MAT455 - PROBLEM SET 11 - DUE ON FRIDAY, DECEMBER 16, 2011

Problem 1 – [tensor representations]

Let $\mathfrak{g}$ be a Lie algebra. Let $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ be a $\mathfrak{g}$-invariant tensor. Show that for arbitrary representations V, W of $\mathfrak{g}$ the element Ω defines an endomorphism $\Omega \in End_{\mathfrak{g}}(V \otimes W)$.

Problem 2 – [Complete reducibility]

(1) Consider the representation $\mathbb{C} \to \mathfrak{gl}(2,\mathbb{C})$, $x \mapsto \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. Show that this representation is not completely reducible.
(2) Let $\mathfrak{g}$ be a complex Lie algebra. Show that if every representation of $\mathfrak{g}$ is completely reducible, then $\mathfrak{g}$ is semi-simple.

Problem 3 – [$\mathfrak{sl}(2,\mathbb{C})$-representations]

Let $V(m)$ be the m-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Show that $V(m) \otimes V(n)$ and $Hom(V(m), V(n))$ are (as representations of $\mathfrak{sl}(2,\mathbb{C})$) isomorphic to

$$V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(|m-n|).$$

Problem 4 – [Cartan subalgebras]

Determine the Cartan subalgebras and the root system for $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$.

Problem 5 – [Invariant bilinear forms]

Let $\mathfrak{g} = \oplus_{i=1}^{n}\mathfrak{g}_i$ be a complex semisimple Lie algebra, where $\mathfrak{g}_i$ are the simple ideals. Let $B_{\mathfrak{g}}$ be the Killing form on $\mathfrak{g}$. Let $\mathcal{B}$ denote the vector space of all invariant bilinear forms B on $\mathfrak{g}\times\mathfrak{g}$, i.e. $B([X,Y],Z)+B(Y,[X,Z]) = 0$ for all $X, Y, Z \in \mathfrak{g}$. Let $B_i(X,Y) = B_{\mathfrak{g}}(\pi_i(X), \pi_i(Y))$, where $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ is the projection relative to the orthogonal decomposition above, $i = 1, \cdots, n$. Prove that $\{B_1, \cdots, B_n\}$ is a basis for $\mathcal{B}$.

What can you say when $\mathfrak{g}$ is only a real semi simple Lie algebra?