MAT455 - PROBLEM SET 3 - DUE ON FRIDAY, OCTOBER 14, 2011

Problem 1 – [Some Lie algebras]

(1) Let $M, N$ be smooth manifolds and $f : M \to N$ a smooth map such that the differential $df$ has constant rank equal to $r$. Then for every $q \in N$ the level set $M' = f^{-1}(q)$ is a closed regular submanifold of $M$ of dimension $\dim(M) - r$.

Show that for every $p \in M'$ we have a canonical identification $T_pM' = \ker(df_p)$.

(2) Compute the Lie algebra of the Lie groups $U(p,q)$, $Sp(2n)$, $B(n)$ and $N(n)$, where $B(n)$ is the group of real invertible upper triangular $n \times n$ matrices, and $N(n)$ is the group of real invertible upper triangular $n \times n$ matrices with 1’s on the diagonal.

Problem 2 – [$\phi$-related vector fields]

Let $M$ and $M'$ be smooth manifolds and $\phi : M \to M'$ a smooth map.

(1) The vector fields $X \in Vect(M)$ and $X' \in Vect(M')$ are said to be $\phi$-related if $d\phi \circ X = X' \circ \phi$. Show that if $X_i \in Vect(M)$ is $\phi$-related to $X'_i \in Vect(M')$ ($i = 1, 2$), then $[X_1, X_2]$ is $\phi$-related to $[X'_1, X'_2]$.

(2) Let $\phi$ be a diffeomorphism. The push-forward $\phi_*(X)$ of a vector field $X \in Vect(M)$ is defined by $(\phi_*(X))_{m'} = d\phi^{-1}(m') (X_{\phi^{-1}(m')})$ for $m' \in M'$. Show that $\phi_* : Vect(M) \to Vect(M')$ is a Lie algebra isomorphism.

Problem 5 – [More on Lie algebra]

(1) Let $\mathfrak{g}$ be a Lie algebra. Show that $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), X \mapsto [X, Y]$ is a Lie algebra homomorphism.

(2) Let $G$ be a Lie group. Define $c_g : G \to G$ by $c_g(h) = ghg^{-1}$ and let $Ad(g) = (dc_g)_e : \mathfrak{g} \to \mathfrak{g}$ be its differential. Show that $Ad : G \to GL(\mathfrak{g})$ is a homomorphism of Lie groups, whose image consists of Lie algebra automorphisms of $\mathfrak{g}$. Show that the Lie algebra homomorphism associated to $Ad$ is $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

(3) Let $H < G$ be a Lie subgroup of $G$. Assume that $H$ is normal. Show that the associated Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal.

(4) A Lie algebra $\mathfrak{g}$ is said to be simple if it is non-abelian and has no ideals except $\{0\}$ and itself. Find a basis $\{e, f, h\}$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Show that $\mathfrak{sl}(2, \mathbb{C})$ is simple.
**Problem 4 – [Frobenius theorem - easy direction]**

Prove that if $\mathcal{D}$ is a smooth distribution on a smooth manifold $M$ which admits maximal integral submanifolds through each point of $M$, then $\mathcal{D}$ is involutive.

**Problem 5 – [Distributions and surfaces]**

Consider the system of Partial Differential Equations

$$\frac{\partial z}{\partial x} = h(x, y, z)$$
$$\frac{\partial z}{\partial y} = g(x, y, z).$$

Set $X := \frac{\partial}{\partial x} + h \frac{\partial}{\partial z}$ and $Y := \frac{\partial}{\partial y} + g \frac{\partial}{\partial z}$.

1. Show that if $z = f(x, y)$ is a solution of the system which describes a surface $\Sigma = \text{graph}(f) \subset \mathbb{R}^3$, then for every $p \in \Sigma$ the tangent space $T_p \Sigma$ is spanned by $X_p$ and $Y_p$.

2. The order of differentiation of $f$ can be exchanged if and only if the distribution defined by $X$ and $Y$ is involutive.