CHAPTER 1

DIFFERENTIABLE AND ANALYTIC MANIFOLDS

1.1. Differentiable Manifolds

We shall devote this chapter to a summary of those concepts and results from the theory of differentiable and analytic manifolds which are needed for our work in the rest of the book. Most of these results are standard and adequately treated in many books (see for example Chevalley [1], Helgason [1], Kobayashi and Nomizu [1], Bishop and Crittenden [1], Narasimhan [1]).

Differentiable structures. For technical reasons we shall permit our differentiable manifolds to have more than one connected component. However, all the manifolds that we shall encounter are assumed to satisfy the second axiom of countability and to have the same dimension at all points. More precisely, let M be a Hausdorff topological space satisfying the second axiom of countability. By a *(C^∞) differentiable structure* on M we mean an assignment

$$\mathfrak{D} : U \mapsto \mathfrak{D}(U) \quad (U \text{ open}, \subseteq M)$$

with the following properties:

(i) for each open $U \subseteq M$, $\mathfrak{D}(U)$ is an algebra of complex-valued functions on U containing 1 (the function identically equal to unity)

(ii) if V, U are open, $V \subseteq U$ and $f \in \mathfrak{D}(U)$, then $f|V \in \mathfrak{D}(V)$;[1] if V_i ($i \in J$) are open, $V = \cup_i V_i$, and f is a complex-valued function defined on V such that $f|V_i \in \mathfrak{D}(V_i)$ for all $i \in J$, then $f \in \mathfrak{D}(V)$

(iii) there exists an integer $m > 0$ with the following property: for any x $y \in M$, one can find an open set U containing x, and m *real* functions $x_1, \ldots, x_m$ from $\mathfrak{D}(U)$ such that (a) the map

$$\xi : y \mapsto (x_1(y), \ldots, x_m(y))$$

is a homeomorphism of U onto an open subset of $\mathbf{R}^m$ (real m-space), and (b)

[1] If F is any function defined on a set A, and $B \subseteq A$, then $F|B$ denotes the restriction of F to B.

for any open set $V \subseteq U$ and any complex-valued function f defined on V, $f \in \mathfrak{D}(V)$ if and only if $f \circ \xi^{-1}$ is a C^∞ function on $\xi[V]$.

Any open set U for which there exist functions $x_1, \ldots, x_m$ having the property described in (iii) is called a *coordinate patch;* $\{x_1, \ldots, x_m\}$ is called a *system of coordinates on U*. Note that for any open $U \subseteq M$, the elements of $\mathfrak{D}(U)$ are continuous on U.

It is not required that M be connected; it is, however, obviously locally connected and metrizable. The integer m in (iii) above, which is the same for all points of M, is called the *dimension* of M. The pair $(M, \mathfrak{D})$ is called *differentiable* (C^∞) *manifold.* By abuse of language, we shall often refer to M itself as a differentiable manifold. It is usual to write $C^\infty(U)$ instead of $\mathfrak{D}(U)$ for any open set $U \subseteq M$ and to refer to its elements as (C^∞) *differentiable functions* on U. If U is any open subset of M, the assignment $V \mapsto C^\infty(V)$ ($V \subseteq U$, open) gives a C^∞ structure on U. U, equipped with this structure, is a C^∞ manifold having the same dimension as M; it is called the *open submanifold* defined by U. The connected components of M are all open submanifolds of M, and there can be at most countably many of these.

Let k be an integer ≥ 0, $U \subseteq M$ any open set. A complex-valued function f defined on U is said to be of *class* C^k on U if, around each point of U, f is a k-times continuously differentiable function of the local coordinates. It is easy to see that this property is independent of the particular set of local coordinates used. The set of all such f is denoted by $C^k(U)$. (We omit k when $k = 0$: $C(U) = C^0(U)$. $C^k(U)$ is an algebra over the field of complex numbers $\mathbf{C}$ and contains $C^\infty(U)$.

Given any complex-valued function f on M, its *support*, supp f, is defined as the complement in M of the largest open set on which f is identically zero. For any open set U and any integer k with $0 \leq k \leq \infty$, we denote by $C_c^k(U)$ the subspace of all $f \in C^k(M)$ for which supp f is a compact subset of U.

