial g/\partial u_i$ defined on D to obtain the function $h \circ x$ which is in $F(p)$. Sometimes the notations

$$\partial_i = \partial/\partial x_i \quad \text{and} \quad \partial_i f = \partial f/\partial x_i$$

are used. The mapping ∂_i has the following properties for $f, g \in F(p)$ and $a, b \in R$,

(1) $\partial_i(af + bg) = a\, \partial_i f + b\, \partial_i g$;
(2) $\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$.

Note that for the coordinate functions $x_j = u_j \circ x$ on U we have from the above definition

$$\partial_i x_j = \partial u_j/\partial u_i = \delta_{ij}$$

and for the constant function $f(q) = c$ for $q \in U$, we have $\partial_i f = 0$.

Next we define an element $\partial_i(p) \in T(M, p)$ as follows: For $f \in F(p)$ we obtain $\partial_i f \in F(p)$, then evaluate $(\partial_i f)(p) \in R$. Thus

$$\partial_i(p)f = (\partial_i f)(p)$$

and from (1), (2) above we see $\partial_i(p)$ satisfies the definition of a tangent at p.

Proposition 2.18 Let (U, x) be a fixed chart at $p \in M$ where $x = (x_1, \ldots, x_m)$. Then the vector space $T(M, p)$ has basis

$$\partial_1(p), \ldots, \partial_m(p)$$

and any $L \in T(M, p)$ has the unique representation

$$L = \sum_{i=1}^{m} a_i \partial_i(p),$$

where $a_i = L(x_i) \in R$. Thus the manifold dimension of M equals the vector space dimension of $T(M, p)$.

PROOF We can assume that $x(p) = 0$ since a translation $x_i = y_i + t$ yields $\partial/\partial x_i = \partial/\partial y_i$. Now from Section 1.4 it is easy to see that any real-valued C^∞-function g defined on $D = x(U)$ has the Taylor's formula expansion about the point $\theta = (0, \ldots, 0) \in D \subset R^m$

$$g = g(\theta) + \sum_{i=1}^{m} u_i g_i,$$

where u_i are coordinates on R^m and g_i are C^∞ at $\theta \in D$. Thus for the real valued function $f = g \circ x \in F(p)$ as previously discussed we obtain on U

$$\begin{aligned} f = g \circ x &= g(\theta) + \textstyle\sum (u_j \circ x)(g_j \circ x) \\ &= g(\theta) + \textstyle\sum x_j f_j, \end{aligned}$$

where $f_j = g_j \circ x \in F(p)$. We apply $\partial_i(p)$ to this equation

$$\begin{aligned} \partial_i(p)f &= (\partial_i f)(p) \\ &= 0 + \textstyle\sum \partial_i(x_j f_j)(p) \\ &= \textstyle\sum [(\partial_i x_j)(p) f_j(p) + x_j(p)\, \partial_i f_j(p)] = f_i(p) \end{aligned}$$

using $\partial_i x_j = \delta_{ij}$ and $x_j(p) = 0$. Next we apply L to the same equation

$$\begin{aligned} L(f) &= L(g(\theta)) + \textstyle\sum L(x_j f_j) \\ &= 0 + \textstyle\sum [(Lx_j) f_j(p) + x_j(p) L(f_j)] = \sum a_j \partial_j(p) f \end{aligned}$$

using the preceding equation. Thus $L = \sum a_j \partial_j(p)$. The elements $\partial_1(p), \ldots, \partial_m(p)$ in $T(M, p)$ are linearly independent. For if $\sum a_j \partial_j(p) = 0$, then applying to coordinate functions,

$$0 = 0(x_i) = \sum a_j \partial_j(p)(x_i) = a_i$$

using $\partial_j(p)(x_i) = \partial_j x_i(p) = \delta_{ij}$.

REMARK Let (U, x) and (V, y) be charts at $p \in M$. Then on $U \cap V$ we have the coordinate functions x_i and y_j defined. Then $\partial/\partial x_i(p)$ and $\partial/\partial y_j(p) = L$ are in $T(M, p)$ and according to Proposition 2.18 we can represent L by

$$\partial/\partial y_j(p) = \sum_{i=1}^{m} \partial x_i/\partial y_j(p)\; \partial/\partial x_i(p),$$

where the matrix $((\partial x_i/\partial y_k)(p))$ is the nonsingular Jacobian matrix obtain by writing $x_i = x_i(y_1, \ldots, y_m)$. Thus we have the matrix for the change of basis in $T(M, p)$ when we change charts at $p \in M$.

