MAT455 - PROBLEM SET 11 - DUE ON FRIDAY, DECEMBER 16, 2011

Problem 1 – [Tensor representations]
Let \( g \) be a Lie algebra. Let \( \Omega \in g \otimes g \) be a \( g \)-invariant tensor. Show that for arbitrary representations \( V, W \) of \( g \) the element \( \Omega \) defines an endomorphism \( \Omega \in \text{End}_g(V \otimes W) \).

Problem 2 – [Complete reducibility]
(1) Consider the representation \( \mathbb{C} \to \mathfrak{gl}(2, \mathbb{C}), x \mapsto \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \). Show that this representation is not completely reducible.
(2) Let \( g \) be a complex Lie algebra. Show that if every representation of \( g \) is completely reducible, then \( g \) is semi-simple.

Problem 3 – [\( \mathfrak{sl}(2, \mathbb{C}) \)-representations]
Let \( V(m) \) be the \( m \)-dimensional irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \). Show that \( V(m) \otimes V(n) \) and \( \text{Hom}(V(m), V(n)) \) are (as representations of \( \mathfrak{sl}(2, \mathbb{C}) \)) isomorphic to \( V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(|m-n|) \).

Problem 4 – [Cartan subalgebras]
Determine the Cartan subalgebras and the root system for \( \mathfrak{sl}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}) \) and \( \mathfrak{so}(n, \mathbb{C}) \).

Problem 5 – [Invariant bilinear forms]
Let \( g = \bigoplus_{i=1}^n g_i \) be a complex semisimple Lie algebra, where \( g_i \) are the simple ideals. Let \( B_g \) be the Killing form on \( g \). Let \( \mathcal{B} \) denote the vector space of all invariant bilinear forms \( B \) on \( g \times g \), i.e. \( B([X, Y], Z) + B(Y, [X, Z]) = 0 \) for all \( X, Y, Z \in g \). Let \( B_i(X, Y) = B_g(\pi_i(X), \pi_i(Y)) \), where \( \pi_i : g \to g_i \) is the projection relative to the orthogonal decomposition above, \( i = 1, \cdots, n \). Prove that \( \{B_1, \cdots, B_n\} \) is a basis for \( \mathcal{B} \).

What can you say when \( g \) is only a real semi simple Lie algebra?