MAT455 - PROBLEM SET 6 - DUE ON FRIDAY, OCTOBER 28, 2011

Problem 1 – [Complete Vector fields]

Show that the vector field $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is a complete vector field on $\mathbb{R}^2$. What about $X = \exp(-x)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$?

Problem 2 – [Surjectivity of the exponential map]

For $n = 2, 3$ show that the exponential map $\exp : \mathfrak{so}(n, \mathbb{R}) \to SO(n, \mathbb{R})$ is surjective.

Problem 3 – [Exponential Map]

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then for $t \in \mathbb{R}$ small enough:

$$\exp(tX)\exp(tY) = \exp(t(X+Y)) + O(t^2),$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Problem 4 – [Adjoint representation]

Let $G \subset GL(n, \mathbb{R})$ be a Lie subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. Show that in this case the adjoint representation $Ad : G \to GL(\mathfrak{g})$ (see previous problem sets) is given by conjugation of G on $\mathfrak{g}$, i.e. for $g \in G$ and $X \in \mathfrak{g}$ we have

$$Ad(g)(X) = gXg^{-1}.$$

Problem 5 – [A universal covering map]

The aim of this exercise is to show that the group $SU(2)$ is the universal covering group of the rotation group $SO(3, \mathbb{R})$.

Consider

$$SU(2) := \{A \in SL(2, \mathbb{C}) \mid A^* = A^{-1}\} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1\}$$

and its Lie algebra

$$\mathfrak{su}(2) = \{X \in \mathfrak{sl}(2, \mathbb{C}) \mid X^* + X = 0\}.$$

(1) Construct a Lie algebra homomorphism $\phi : SU(2) \to SO(3,\mathbb{R})$ whose kernel is $\{\pm Id_2\}$, where $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

[Hint: Use the adjoint representation of $SU(2)$ on its Lie algebra. Show that $b(X,Y) = -\frac{1}{2}\mathrm{tr}(XY)$ defines a positive definite bilinear form on the real vector space underlying $\mathfrak{su}(2)$.]

(2) Show that the differential of ϕ at the identity satisfies $(d\phi)_e : \mathfrak{su}(2) \to \mathfrak{so}(3,\mathbb{R})$ is a Lie algebra isomorphism and deduce that ϕ is a covering map.

(3) Show that $SU(2)$ is homeomorphic to the 3-sphere S^3 and deduce that $SU(2)$ is simply connected. Show that $SO(3,\mathbb{R})$ is homeomorphic to the projective space $\mathbb{RP}^3$. What is the fundamental group of $SO(3,\mathbb{R})$?

(4) Are there any other Lie groups with Lie algebra isomorphic to $\mathfrak{su}(2)$?