MAT455 - PROBLEM SET 10 - DUE ON FRIDAY, DECEMBER 9, 2011

Problem 1 – [Simple Lie algebras]

Show that the Lie algebra $\mathfrak{sl}(n, \mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, is simple.

Problem 2 – [Examples]

(1) Give an example of a Lie algebra $\mathfrak{g}$ with a non-zero radical and a non-degenerate invariant symmetric bilinear form.
(2) Give an example of a solvable Lie algebra whose Killing form is not identically zero.
(3) Give an example of a solvable but non-nilpotent Lie algebra whose Killing form is identically zero.
(4) Let $\mathfrak{g}$ be a Lie algebra over some field k and let $F = End_{\mathfrak{g}}(V)$ be the space of endomorphisms of V which commute with the action of $\mathfrak{g}$. Show that F is a field. Give an example where F is not commutative.

Problem 3 – [Real forms]

Let G be a complex connected Lie group, let K be a real submanifold of G which is also a subgroup. Let $\mathfrak{g}$ (defined over $\mathbb{C}$) and $\mathfrak{k}$ (defined over $\mathbb{R}$) denote the corresponding Lie algebras.

(1) Assume $\mathfrak{k} + i\mathfrak{k} = \mathfrak{g}$. Show that if a complex submanifold H of G which is also a subgroup contains K, then $H = G$.
(2) Show that the property $\mathfrak{k} + i\mathfrak{k} = \mathfrak{g}$ is satisfied in the following examples:
   - $G = SL(n, \mathbb{C})$, $K = SU(n)$
   - $G = SO(n, \mathbb{C})$, $K = SO(n)$
   - $G = Sp(2n, \mathbb{C})$, $K = SU(2n) \cap G = Sp(2n)$.

Problem 4 – [Reductive part of a Lie algebra]

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. Prove that $[\mathfrak{r}, \mathfrak{g}]$ is the smallest of the ideals $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is reductive.