MAT455 - PROBLEM SET 3 - DUE ON FRIDAY, OCTOBER 21, 2011

Problem 1 – [Covering groups]

(1) Let G be a connected Lie group, H a topological space and $p : H \to G$ a covering map. Then there exists a unique Lie group structure on H such that p is a smooth group homomorphism and the kernel of p is a discrete subgroup of H.
(2) Let G be a connected Lie group and (H, p) a covering of G equipped with the Lie group structure from above. Show that p is a local isomorphism of Lie groups and $d\phi_e$ is an isomorphism of Lie algebras.

Problem 2 – [Abelian Lie groups]

(1) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that G is abelien if and only if G is abelien.
(2) Show that D is a discrete subgroup of $\mathbb{R}^n$ if and only if D is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq n$.
(3) Deduce that any connected abelien Lie group is isomorphic to $\mathbb{R}^n \times \mathbb{T}^k$ for some n and k. (Here $\mathbb{T}^k$ denotes the k-dimensional torus $\mathbb{R}^k/\mathbb{Z}^k$.)

Problem 3 – [Exponential Map]

Let $G = S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Determine the Lie algebra $\mathfrak{g}$ of G and the exponential map $\exp : \mathfrak{g} \to G$.

Problem 4 – [One-parameter subgroups]

Let $G = GL(n, \mathbb{R})$, let $X_e \in T_eG = M_n(\mathbb{R})$ and let X be the unique invariant vector field on G defined by X_e. Show that for any $f \ni C^\infty(G)$ and every $g \in G$

$$(Xf)(g) = \frac{d}{dt}|_{t=0} f(exp(tX)g).$$

Problem 5 – [The universal cover of $SL(2, \mathbb{R})$]

(1) Construct an explicit diffeomorphism between $S^1 \times \mathbb{R} \times \mathbb{R}$ and $SL(2, \mathbb{R})$. (Hint: Use the so called Iwasawa decomposition $SL(2, \mathbb{R}) = SO(2, \mathbb{R}) \cdot A \cdot N$, where $A \cdot N$ denotes the subgroup group of upper triangular matrices in $SL(2, \mathbb{R})$.)
(2) Construct a differentiable group structure on the manifold $S^1 \times \mathbb{R} \times \mathbb{R}$ such that the resulting Lie group is isomorphic to $SL(2, \mathbb{R})$. (Hint: Use the coordinates constructed in (1).)

(3) Construct on $\mathbb{R}^3$ a Lie group structure and a surjective Lie group homomorphism $\mathbb{R}^3 \to SL(2, \mathbb{R})$ (with respect to the Lie group structure you constructed).
(4) Determine the center of $\mathbb{R}^3$ with the above Lie group structure and the kernel of ϕ.
(5) Deduce that $\mathbb{R}^3$ with this Lie group structure is the universal covering group of $SL(2, \mathbb{R})$, which is also denoted by $\widetilde{SL(2, \mathbb{R})}$.