PROBLEM 1 – [ISOMORPHISM]
Let $\mathfrak{g}$ be a Lie algebra.

(1) Show that if $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then $\mathfrak{g}/\ker(\alpha)$ and $\alpha(\mathfrak{g})$ are isomorphic as Lie algebras.

(2) Show that if $\mathfrak{h} \subset \mathfrak{k}$ are ideals of $\mathfrak{g}$ then $\mathfrak{k}/\mathfrak{h}$ is an ideal in $\mathfrak{g}/\mathfrak{h}$ and

$$(\mathfrak{g}/\mathfrak{h})/(\mathfrak{k}/\mathfrak{h})$$

is isomorphic to $\mathfrak{g}/\mathfrak{k}$ as Lie algebras.

(3) Show that if $\mathfrak{h}, \mathfrak{k}$ are ideals of $\mathfrak{g}$, then so are $\mathfrak{h} + \mathfrak{k}$ and $\mathfrak{h} \cap \mathfrak{k}$. Moreover

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k})$$

is isomorphic to $(\mathfrak{h} + \mathfrak{k})/\mathfrak{k}$ as Lie algebras.

PROBLEM 2 – [SOLVABLE LIE ALGEBRAS I]

(1) Show that subalgebras and homomorphic images of solvable Lie algebras are solvable.

(2) Show that $\mathfrak{g}$ is solvable if and only if there exists a descending chain

of ideal $\mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{0\}$ such that

- $\mathfrak{g}_{i+1} \subset \mathfrak{g}_i$ is an ideal for all $i$.
- $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian for all $i$.

(3) Show that if $\mathfrak{h}, \mathfrak{k}$ are solvable ideals of a Lie algebra $\mathfrak{g}$ then $\mathfrak{h} + \mathfrak{k}$ is a solvable ideal.

(4) Deduce that every Lie algebra contains a unique maximal solvable ideal.

PROBLEM 3 – [COMPLEXIFICATION]

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_C$ its complexification.

(1) Show that a real Lie algebra $\mathfrak{g}$ is solvable if and only if its complexification $\mathfrak{g}_C$ is solvable.

(2) Show that a real Lie algebra $\mathfrak{g}$ is nilpotent if and only if its complexification $\mathfrak{g}_C$ is nilpotent.

(3) Show that a real Lie algebra $\mathfrak{g}$ is simple (over $\mathbb{R}$) if and only if its complexification $\mathfrak{g}_C$ is simple over $\mathbb{C}$. [Recall: A Lie algebra is simple if it is of dimension $\geq 2$ and it contains no ideal except $\{0\}$ and itself.]

PROBLEM 4 – [SOLVABLE LIE ALGEBRAS II]

(1) Every finite dimensional irreducible complex representations of a solvable Lie algebra has dimension 1.

(2) Every finite dimensional irreducible real representations of a solvable Lie algebra has dimension $\leq 2$. 
Problem 5 – [Killing form]

Let $\mathfrak{g}$ be a Lie algebra and $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint representation. The Killing form on $\mathfrak{g}$ is defined as $B_{\mathfrak{g}}(X,Y) := \text{tr}(ad(X) \circ ad(Y))$ for all $X, Y \in \mathfrak{g}$.

Compute the Killing forms of the following Lie algebras:

1. $\mathfrak{sl}(2, \mathbb{R})$
2. $\mathfrak{t} \subseteq \mathfrak{sl}(2, \mathbb{R})$ the subalgebra of upper triangular matrices
3. $\mathfrak{n} \subseteq \mathfrak{sl}(2, \mathbb{R})$ the sub algebra of strictly upper triangular matrices
4. $\mathfrak{su}(2)$

Which of the Killing forms are non-degenerate? Which are degenerate? Which ones are definite?