MAT455 - PROBLEM SET 7 - DUE ON FRIDAY, NOVEMBER 11, 2011

Problem 1 – [Connected Lie groups]

Classify all connected Lie groups of dimension $d \leq 2$.

Problem 2 – [Subgroups of Lie groups]

Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$. Let $U \subset \mathfrak{g}$ be a neighborhood of $0 \in \mathfrak{g}$ and $V \subset G$ a neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ us a diffeomorphism. Suppose that

$$\exp(U \cap \mathfrak{h}) = V \cap H.$$

Show that

(1) H is a Lie subgroup of G with the relative topology.
(2) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.
(3) $\mathfrak{h}$ is the Lie algebra of H.

Problem 3 – [Lie derivative]

Let X and Y be smooth vector fields on a manifold M. Recall that the Lie dervative $(L_X Y)_m$ of Y with respect to X at $m \in M$ is defined by

$$(L_X Y)_m = \lim_{t \to 0} \frac{dX_{-t}(Y_{X_t(m)}) - Y_m}{t} = \frac{d}{dt}|_{t=0}(dX_{-t}(Y_{X_t(m)})),$$

where X_t is the (local) one -parameter group of transformations associated to X. Show that $L_X Y = [X, Y]$, where $[X, Y]$ is the Lie bracket of vector fields, defined by

$$[X, Y](f) = X_m(Yf) - Y_m(Xf),$$

for all $f \in C^\infty(M)$.

Problem 4 – [$SL(2, \mathbb{C})$]

Let $SL(2, \mathbb{C})$ act on itself by conjugation. Determine all the orbits.

Problem 5 – [Haar measure]

(1) Let G be a real Lie group of dimension n. Let V be the real vector space of all smooth n-forms which are invariant under left translation by G. Show that $dim(V) = 1$. Prove that for any $\omega \in V$, the corresponding Borel measure on G is a left Haar measure.

(2) Let μ be a left Haar measure for G and Δ the modular function. Prove that $\Delta(y) = det Ad(y)^{-1}$ for all $y \in G$. Show that G is unimodular if an only of $\mathrm{tr} ad(X) = 0$ for all $X \in \mathfrak{g}$. Deduce that all classical Lie groups are unimodular. Give an example of a group which is not unimodular.

(3) Let $G = GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$. Let dX denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Show that $d\mu = \frac{dX}{|det(X)|^n}$ is a left Haar measure on G. Obtain the corresponding formula for $GL(n, \mathbb{C})$. [Note: what I wrote in class today $d\mu = \frac{dX}{|det(X)|}$ was wrong.]