Problem 1 – [Complete Vector fields]

Show that the vector field $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ is a complete vector field on $\mathbb{R}^2$. What about $X = \exp(-x) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$?

Problem 2 – [Surjectivity of the exponential map]

For $n = 2, 3$ show that the exponential map $\exp : \mathfrak{so}(n, \mathbb{R}) \to SO(n, \mathbb{R})$ is surjective.

Problem 3 – [Exponential Map]

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then for $t \in \mathbb{R}$ small enough:

$$\exp(tX) \exp(tY) = \exp(t(X + Y)) + O(t^2),$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Problem 4 – [Adjoint representation]

Let $G \subset GL(n, \mathbb{R})$ be a Lie subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. Show that in this case the adjoint representation $Ad : G \to GL(\mathfrak{g})$ (see previous problem sets) is given by conjugation of $G$ on $\mathfrak{g}$, i.e. for $g \in G$ and $X \in \mathfrak{g}$ we have

$$Ad(g)(X) = gXg^{-1}.$$

Problem 5 – [A universal covering map]

The aim of this exercise is to show that the group $SU(2)$ is the universal covering group of the rotation group $SO(3, \mathbb{R})$.

Consider

$$SU(2) := \{ A \in SL(2, \mathbb{C}) \mid A^* = A^{-1} \} = \{ \begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \}$$

and its Lie algebra

$$\mathfrak{su}(2) = \{ X \in \mathfrak{sl}(2, \mathbb{C}) \mid X^* + X = 0 \}.$$
(1) Construct a Lie algebra homomorphism \( \phi : SU(2) \rightarrow SO(3, \mathbb{R}) \) whose kernel is \( \{ \pm \text{Id}_2 \} \), where \( \text{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

[Hint: Use the adjoint representation of \( SU(2) \) on its Lie algebra. Show that \( b(X,Y) = -\frac{1}{2} \text{tr}(XY) \) defines a positive definite bilinear form on the real vector space underlying \( \text{su}(2) \).]

(2) Show that the differential of \( \phi \) at the identity satisfies \( (d\phi)_e : \text{su}(2) \rightarrow \text{so}(3, \mathbb{R}) \) is a Lie algebra isomorphism and deduce that \( \phi \) is a covering map.

(3) Show that \( SU(2) \) is homeomorphic to the 3-sphere \( S^3 \) and deduce that \( SU(2) \) is simply connected. Show that \( SO(3, \mathbb{R}) \) is homeomorphic to the projective space \( \mathbb{R}P^3 \). What is the fundamental group of \( SO(3, \mathbb{R}) \)?

(4) Are there any other Lie groups with Lie algebra isomorphic to \( \text{su}(2) \)?