There is no difficulty in constructing nontrivial elements of $C^\infty(M)$. We mention the following results, which are often useful.

(i) Let $U \subseteq M$ be open and $K \subseteq U$ be compact; then we can find $\varphi \in C^\infty(M)$ such that $0 \leq \varphi(x) \leq 1$ for all x, with $\varphi = 1$ in an open set containing K, and $\varphi = 0$ outside U.

(ii) Let $\{V_i\}_{i \in J}$ be a locally finite[2] open covering of M with $Cl(V_i)$ (Cl denoting closure) compact for all $i \in J$; then there are $\varphi_i \in C^\infty(M) (i \in J)$ such that

(a) for each $i \in J$ $\varphi_i \geq 0$ and supp φ_i is a (compact) subset of V_i

(b) $\sum_{i \in J} \varphi_i(x) = 1$ for all $x \in M$ (this is a finite sum for each x, since $\{V_i\}_{i \in J}$ is locally finite).

$\{\varphi_i\}_{i \in J}$ is called a *partition of unity subordinate to the covering* $\{V_i\}_{i \in J}$.

[2]A family $\{E_i\}_{i \in J}$ of subsets of a topological space S is called *locally finite* if each point of X has an open neighborhood which meets E_i for only finitely many $i \in J$.

Tangent vectors and differential expressions. Let M be a C^∞ manifold of dimension m, fixed throughout the rest of this section. Let $x \in M$. Two C^∞ functions defined around x are called *equivalent if* they coincide on an open set containing x. The equivalence classes corresponding to this relation are known as *germs of C^∞ functions at x*. For any C^∞ function f defined around x we write $\mathbf{f}_x$ for the corresponding germ at x. The algebraic operations on the set of differentiable functions give rise in a natural and obvious fashion to algebraic operations on the set of germs at x, converting the latter into an algebra over $\mathbf{C}$; we denote this algebra by $\mathbf{D}_x$. A germ is called *real* if it is defined by a real C^∞ function. The real germs form an algebra over $\mathbf{R}$. For any germ $\mathbf{f}$ at x we write $\mathbf{f}(x)$ to denote the common value at x of all the C^∞ functions belonging to $\mathbf{f}$. It is easily seen that any germ at x is determined by a C^∞ function defined on all of M.

Let $\mathbf{D}_x^*$ be the algebraic dual of the complex vector space $\mathbf{D}_x$, i.e., the complex vector space of all linear maps of $\mathbf{D}_x$ into $\mathbf{C}$. An element of $\mathbf{D}_x^*$ is said to be *real* if it is real-valued on the set of real germs. A *tangent vector to M at x* is an element v of $\mathbf{D}_x^*$ such that

$$(1.1.1) \qquad \begin{cases} \text{(i)} & v \text{ is real} \\ \text{(ii)} & v(\mathbf{fg}) = \mathbf{f}(x)v(\mathbf{g}) + \mathbf{g}(x)v(\mathbf{f}) \text{ for all } \mathbf{f}, \mathbf{g} \in \mathbf{D}_x. \end{cases}$$

The set of all tangent vectors to M at x is an $\mathbf{R}$-linear subspace of $\mathbf{D}_x^*$, and is denoted by $T_x(M)$; it is called the *tangent space to M at x*. Its complex linear span $T_{xc}(M)$ is the set of all elements of $\mathbf{D}_x^*$ satisfying (ii) of (1.1.1). Let U be a coordinate patch containing x with $x_1, \ldots, x_m$ a system of coordinates on U, and let

$$\tilde{U} = \{(x_1(y), \ldots, x_m(y)) : y \in U\}.$$

For any $f \in C^\infty(U)$ let $\tilde{f} \in C^\infty(\tilde{U})$ be such that $\tilde{f} \circ (x_1, \ldots, x_m) = f$. Then the maps

$$f \mapsto \left(\frac{\partial \tilde{f}}{\partial t_j}\right)_{t_1 = x_1(x), \ldots, t_m = x_m(x)}$$

for $1 \leq j \leq m$ ($t_1, \ldots, t_m$ being the usual coordinates on $\mathbf{R}^m$) induce linear maps of $\mathbf{D}_x$ into $\mathbf{C}$ which are easily seen to be tangent vectors; we denote these by $(\partial/\partial x_j)_x$. They form a basis for $T_x(M)$ over $\mathbf{R}$ and hence of $T_{xc}(M)$ over $\mathbf{C}$.

