MAT455 - PROBLEM SET 9 - DUE ON MONDAY, NOVEMBER 28, 2011

PROBLEM 1 – [KILLING FORM I]

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $B_{\mathfrak{g}}(X,Y) = tr(ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(Y))$ be its Killing form. Show that for any $X, Y, Z \in \mathfrak{g}$

$$B_{\mathfrak{g}}(X,[Y,Z]) = B_{\mathfrak{g}}([X,Y],Z)$$

and deduce that for any $g \in G$

$$B_{\mathfrak{g}}(Ad(g)X, Ad(g)Y) = B_{\mathfrak{g}}(X,Y).$$

PROBLEM 2 – [KILLING FORM AND COMPLEXIFICATION]

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Show that the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are related by

$$B_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{g}\times\mathfrak{g}} = B_{\mathfrak{g}}.$$

PROBLEM 3 – [RADICAL]

Let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ be a classical complex Lie algebra, i.e. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C}), \mathfrak{o}(n,\mathbb{C})$. Show that the maximal solvable radical of $\mathfrak{g}$ is equal to the center of $\mathfrak{g}$.

PROBLEM 4 – [JORDAN DECOMPOSITION]

Let V be a complex vector space. Let $x, y \in End(V)$. Let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions. Show that if x and y commute, then the Jordan decomposition is $(x+y) = (x+y)_s + (x+y)_n$ with $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y_n$.

Give an example that this can fail if x and y do not commute.

PROBLEM 5 – [KILLING FORM II]

Give examples of solvable Lie algebras $\mathfrak{g}$ of dimension two and of dimension three, which have a nontrivial Killing form.