MAT455 - PROBLEM SET 8 - DUE ON FRIDAY, NOVEMBER 18, 2011

Problem 1 – [Isomorphism]

Let $\mathfrak{g}$ be a Lie algebra.

(1) Show that if $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then $\mathfrak{g}/\ker(\alpha)$ and $\alpha(\mathfrak{g})$ are isomorphic as Lie algebras.
(2) Show that if $\mathfrak{h} \subset \mathfrak{k}$ are ideals of $\mathfrak{g}$ then $\mathfrak{k}/\mathfrak{h}$ is an ideal in $\mathfrak{g}/\mathfrak{h}$ and $(\mathfrak{g}/\mathfrak{h})/(\mathfrak{k}/\mathfrak{h})$ is isomorphic to $\mathfrak{g}/\mathfrak{k}$ as Lie algebras.
(3) Show that if $\mathfrak{h}, \mathfrak{k}$ are ideals of $\mathfrak{g}$, then so are $\mathfrak{h} + \mathfrak{k}$ and $\mathfrak{h} \cap \mathfrak{k}$. Moreover $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k})$ is isomorphic to $(\mathfrak{h} + \mathfrak{k})/\mathfrak{k}$ as Lie algebras.

Problem 2 – [Solvable Lie algebras I]

(1) Show that subalgebras and homomorphic images of solvable Lie algebras are solvable.
(2) Show that $\mathfrak{g}$ is solvable if and only if there exists a descending chain of ideal $\mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{0\}$ such that
   - $\mathfrak{g}_{i+1} \subset \mathfrak{g}_i$ is an ideal for all i.
   - $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian for all i.
(3) Show that if $\mathfrak{k}, \mathfrak{h}$ are solvable ideals of a Lie algebra $\mathfrak{g}$ then $\mathfrak{h} + \mathfrak{k}$ us a solvable ideal.
(4) Deduce that every Lie algebra contains a unique maximal solvable ideal.

Problem 3 – [Complexification]

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification.

(1) Show that a real Lie algebra $\mathfrak{g}$ is solvable if and only if its complexification $\mathfrak{g}_{\mathbb{C}}$ is solvable.
(2) Show that a real Lie algebra $\mathfrak{g}$ is nilpotent if and only if its complexification $\mathfrak{g}_{\mathbb{C}}$ is nilpotent.
(3) Show that a real Lie algebra $\mathfrak{g}$ is simple (over $\mathbb{R}$) if and only if its complexification $\mathfrak{g}_{\mathbb{C}}$ is simple over $\mathbb{C}$. [Recall: A Lie algebra is simple if it is of dimension ≥ 2 and it contains no ideal except $\{0\}$ and itself.]

Problem 4 – [Solvable Lie algebras II]

(1) Every finite dimensional irreducible complex representations of a solvable Lie algebra has dimension 1.
(2) Every finite dimensional irreducible real representations of a solvable Lie algebra has dimension ≤ 2.

PROBLEM 5 – [KILLING FORM]

Let $\mathfrak{g}$ be a Lie algebra and $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint representation. The Killing form on $\mathfrak{g}$ is defined as $B_{\mathfrak{g}}(X, Y) := \operatorname{tr}(ad(X) \circ ad(Y))$ for all $X, Y \in \mathfrak{g}$.

Compute the Killing forms of the following Lie algebras:

(1) $\mathfrak{sl}(2, \mathbb{R})$
(2) $\mathfrak{t} \subset \mathfrak{sl}(2, \mathbb{R})$ the subalgebra of upper triangular matrices
(3) $\mathfrak{n} \subset \mathfrak{sl}(2, \mathbb{R})$ the sub algebra of strictly upper triangular matrices
(4) $\mathfrak{su}(2)$

Which of the Killing forms are non-degenerate? Which are degenerate? Which ones are definite?