Examples (2) For $p \in M = R^m$ and $X \in R^m$, let $L_X \in T(M, p)$ be the tangent given in example (1) of this section. Then the map $R^m \to T(M, p): X \to L_X$ is linear because $L_{aX+bY}(f) = Df(p)(aX + bY) = (aL_X + bL_Y)(f)$. Also this map is an isomorphism. (Why?) Thus at each point $p \in M = R^m$ we can attach the tangent space which is isomorphic to M itself.

(3) Let N be a group with identity e and let $(x, y) = xyx^{-1}y^{-1}$ be in N and for A, B subsets of N let (A, B) be the subgroup generated by all commutators (x, y) with $x \in A$, $y \in B$. Then for $N_1 = (N, N)$, $N_{k+1} = (N_k, N)$ we have

$$N \supset N_1 \supset \cdots \supset N_k \supset \cdots$$

and call N **nilpotent** if there exists k with $N_k = \{e\}$. Now let N be the submanifold of $SL(V)$ consisting of the nilpotent subgroup given by the set of triangular matrices

$$\begin{bmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}$$

where $*$ denotes arbitrary elements from R. We shall now show that the vector space $T_I(N)$ is isomorphic to the vector space of all triangular (nilpotent) matrices

$$\begin{bmatrix} 0 & & * \\ & 0 & \\ & & \ddots \\ 0 & & & 0 \end{bmatrix}$$

and denote this vector space of matrices by n. To show the isomorphism we shall use the exp mapping by showing $\exp: n \to N$ is locally invertible. Now for $A \in n$ we see that the associative products $A^2, A^3, \ldots, A^k$ are all in n and since A is a nilpotent matrix $A^m = 0$. Thus we see that

$$\exp A = I + A + \cdots + A^{m-1}/(m-1)!$$

is in N so that $\exp : n \to N$. Also $D\exp(0)$ is invertible. Thus as before, there exist a neighborhood U_0 of 0 in n and a neighborhood of U_I of I in N so that $(U_I, \log)$ is a chart in N at I. Consequently we have, since U_0 is open in n,

$$\dim U_I = \dim U_0 = \dim n.$$

However $\dim T_I(N) = \dim U_I$ and since the vector spaces $T_I(N)$ and n have the same dimension, they are isomorphic.

Exercises (1) Show that n is a **nilpotent** Lie algebra. Thus first show $[n, n] = \{\sum [A_i, B_i] : A_i, B_i \in n\} \subset n$. Next define

$$n^1 = [n, n] \qquad \text{and} \qquad n^{k+1} = [n^k, n]$$

and note that $n \supset n^1 \supset \cdots \supset n^k \supset \cdots$. So finally show $n^p = 0$ for some p. This will show that the nilpotent (Lie) group N is such that the tangent space $T_I(N)$ is vector space isomorphic to a nilpotent Lie algebra n.

(2) Let M, N be C^∞-manifolds and let $p \in M$, $q \in N$. Then show

$$T(M \times N, (p, q)) \cong T(M, p) \times T(N, q) \cong T_p(M) \oplus T_q(N).$$

Recall that if V is an m-dimensional vector space over R, then its **dual space** $V^* = \text{Hom}(V, R)$. Elements of V^* are called **linear functionals** and the map

$$V \times V^* \to R : (X, f) \to f(X)$$

is bilinear and is frequently written

$$f(X) = \langle X, f \rangle.$$

Now for any basis $X_1, \ldots, X_m$ of V we have the **dual basis** $f_1^*, \ldots, f_m^*$ given by

$$f_j^*(X_i) = \langle X_i, f_j^* \rangle = \delta_{ij}.$$

From this we see any $X \in V$ can be written in the form

$$X = \sum f_i^*(X) X_i.$$

Definition 2.19 The **cotangent space** at $p \in M$ is the dual space of $T_p(M)$ and is denoted by $T_p^*(M)$, or $T^*(M, p)$, or M_p^*. The elements of $T^*(M, p)$ are frequently called **differentials** at p and $T^*(M, p)$ is also called the **space of differentials** at p.

