MAT455 - PROBLEM SET 1 - DUE ON WEDNESDAY, SEPTEMBER 21, 2011

Problem 1

Let G_1 and G_2 be two groups, and $\phi : G_2 \to \mathrm{Aut}(G_1)$ a homomorphism. The semidirect product $G = G_1 \times_\phi G_2$ is the product $G_1 \times G_2$ with multiplication defined by $(x_1, x_2) \cdot (y_1, y_2) = (x_1\phi(x_2)y_1, x_2y_2)$.

(1) Show that the multiplication is associative.
(2) Determine the inverse of (x_1, x_2).
(3) Show that $H_1 = G_1 \times \{e\}$ is a normal subgroup. Here $e \in G_2$ is the identity element.
(4) Is $H_2 = \{e\} \times G_2$ a normal subgroup? Again, $e \in G_1$ denotes the identity element.

Problem 2

Let G be a topological group.

(1) Let H be a closed subgroup. Show that H is a topological group.
(2) Let $H < G$ be a normal subgroup. Show that G/H is a topological group.

Problem 3

Let G be a topological group and X a topological space. Let $\mu : G \times X \to X$ be a continuous transitive group action, that is μ is a continuous map such that

- (Associativity) $\mu(g, \mu(h, x)) = \mu(gh, x)$ for all $g, h \in G$ and all $x \in X$.
- (Identity) $\mu(e, x) = x$ for all $x \in X$.
- (Transitivity) $\mu(G, x) = X$ for all $x \in X$.

Show the following:

(1) If G is compact, then X is compact.
(2) If G is connected, then X is connected.

Problem 4

On $\mathbb{R}^{p+q}$ consider the non-degenerate symmetric bilinear form $B_{p,q}$ given by

$$B_{p,q}(x, y) := -\sum_{i=1}^{p} x_iy_i + \sum_{j=p+1}^{p+q} x_jy_j.$$

Define $O(p,q) := \{A \in GL_{p+q}(\mathbb{R}) \mid \forall x, y \in \mathbb{R}^{p+q} : B_{p,q}(Ax, Ay) = B_{p,q}(x,y)\}$ and $SO(p,q) = O(p,q) \cap SL_{p+q}(\mathbb{R})$.

(1) Show that $O(p,q)$ is compact if an only if $p = 0$ or $q = 0$.
(2) Show that $O(p,q)$ is not connected for any p, q. Show that $SO(p,q)$ is connected if and only if $p = 0$ or $q = 0$.
(3) Construct an embedding $O(p,q) \to O(p+q, \mathbb{C})$.

[Hint: Use Problem 3 for (1) and (2). For (3) use the fact that over $\mathbb{C}$ all non-degenerate symmetric bilinear forms are equivalent.]

Problem 5

Define a topology on $\mathbb{R}$ such that $(\mathbb{R}, +)$ is not a topological group.