Define the element $1_x \in \mathbf{D}_x^*$ by

$$(1.1.2) \qquad 1_x(\mathbf{f}) = \mathbf{f}(x) \quad (\mathbf{f} \in \mathbf{D}_x).$$

1_x is real and linearly independent of $T_x(M)$. It is easy to see that for an element $v \in \mathbf{D}_x^*$ to belong to the complex linear span of 1_x and $T_x(M)$ it is necessary and sufficient that $v(\mathbf{f}_1\mathbf{f}_2) = 0$ for all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{D}_x$ which vanish at x. This leads naturally to the following generalization of the concept of a tangent vector. Let

$$\mathbf{J}_x = \{\mathbf{f} : \mathbf{f} \in \mathbf{D}_x, \mathbf{f}(x) = 0\} \tag{1.1.3}$$

Then $\mathbf{J}_x$ is an ideal in $\mathbf{D}_x$. For any integer $p \geq 1$, $\mathbf{J}_x^p$ is defined to be the linear span of all elements which are products of p elements from $\mathbf{J}_x$; $\mathbf{J}_x^p$ is also an ideal in $\mathbf{D}_x$. For any integer $r \geq 0$ we define a *differential expression of order* $\leq r$ to be any element of $\mathbf{D}_x^*$ which vanishes on $\mathbf{J}_x^{r+1}$; the set of all such is a linear subspace of $\mathbf{D}_x^*$ and is denoted by $T_{xc}^{(r)}(M)$. The real elements in $T_{xc}^{(r)}(M)$ from an **R**-linear subspace of $T_{xc}^{(r)}(M)$, spanning it (over **C**), and is denoted by $T_x^{(r)}(M)$. We have $T_x^{(0)}(M) = \mathbf{R}\cdot 1_x$, $T_x^{(1)}(M) = \mathbf{R}\cdot 1_x + T_x(M)$, and $T_x^{(r)}(M)$ increases with increasing r. Put

$$\begin{aligned} T_x^{(\infty)}(M) &= \bigcup_{r\geq 0} T_x^{(r)}(M) \\ T_{xc}^{(\infty)}(M) &= \bigcup_{r\geq 0} T_{xc}^{(r)}(M). \end{aligned} \tag{1.1.4}$$

$T_{xc}^{(\infty)}(M)$ is a linear subspace of $\mathbf{D}_x^*$, and $T_x^{(\infty)}(M)$ is an **R**-linear subspace spanning it over **C**.

It is easy to construct natural bases of the $T_x^{(r)}(M)$ in local coordinates. Let U be a coordinate patch containing x and let $\tilde{U}$ and $x_1, \ldots, x_m$ be as in the discussion concerning tangent vectors. Let (α) be any multiindex, i.e., $(\alpha) = (\alpha_1, \ldots, \alpha_m)$ where the α_j are integers ≥ 0; put $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Then the map

$$f \mapsto \left(\frac{\partial^{|\alpha|}\tilde{f}}{\partial t_1^{\alpha_1}\cdots\partial t_m^{\alpha_m}}\right)_{t_1 = x_1(x), \ldots, t_m = x_m(x)} \qquad (f \in C^\infty(U))$$

induces a linear function on $\mathbf{D}_x$ which is real. Let $\partial_x^{(\alpha)}$ denote this (when $(\alpha) = (0)$, $\partial_x^{(\alpha)} = 1_x$). Clearly, $\partial_x^{(\alpha)} \in T_x^{(r)}(M)$ if $|\alpha| \leq r$.