Now let $f \in F(p)$ and define the element $df \in T^*(M, p)$ by

$$df : T(M, p) \to R : L \to L(f);$$

that is, $df(L) = \langle L, df \rangle = L(f)$. Sometimes a more specific notation $df(p)$ or df_p will be used. In particular if (U, x) is a chart with x_i the coordinate functions, then a basis for $T(M, p)$ is given by $\partial_1(p), \ldots, \partial_m(p)$ and a dual basis for $T^*(M, p)$ is given by $dx_1(p), \ldots, dx_m(p)$ because they satisfy

$$\langle \partial_i(p), dx_j(p) \rangle = \partial(x_j)/\partial x_i(p) = \delta_{ij}.$$

Now for any $L \in T(M, p)$ and any $f \in F(p)$ we have from Proposition 2.18 that $L = \sum L(x_i)\, \partial_i(p)$ and therefore

$$\begin{aligned} df(L) = L(f) &= \sum L(x_i)(\partial_i f)(p) \\ &= \sum (\partial_i f)(p) L(x_i) \\ &= \sum (\partial_i f)(p)\, dx_i(L); \end{aligned}$$

that is,

$$df(p) = \sum (\partial_i f)(p)\, dx_i(p).$$

Combining various facts we have the following result.

Proposition 2.20 Let M be an m-dimensional manifold and let $f_1, \ldots, f_r \in F(p)$ for $p \in M$.

(a) Each $f \in F(p)$ equals $g(f_1, \ldots, f_r)$ on a suitable neighborhood $V = V(f)$ of p, where $g : R^r \to R$ is of class C^∞ if and only if $df_1(p), \ldots, df_r(p)$ generate the cotangent space $T^*(M, p)$.

(b) The functions $f_1, \ldots, f_m$ (that is, $r = m$) are the coordinates of some chart (U, f) at p where $f = (f_1, \ldots, f_m)$ if and only if the set $df_1(p), \ldots, df_m(p)$ is a basis of $T^*(M, p)$.

PROOF (a) Let (U, x) be a chart at p and suppose each $f \in F(p)$ equals $g(f_1, \ldots, f_r)$ on $V \cap U$. Then each of the coordinate functions

$$x_i = g_i(f_1, \ldots, f_r)$$

and therefore $dx_i = \sum \partial_k g_i(p)\, df_k$. However, since the dx's generate $T^*(M, p)$, the df's also generate $T^*(M, p)$. Conversely, assume the df's generate $T^*(M, p)$ and represent f_i in coordinates

$$f_i = h_i(x_1, \ldots, x_m).$$

Then we obtain

$$df_i = \sum \partial_k h_i(p)\, dx_k,$$

$i = 1, \ldots, r$. Now since the df's generate $T^*(M, p)$ the $m \times r$ matrix $(\partial_k h_i(p))$ has rank $m \le r$. Thus we can assume that there exists a system of m functions

$$f_{i_j} = h_{i_j}(x_1, \ldots, x_m) \qquad \text{for} \quad j = 1, \ldots, m$$

which define a function $F: R^m \to R^m : x = (x_1, \ldots, x_m) \to (F_1(x), \ldots, F_m(x))$ where $F_j(x) = f_{i_j}(x)$. Also $DF(x(p))$ is invertible so that by the inverse function theorem we can write locally

$$x_i = k_i(f_{i_1}, \ldots, f_{i_m}),$$

where $k_i : R^m \to R$ are C^∞. However, each $f \in F(p)$ equals $G(x_1, \ldots, x_m)$ locally, where G is C^∞, and using the above expression for x's in terms of f_j's we have the results.

To show (b) just note that for $r = m$ we have $df_1, \ldots, df_m$ generate $T_p^*(M)$ if and only if they form a basis. Then we can use the above equations expressing $x_i = k_i(f_1, \ldots, f_m)$ and $f_j = h_j(x_1, \ldots, x_m)$ to see $f_1, \ldots, f_m$ are coordinates for some chart at $p \in M$.

Exercise (3) Let U be open in R^m and let $f: U \to R$ be of class C^∞. Compare $Df(p)$ and $df(p)$ for $p \in U$.

