MAT455 - PROBLEM SET 3 - DUE ON WEDNESDAY, OCTOBER 5, 2011

Problem 1 – [Topological groups - local to global]

(1) Let X be a topological space with admits a transitive continuous action of a topological group. Let $x \in X$ be an arbitrary point. Show that the topology of X is uniquely determined by the collection of neighborhoods of x.
(2) Let G, H be connected topological groups and $f : G \to H$ a group homomorphism. Suppose that the restriction of f to some neighborhood U of the identity element $e \in G$ is a homeomorphism onto its image. Show that then f is a homeomorphism (and thus an isomorphism of topological groups).

Problem 2 – [One-parameter subgroups]

Let G be a Lie group. A *one-parameter subgroup* is a continuous homomorphism $\mathbb{R} \to G$.

Show that the differentiable one-parameter subgroup of $GL(n, \mathbb{R})$ are precisely the maps

$$\gamma_X : \mathbb{R} \to GL(n, \mathbb{R}), \quad t \mapsto \exp(tX)$$

for $X \in M(n, \mathbb{R})$. Conclude that every differentiable one parameter subgroup of $GL(n, \mathbb{R})$ is analytic.

Problem 3 – [Linear Actions]

Let G be a group and V, V_1, V_2 vector spaces. An action $\mu : G \times V \to V$ is said to be *linear* if the maps $\mu(g, \cdot)$ are linear for all $g \in G$.

(1) Show that linear actions $\mu : G \times V \to V$ are in 1-1 correspondence with group homomorphisms $G \to GL(V)$.
(2) Let $\mu_i : G \times V_i \to V_i$ be linear actions. Define a linear action $(\mu_1 \oplus \mu_2)$ of G on $V_1 \oplus V_2$, and a linear action $(\mu_1 \otimes \mu_2)$ of G on $V_1 \otimes V_2$.
(3) Show that if $\mu : G \times V \to V$ is a linear action and $V^* = \mathrm{Hom}(V, \mathbb{R})$ denotes the dual space, then there is a unique linear action $\mu^* : G \times V^* \to V^*$ satisfying

$$\mu^*(g, a)(\mu(g, v)) = \alpha(v).$$

Problem 4 – [Homogeneous Spaces]

Let G be a topological group.

(1) If $H < G$ is a subgroup then the projection $G \to G/H$ is open, where G/H is endowed with the quotient topology.
(2) Let X be a Hausdorff topological space and $\mu : G \times X \to X$ a continuous transitive group action. If $\mu_{x_0} : G \to X$, $g \mapsto \mu(g, x_0)$ is open then $X \cong G/G_{x_0}$, where G_{x_0} denotes the stabilizer of x_0 in G.
(3) If G is a Lie group, X a smooth manifold and $\mu : G \times X \to X$ a smooth transitive action then $d\mu_{x_0}$ has constant rank for every $x_0 \in X$ and $X \cong G/G_{x_0}$ as topological spaces.
(4) Find Lie groups G and H such that G/H is homeomorphic to
   - the projective space $\mathbb{RP}^n$,
   - the unit disc $\mathbb{D} \subset \mathbb{R}^2$,
   - the quadric $Q_{p,q} = \{v \in \mathbb{R}^{p+q} \backslash \{0\} \,|\, B_{p,q}(v, v) = 0\}$, where $B_{p,q}$ is the non-degenerate symmetric bilinear form of signature (p, q) on $\mathbb{R}^{p+q}$ given by $B_{p,q}(x, y) = -\sum_{i=1}^{p} x_i y_i + \sum_{j=1}^{q} x_{p+j} y_{p+j}$.

   Try to find more than one pair of groups G, H if you can.

Problem 5 – [The symplectic group]

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or more generally a field of characteristic zero). Consider $\mathbb{K}^{2n}$ with the non-degenerate skew-symmetric form F, determined by $F(x, y) = \sum_{i=1}^{n} x_i y_{n+i} - x_{n+i} y_i$. The symplectic group $Sp(2n, \mathbb{K})$ is defined as

$$Sp(2n, \mathbb{K}) := \{A \in GL(2n, \mathbb{K}) \,|\, F(Ax, Ay) = F(x, y) \quad \forall x, y \in \mathbb{K}^{2n}\}.$$

Expressing F by the matrix $F = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$ one gets

$$Sp(2n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \,|\, A^T F A = J\}.$$

(1) A complex structure on $\mathbb{R}^{2n}$ is a linear endomorphism $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $J^2 = -id$, where id is the identity. A complex structure J is said to be compatible with F if $F(Jx, Jy) = F(x, y)$ for all $x, y \in \mathbb{R}^{2n}$ and $F(x, Jx) > 0$ for all $x \in \mathbb{R}^{2n}$. Denote the space of compatible complex structures on $\mathbb{R}^{2n}$ by $\mathcal{J}$. Show that $Sp(2, \mathbb{R})$ acts transtively on $\mathcal{J}$ and determine the stabilizer group of $J_0 = \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}$.
(2) A linear subspace $V \subset \mathbb{K}^{2n}$ is said to be isotropic (with respect to F) if $F(v, w) = 0$ for all $v, w \in V$. A subspace $L \subset \mathbb{K}^{2n}$ is called a Lagrangian subspace if L is isotropic and $\dim(L) = n$. Denote the space of all Lagrangian subspaces of $\mathbb{K}^{2n}$ by $\mathcal{L}(\mathbb{K}^{2n})$. Show that $Sp(2n, \mathbb{K})$ acts transitively on $\mathcal{L}(\mathbb{K}^{2n})$. Determine the stabilizer of the Lagrangian subspace L_0, which is spanned by the first n basis vectors of $\mathbb{K}^{2n}$.