Problem 1 – [Killing form I]

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $B_\mathfrak{g}(X, Y) = tr(ad_\mathfrak{g}(X)ad_\mathfrak{g}(Y))$ be its Killing form. Show that for any $X, Y, Z \in \mathfrak{g}$

$$B_\mathfrak{g}(X, [Y, Z]) = B_\mathfrak{g}([X, Y], Z)$$

and deduce that for any $g \in G$

$$B_\mathfrak{g}(Ad(g)X, Ad(g)Y) = B_\mathfrak{g}(X, Y).$$

Problem 2 – [Killing form and complexification]

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_\mathbb{C}$ its complexification. Show that the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$ are related by

$$B_{\mathfrak{g}_\mathbb{C}}|_{\mathfrak{g} \times \mathfrak{g}} = B_{\mathfrak{g}}.$$

Problem 3 – [Radical]

Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be a classical complex Lie algebra, i.e. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}), \mathfrak{o}(n, \mathbb{C})$. Show that the maximal solvable radical of $\mathfrak{g}$ is equal to the center of $\mathfrak{g}$.

Problem 4 – [Jordan decomposition]

Let $V$ be a complex vector space. Let $x, y \in End(V)$. Let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions. Show that if $x$ and $y$ commute, then the Jordan decomposition is $(x + y) = (x + y)_s + (x + y)_n$ with $(x + y)_s = x_s + y_s$ and $(x + y)_n = x_n + y_n$.

Give an example that this can fail if $x$ and $y$ do not commute.

Problem 5 – [Killing form II]

Give examples of solvable Lie algebras $\mathfrak{g}$ of dimension two and of dimension three, which have a nontrivial Killing form.