Lemma 1.1.1. *Let $r \geq 0$ be an integer and let $x \in M$. Then the differential expressions $\partial_x^{(\alpha)}$ $(|\alpha| \leq r)$ form a basis for $T_x^{(r)}(M)$ over* **R** *and for $T_{xc}^{(r)}(M)$ over* **C**.

Proof. Since this is a purely local result, we may assume that M is the open cube $\{(y_1, \ldots, y_m) : |y_j| < a \text{ for } 1 \leq j \leq m\}$ in $\mathbf{R}^m$ with x as the origin. Let $t_1, \ldots, t_m$ be the usual coordinates, and for any multiindex $(\beta) = (\beta_1, \ldots, \beta_m)$ let $\mathbf{t}^{(\beta)}$ denote the germ at the origin defined by $t_1^{\beta_1} \ldots t_m^{\beta_m}/\beta_1! \cdots \beta_m!$

Let f be a real C^∞ function on M and let $g_{x_1,\ldots,x_m}(t) = f(tx_1, \ldots, tx_m)$ $(-1 \leq t \leq 1, (x_1, \ldots, x_m) \in M)$. By expanding $g_{x_1,\ldots,x_m}$ about $t = 0$ in its Taylor series, we get

$$g_{x_1,\ldots,x_m}(t) = \sum_{0\leq s\leq r} \frac{t^s}{s!} g_{x_1,\ldots,x_m}^{(s)}(0) + \frac{1}{r!}\int_0^t (t-u)^r g_{x_1,\ldots,x_m}^{(r+1)}(u)\,du$$

for $0 \leq t \leq 1$. Putting $t = 1$ and evaluating the t-derivatives of $g_{x_1,\ldots,x_m}$ in terms of the partial derivatives of f, we get, for all $(x_1, \ldots, x_m) \in M$,

$$\begin{aligned} f(x_1, \ldots, x_m) &= \sum_{|\beta|\leq r} \frac{x_1^{\beta_1}\cdots x_m^{\beta_m}}{\beta_1!\cdots\beta_m!}\partial_x^{(\beta)}(\mathbf{f}) \\ &\quad + \sum_{|\alpha| = r+1} \frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!} h^{(\alpha)}(x_1, \ldots, x_m), \end{aligned}$$

where

$$h^{(\alpha)}(x_1, \ldots, x_m) = (r+1)\int_0^1 (1-u)^r \left(\frac{\partial^{\alpha_1+\cdots+\alpha_m} f}{\partial t_1^{\alpha_1}\cdots\partial t_m^{\alpha_m}}\right)(ux_1, \ldots, ux_m)\,du.$$

Clearly, the $h^{(\alpha)}$ are real C^∞ functions on M. Passing to the germs at the origin, we get

$$\mathbf{f} = \sum_{|\beta|\leq r} \partial_x^{(\beta)}(f)\mathbf{t}^{(\beta)} + \sum_{|\alpha| = r+1} \mathbf{t}^{(\alpha)}\mathbf{h}^{(\alpha)}.$$

Since $\mathbf{t}^{(\alpha)} \in \mathbf{J}_x^{r+1}$ for any (α) with $|\alpha| = r+1$, we get, for any $\lambda \in T_x^{(r)}(M)$,

$$\lambda = \sum_{|\beta|\leq r} \lambda(\mathbf{t}^{(\beta)})\partial_x^{(\beta)}$$

This shows that the $\partial_x^{(\beta)}(|\beta| \leq r)$ span $T_x^{(r)}(M)$ over **R**. On the other hand, the $\partial_x^{(\beta)}$ are linearly independent over **R** or **C**, since

$$\partial_x^{(\beta)}(\mathbf{t}^{(\gamma)}) = \begin{cases} 0 & (\gamma) \neq (\beta) \\ 1 & (\gamma) = (\beta) \end{cases}$$

This proves the lemma.

