MAT455 - PROBLEM SET 3 - DUE ON FRIDAY, OCTOBER 14, 2011

Problem 1 – [Some Lie algebras]

(1) Let M, N be smooth manifolds and $f : M \to N$ a smooth map such that the differential df has constant rank equal to r. Then for every $q \in N$ the level set $M' = f^{-1}(q)$ is a closed regular submanifold of M of dimension $\dim(M) - r$.

Show that for every $p \in M'$ we have a canonical identification

$$T_pM' = ker(df_p).$$

(2) Compute the Lie algebra of the Lie groups $U(p, q)$, $Sp(2n)$, $B(n)$ and $N(n)$, where $B(n)$ is the group of real invertible upper triangular $n\times n$ matrices, and $N(n)$ is the group of real invertible upper triangular $n \times n$ matrices with 1's on the diagonal.

Problem 2 – [ϕ-related vector fields]

Let M and M' be smooth manifolds and $\phi : M \to M'$ a smooth map.

(1) The vector fields $X \in Vect(M)$ and $X' \in Vect(M')$ are said to be ϕ-related if $d\phi \circ X = X' \circ \phi$. Show that if $X_i \in Vect(M)$ is ϕ-related to $X'_i \in Vect(M')$ $(i = 1, 2)$, then $[X_1, X_2]$ is ϕ-related to $[X'_1, X'_2]$.

(2) Let ϕ be a diffeomorphism. The push-forward $\phi_*(X)$ of a vector field $X \in Vect(M)$ is defined by $(\phi_*X)_{m'} = d\phi_{\phi^{-1}(m')}(X_{\phi^{-1}(m')})$ for $m' \in M'$. Show that $\phi_* : Vect(M) \to Vect(M')$ is a Lie algebra isomorphism.

Problem 5 – [More on Lie algebra]

(1) Let $\mathfrak{g}$ be a Lie algebra. Show that $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $X \mapsto [X, Y]$ is a Lie algebra homomorphism.

(2) Let G be a Lie group. Define $c_g : G \to G$ by $c_g(h) = ghg^{-1}$ and let $Ad(g) = (dc_g)_e : \mathfrak{g} \to \mathfrak{g}$ be its differential. Show that $Ad : G \to GL(\mathfrak{g})$ is a homomorphism of Lie groups, whose image consists of Lie algebra automorphisms of $\mathfrak{g}$. Show that the Lie algebra homomorphism associated to Ad is $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

(3) Let $H < G$ be a Lie subgroup of G. Assume that H is normal. Show that the associated Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal.

(4) A Lie algebra $\mathfrak{g}$ is said to be simple if it is non-abelien and has no ideals except $\{0\}$ and itself. Find a basis $\{e, f, h\}$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Show that $\mathfrak{sl}(2, \mathbb{C})$ is simple.

Problem 4 – [Frobenius theorem - easy direction]

Prove that if $\mathcal{D}$ is a smooth distribution on a smooth manifold M which admits maximal integral submanifolds through each point of M, then $\mathcal{D}$ is involutive.

Problem 5 – [Distributions and surfaces]

Consider the system of Partial Differential Equations

$$\frac{\partial z}{\partial x} = h(x, y, z)$$
$$\frac{\partial z}{\partial y} = g(x, y, z).$$

Set $X := \frac{\partial}{\partial x} + h\frac{\partial}{\partial z}$ and $Y := \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}$.

(1) Show that if $z = f(x, y)$ is a solution of the system which describes a surface $\Sigma = graph(f) \subset \mathbb{R}^3$, then for every $p \in \Sigma$ the tangent space $T_p\Sigma$ is spanned by X_p and Y_p.

(2) The order of differentiation of f can be exchanged if and only if the distribution defined by X and Y is involutive.