MAT455 - PROBLEM SET 2 - DUE ON WEDNESDAY, SEPTEMBER 28, 2011

Problem 1 – [Discrete normal subgroups]

Let G be a connected Lie group. Let $N < G$ be a discrete normal subgroup of G. Show that N is central.

Recall that a subgroup $N < G$ is central if $gn = ng$ for all $n \in N$ and all $g \in G$.

[Hint: For a fixed element $n \in N$ consider the map $G \to N$, $g \mapsto gng^{-1}$.]

Problem 2 – [Fundamental group]

Let G be a connected Lie group. Show that the fundamental group $\pi_1(G)$ is commutative.

[Hint: Consider the universal cover $p : \widetilde{G} \to G$ an apply Problem 1]

Problem 3 – [Trace and Determinant]

Let $M_n(\mathbb{R})$ denote the space of $n \times n$ matrices with real entries. Consider the determinant $\det : M_n(\mathbb{R}) \to \mathbb{R}$ and the trace $\mathrm{tr} : M_n(\mathbb{R}) \to \mathbb{R}$.

(1) Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$ for any $A \in M_n(\mathbb{R})$. Conclude that e^A is invertible for any $A \in M_n(\mathbb{R})$. Recall that the exponential map is defined by $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

(2) Let V be a finite dimensional real vector space and $m : V^n \to \mathbb{R}$ a multilinear map. Show that

$$(dm)_{(v_1,\cdots,v_n)}(w_1,\cdots,w_n) = \sum_{i=1}^{n} m(v_1,\cdots,v_{i-1},w_i,v_{i+1},\cdots,v_m).$$

[Recall that $(dm_v)(w) = \frac{d}{dt}_{t=0}(m(v+tw))$.

(3) Deduce that $(d\det)_{Id_n}(A) = tr(A)$, where Id_n denotes the identity matrix in $M_n(\mathbb{R})$.

Problem 4 – [Implicit Function Theorem]

Prove the implicit function theorem: Let M and N be C^∞-manifolds of dimension m and n respectively. Show that if $f : M \to N$ is smooth and the differential df has constant rank k, then for all $q \in f(M)$ the level set $f^{-1}(q)$ is a closed regular submanifold of M of dimension $m - k$.

The rank of the differential df of f is constant if the linear map $(df)_p : T_pM \to T_pN$ has the same rank for all $p \in M$.

Problem 5 – [Matrix Lie groups]

(1) Show that the Lie group topology on $GL(n,\mathbb{R})$ is the unique topology on $GL(n,\mathbb{R})$ such that the matrix exponential map $\exp : M_n(\mathbb{R}) \to GL(n,\mathbb{R})$, given by $\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}$ restricts to a diffeomorphism from some neighborhood of 0 in $M_n(\mathbb{R})$ to some neighborhood of the identity $Id_n \in GL(n,\mathbb{R})$.

(2) Consider $\mathbb{C}^{p+q}$ with the nondegenerate Hermitian form $B(x,y) = -\sum_{i=1}^{p} x_i\overline{y_i} + \sum_{j=p+1}^{q} x_j\overline{y_j}$. Then the group $U(p,q)$ is defined as

$$U(p,q) := \{A \in GL(p+q,\mathbb{C}) \,|\, B(Ax,Ay) = B(x,y) \quad \forall x,y \in \mathbb{C}^{p+q}\}.$$

Note that $U(p,q) = \{A \in GL(p+q,\mathbb{C}) \,|\, \overline{A}^T I_{p,q} A = I_{p,q}\}$, where $I_{p,q}$ is the matrix $\begin{pmatrix} -Id_p & 0 \\ 0 & Id_q \end{pmatrix}$.

Show that the matrix groups $U(p,q)$ are Lie groups for all $p,q \geq 0$.

[Hint: Use Problem 4 and adapt the argument given in class for $SL(n,\mathbb{R})$.]