Vector fields. Let X $(x \mapsto X_x)$ be any assignment such that $X_x \in T_{xc}(M)$ for all $x \in M$. Then for any function $f \in C^\infty(M)$, the function $Xf : x \mapsto X_x(\mathbf{f}_x)$ is well defined on M, $\mathbf{f}_x$ being the germ at x defined by f. If U is any coordinate patch and $x_1, \ldots, x_m$ are coordinates on U, there are unique complex-valued functions $a_1, \ldots, a_m$ on U such that

$$X_y = \sum_{1\leq j\leq m} a_j(y)\left(\frac{\partial}{\partial x_j}\right)_y \qquad (y \in U).$$

X is called a *vector field on* M if $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$, or equivalently, if for each $x \in M$ there exist a coordinate patch U containing x and coordinates $x_1, \ldots, x_m$ on U such that the a_j defined above are C^∞ functions on U. A vector field X is said to be *real* if $X_x \in T_x(M)$ $\forall$ $x \in M$; X is real if and only if Xf is real for all real $f \in C^\infty(M)$. Given a vector field X, the mapping $f \longrightarrow Xf$ is a derivation of the algebra $C^\infty(M)$; i.e., for all f and

$g \in C^\infty(M)$,

(1.1.5) $$X(fg) = f \cdot Xg + g \cdot Xf.$$

This correspondence between vector fields and derivations is one to one and maps the set of all vector fields onto the set of all derivations of $C^\infty(M)$. Denote by $\mathfrak{I}(M)$ the set of all vector fields on M. If $X \in \mathfrak{I}(M)$ and $f \in C^\infty(M)$, $fX: x \mapsto f(x)X_x$ is also a vector field. In this way, $\mathfrak{I}(M)$ becomes a module over $C^\infty(M)$. We make in general no distinction between a vector field and the corresponding derivation of $C^\infty(M)$.

Let X and Y be two vector fields. Then $X \circ Y - Y \circ X$ is an endomorphism of $C^\infty(M)$ which is easily verified to be a derivation. The associated vector field is denoted by $[X, Y]$ and is called the *Lie bracket* of X with Y. The map

$$(X, Y) \mapsto [X, Y]$$

is bilinear and possesses the following easily verified properties:

(1.1.6) $$\begin{cases} \text{(i)} & [X,X] = 0 \\ \text{(ii)} & [X,Y] + [Y,X] = 0 \\ \text{(iii)} & [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \end{cases}$$

(X, Y, and Z being arbitrary in $\mathfrak{I}(M)$). If X and Y are real, so is $[X, Y]$. The relation (iii) of (1.1.6) is known as the *Jacobi identity*.

Differential operators. Let $r \geq 0$ be an integer and let

(1.1.7) $$D: x \mapsto D_x$$

be an assignment such that $D_x \in T^{(r)}_{xc}(M)$ for all $x \in M$. If $f \in C^\infty(M)$, the function $Df: x \mapsto D_x(\mathbf{f}_x)$ is well defined on M, $\mathbf{f}_x$ being the germ defined by f at x. If U is a coordinate patch and $x_1, \ldots, x_m$ are coordinates on U, then by Lemma 1.1.1 there are unique complex functions $a_{(\alpha)}$ on U such that

$$D_y = \sum_{|\alpha| \leq r} a_{(\alpha)}(y)\partial_y^{(\alpha)} \quad (y \in U).$$

D is called a *differential operator on* M if $Df \in C^\infty(M)$ for all $f \in C^\infty(M)$, or equivalently, if for each $x \in M$ we can find a coordinate patch U containing x with coordinate $x_1, \ldots, x_m$ such that the $a_{(\alpha)}$ defined above are in $C^\infty(U)$. The smallest integer $r \geq 0$ such that $D_x \in T^{(r)}_{xc}(M)$ for all $x \in M$ is called the *order* (ord(D)) or the *degree* (deg(D)) of D. For any differential operator D on M and $x \in M$, D_x is called the *expression* of D at x. If Df is real for any real-valued $f \in C^\infty(M)$, we say that D is *real*. The set of all differential operators on M is denoted by Diff(M). If $f \in C^\infty(M)$ and $D \in$ Diff(M), $fD: x \mapsto f(x)\, D_x$ is again a differential operator; its order cannot exceed the order of D. Thus Diff(M) is a module over $C^\infty(M)$. A *vector field* is a differential operator of order ≤ 1. If $\{V_i\}_{i \in J}$ is an open covering of M and $D_i (i \in J)$ is a differential operator on V_i such that

(a) $\sup_{i \in J} \operatorname{ord}(D_i) < \infty$
(b) if $V_{i_1} \cap V_{i_2} \neq \phi$, the restrictions of D_{i_1} and D_{i_2} to $V_{i_1} \cap V_{i_2}$ are equal,

then there exists exactly one differential operator D on M such that for any $i \in J$ D_i is the restriction of D to V_i.