5. Tangent Maps (Differentials)

In the preceding section we considered a C^∞-map g from the manifold M into the manifold R and noted that the differential $df(p)$ is a linear map from the tangent space $T(M, p)$ into the vector space $R \cong T(R, f(p))$; this isomorphism uses example (2) of Section 2.4. We shall generalize this situation by showing that a C^∞-map $f: M \to N$ between two manifolds induces a linear map $df(p) : T(M, p) \to T(N, f(p))$. However, by means of coordinate functions this generalized situation reduces to that of the preceding section.

Definition 2.21 Let M and N be C^∞-manifolds and let $f: M \to N$ be a C^∞-mapping. The **differential of** f at $p \in M$ is the map

$$df(p) : T(M, p) \to T(N, f(p))$$

given as follows. For $L \in T(M, p)$ and for $g \in F(f(p))$, we define the action of $df(p)(L)$ on g by

$$[df(p)(L)](g) = L(g \circ f).$$

REMARKS (1) We shall frequently use the less specific notation df for $df(p)$ when there should be no confusion. Also we shall use the notation

$$Tf = Tf(p) = df(p)$$

and also call $Tf(p)$ the **tangent map** of f at p. This notation is very useful in discussing certain functors on categories involving manifolds.

(2) We note that for $g \in F(f(p))$ the function $g \circ f$ is in $F(p)$ so the operation $L(g \circ f)$ is defined. We must next show $Tf(L)$ is actually in $T(N, f(p))$ by showing it is a derivation. Thus for $g, h \in C^\infty(f(p))$,

$$\begin{aligned} Tf(L)(ag + bh) &= L(a(g \circ f) + b(h \circ f)) \\ &= aL(g \circ f) + bL(h \circ f) \\ &= a[Tf(L)](g) + b[Tf(L)](h) \end{aligned}$$

and the product rule is also easy.

The following result shows that $df(p)$ is the correct generalization for $Df(p)$ of Section 1.2, where $f: U \to W$ is a C^∞-map of an open set U in R^m and W is some Euclidean space.

Proposition 2.22 Let $f: M \to N$ be a C^∞-map of C^∞-manifolds and let $p \in M$. Then the map

$$Tf(p) : T(M, p) \to T(N, f(p))$$

is a linear transformation; that is, $Tf(p) \in \text{Hom}(T(M, p), T(N, f(p)))$. Furthermore if (U, x) is a chart at p and (V, y) is a chart at $f(p)$, then $Tf(p)$ has a matrix which is the Jacobian matrix of f represented in these coordinates.

PROOF Let $X, Y \in T(M, p)$. Then for $a, b \in R$ and $g \in F(f(p))$ we have

$$\begin{aligned} [Tf(aX + bY)](g) &= (aX + bY)(g \circ f) \\ &= aX(g \circ f) + bY(g \circ f) \\ &= [a\, Tf(X) + b\, Tf(Y)](g) \end{aligned}$$

so that $Tf(aX + bY) = a\, Tf(X) + b\, Tf(Y)$. Next let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be the given coordinate functions so that we can represent f in terms of coordinates in the neighborhood V by

$$f_k = y_k \circ f = f_k(x_1, \ldots, x_m) \qquad \text{for} \quad k = 1, \ldots, n.$$

Now let $\partial/\partial x_i = \partial_i(p)$ and $\partial/\partial y_i = \partial_i(f(p))$ determine a basis for $T(M, p)$ and $T(N, f(p))$, respectively. Thus to determine a matrix for Tf we compute its action on the basis $\partial/\partial x_i$ in $T(M, p)$. Let

$$Tf(\partial/\partial x_i) = \sum_j b_{ji}\, \partial/\partial y_j$$

be in $T(N, f(p))$. Then we evaluate the matrix (b_{ji}) using the fact that $y_k \in F(f(p))$ as follows

$$\begin{aligned} \partial f_k/\partial x_i(p) &= \partial_i(p)(y_k \circ f) \\ &= [Tf(\partial_i(p))](y_k) \\ &= \sum_j b_{ji}\, \partial(y_k)/\partial y_j = b_{ki} \end{aligned}$$

using $\partial(y_k)/\partial y_j = \delta_{kj}$. Thus $(b_{ji}) = (\partial f_j/\partial x_i(p))$ is the desired Jacobian matrix.