Let D $(x \mapsto D_x)$ be a differential operator of order $\leq r$. We also denote by D the endomorphism $f \mapsto Df$ of $C^\infty(M)$. This endomorphism is then easily verified to have the following properties:

(1.1.8)
(i) it is *local;* i.e., if $f \in C^\infty(M)$ vanishes on an open set U, Df also vanishes on U
(ii) if $x \in M$, and $f_1, \ldots, f_{r+1}$ are $r + 1$ functions in $C^\infty(M)$ which vanish at x, then
$$(D(f_1 f_2 \cdots f_{r+1}))(x) = 0.$$

Conversely, it is quicky verified that given any endomorphism E of $C^\infty(M)$ satisfying (ii) of (1.1.8) for some integer $r \geq 0$, E is local and there is exactly one differential operator D on M such that $Df = Ef$ for all $f \in C^\infty(M)$; and ord$(D) \leq r$. In view of this, we make no distinction between a differential operator and the endomorphism of $C^\infty(M)$ induced by it. It follows easily from the expression of a differential operator in local coordinates that if D_1 and D_2 are differential operators of respective orders r_1 and r_2, then $D_1 D_2$ is also a differential operator, and its order is $\leq r_1 + r_2$; moreover, $D_1 D_2 - D_2 D_1$ is a differential operator of order $\leq r_1 + r_2 - 1$. Diff(M) is thus an algebra (not commutative); if Diff$(M)_r$ is the set of elements of Diff(M) of order $\leq r$, $r \mapsto$ Diff$(M)_r$ converts Diff(M) into a filtered algebra. A differential operator of order 0 is just the operator of multiplication by a C^∞ function; if u is in $C^\infty(M)$ we denote again by u the operator $f \mapsto uf$ of $C^\infty(M)$.

If $M = \mathbf{R}^m$ and D is a differential operator of order $\leq r$, there are unique C^∞ functions $a_{(\alpha)}$ $(|\alpha| \leq r)$ on M (*coefficients* of D) such that

$$D = \sum_{|\alpha| \leq r} a_{(\alpha)} \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \cdots \partial t_m^{\alpha_m}},$$

$t_1, \ldots, t_m$ being the linear coordinates on M. It is natural to ask whether such global representations exist on more general manifolds. The following theorem gives one such result.

Theorem 1.1.2. *Let $X_1, \ldots, X_m$ be m vector fields on M such that $(X_1)_x, \ldots, (X_m)_x$ form a basis of $T_{xc}(M)$ for each $x \in M$. For any multiindex $(\alpha) = (\alpha_1, \ldots, \alpha_m)$ let $X^{(\alpha)}$ be the differential operator*

$$X^{(\alpha)} = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_m^{\alpha_m} \tag{1.1.9}$$

(when $(\alpha) = (0) X^{(\alpha)} = 1$, the identity operator). Then the $X^{(\alpha)}$ are linearly independent over $C^\infty(M)$. If D is any differential operator of order $\leq r$, we can find unique C^∞ functions $a_{(\alpha)}$ on M such that

$$D = \sum_{|\alpha| \leq r} a_{(\alpha)} X^{(\alpha)}. \tag{1.1.10}$$

If the X_i are real, then for any real differential operator D the $a_{(\alpha)}$ defined by (1.1.10) are all real.

Proof. For any integer $r \geq 0$, let $\mathfrak{D}_r$ denote the complex vector space of all differential operators on M of the form $\sum_{|\alpha| \leq r} f_{(\alpha)} X^{(\alpha)}$, the $f_{(\alpha)}$ being C^∞ functions on M. Note that $\mathfrak{D}_1$ contains all vector fields. In fact, if Z is any vector field, we can write $Z = \sum_{1 \leq j \leq m} c_j X_j$ for uniquely defined functions c_j. To see that the c_j are in $C^\infty(M)$, let U be a coordinate patch with coordinates $x_1, \ldots, x_m$. Then there are C^∞ functions d_j, a_{jk} on U $(1 \leq j, k \leq m)$ such that $Z_y = \sum_{1 \leq j \leq m} d_j(y)(\partial/\partial x_j)_y$ and $(X_j)_y = \sum_{1 \leq k \leq m} a_{jk}(y)(\partial/\partial x_k)_y$ for all $y \in U$. Since the $(X_j)_y$ $(1 \leq j \leq m)$ are linearly independent for all y, the matrix (a_{jk}) is invertible. If a^{jk} are the entries of the inverse matrix, they are in $C^\infty(U)$ and $c_j = \sum_{1 \leq j \leq m} d_k a^{kj}$ on U.

We begin the proof of the theorem by showing that if l is an integer ≥ 1 and $Z_1, \ldots, Z_l$ are l vector fields, then the product $Z_1 \cdots Z_l$ belongs to $\mathfrak{D}_l$. For $l = 1$, this is just the remark made in the previous paragraph. Proceed by induction on l. Let $l > 1$, and assume that the result holds for any $l - 1$ vector fields. Let $Z_1, \ldots, Z_l$ be l vector fields, and write $E = Z_1 \cdots Z_l$.

Notice first that if $Y_1, \ldots, Y_l$ are any l vector fields, $F = Y_1 \cdots Y_l$, and F' is the product obtained by interchanging two adjacent Y's, then $F - F'$ is a product of $l - 1$ vector fields. So $F - F' \in \mathfrak{D}_{l-1}$ by the induction hypothesis. Since any permutation is a product of such adjacent interchanges, it follows from the induction hypothesis that $Y_1 \cdots Y_l - Y_{i_1} Y_{i_2} \cdots Y_{i_l} \in \mathfrak{D}_{l-1}$ for any permutation $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$. But if $1 \leq j_1 \leq j_2 \leq \cdots \leq j_l \leq m$, then $X_{j_1} \cdots X_{j_l} = X^{(\alpha)}$ for a suitable (α) with $|\alpha| = l$, so that $X_{j_1} \cdots X_{j_l} \in \mathfrak{D}_l$. Hence, from what we proved above, if $(k_1, \ldots, k_m)$ is any permutation of $(1, \ldots, m)$ and (α) is any multi-index with $|\alpha| \leq l$, then $X_{k_1}^{\alpha_1} \cdots X_{k_m}^{\alpha_m} \in \mathfrak{D}_l$.

Now consider E. By the induction hypothesis, there exist C^∞ functions $b_{(\beta)}$ and c_j on M such that $Z_1 = \sum_{1 \leq j \leq m} c_j X_j$ and $Z_2 \cdots Z_l = \sum_{|\beta| \leq l-1} b_{(\beta)} X^{(\beta)}$. So

$$\begin{aligned} E &= \sum_{1 \leq j \leq m} \sum_{|\beta| \leq l-1} c_j (X_j \circ b_{(\beta)}) X^{(\beta)} \\ &= \sum_{1 \leq j \leq m} \sum_{|\beta| \leq l-1} c_j b_{(\beta)} X_j X^{(\beta)} + \sum_{1 \leq j \leq m} \sum_{|\beta| \leq l-1} c_j (X_j b_{(\beta)}) X^{(\beta)}. \end{aligned}$$

Since, for all (β) with $|\beta| \leq l - 1$, $X_j X^{(\beta)} \in \mathfrak{D}_l$ (by what was seen in the preceding paragraph), we have $E \in \mathfrak{D}_l$.

We can now complete the proof of the theorem. Let $r \geq 0$ be any integer. Let U be a coordinate patch with coordinates $x_1, \ldots, x_m$ and let $\partial^{(\alpha)}$ be the differential operators $y \mapsto \partial_y^{(\alpha)}$ on U. By the result of the preceding paragraph (applied to the manifold U), there exist C^∞ functions $a_{(\alpha),(\beta)}$ on U such that

$$\partial^{(\alpha)} = \sum_{|\beta| \leq r} a_{(\alpha),(\beta)} X^{(\beta)} \quad (|\alpha| \leq r) \tag{1.1.11}$$

on U. This shows at once that for any $y \in M$, the $X_y^{(\beta)}$ $(|\beta| \leq r)$ span $T_{yc}^{(r)}(M)$; since their number is exactly the dimension of $T_{yc}^{(r)}(M)$, they must be linearly independent too. Therefore, if D is a differential operator of order $\leq r$, we can find unique functions $a_{(\beta)}$ on M such that

$$D = \sum_{|\beta| \leq r} a_{(\beta)} X^{(\beta)} \tag{1.1.12}$$

To prove that the $a_{(\alpha)}$ are C^∞, we restrict our attention to U and use the above notation. We select C^∞ functions $g_{(\alpha)}$ on U such that $D = \sum_{|\alpha| \leq r} g_{(\alpha)} \partial^{(\alpha)}$ on U. Then by (1.1.11) and (1.1.12) we have, on U,

$$a_{(\beta)} = \sum_{|\alpha| \leq r} g_{(\alpha)} a_{(\alpha),(\beta)} \quad (|\beta| \leq r),$$

proving that the $a_{(\beta)}$ are C^∞. The last statement is obvious. This proves the theorem.

We shall often use Harish-Chandra's notation for denoting the application of differential operators. Thus, if f is a C^∞ function and D a differential operator, $f(x; D)$ denotes the value of Df at $x \in M$.

Exterior differential forms. Let W be a finite-dimensional vector space of dimension m over a field F of characteristic 0. Put $\Lambda_0(W) = F$, and for any integer $k \geq 1$, define $\Lambda_k(W)$ as the vector space of all k-linear skew-symmetric functions on $W \times \cdots \times W$ (k factors) with values in F. $\Lambda_k(W)$ is then 0 if $k > m$, and $\dim \Lambda_k(W) = \binom{m}{k}$, $1 \leq k \leq m$. We write $\Lambda(W)$ for the direct sum of the $\Lambda_k(W)$, $0 \leq k \leq m$ and write $\wedge$ for the operation of exterior multiplication in $\Lambda(W)$ which converts it into an associative algebra over F, its unit being the unit 1 of F. We assume that the reader is, familiar with the defintion of $\wedge$ and the properties of $\Lambda(W)$ (cf. Exercises 9–11). If $\varphi, \varphi' \in \Lambda_1(W)$ (= dual of W), $\varphi \wedge \varphi' = -\varphi' \wedge \varphi$; in particular, $\varphi \wedge \varphi = 0$. More generally, if $\varphi \in \Lambda_r(W)$ and $\varphi' \in \Lambda_{r'}(W)$, then $\varphi \wedge \varphi' \in \Lambda_{r+r'}(W)$, and $\varphi \wedge \varphi' = (-1)^{rr'}\varphi' \wedge \varphi$. If $\{\varphi_1, \ldots, \varphi_m\}$ is a basis for $\Lambda_1(W)$, and $1 \le k \le m$, the $\binom{m}{k}$ elements $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ $(1 \le i_1 < \cdots < i_k \le m)$ form a basis for $\Lambda_k(W)$. Note that $\dim \Lambda_m(W) = 1$ and that $\varphi_1 \wedge \cdots \wedge \varphi_m$ is a basis for it. If $\psi_1, \ldots, \psi_m$ is another basis for $\Lambda_1(W)$, where $\psi_i = \sum_{1 \le j \le m} a_{ij}\varphi_j$ $(1 \le i \le m)$, and if A is the matrix $(a_{ij})_{1 \le i, j \le m}$, then

$$\psi_1 \wedge \cdots \wedge \psi_m = \det(A) \cdot \varphi_1 \wedge \cdots \wedge \varphi_m \tag{1.1.13}$$

A 0-*form* is a C^∞ function on M. Let $1 \le k \le m$ and let

$$\omega : x \mapsto \omega_x$$

be an assignment such that $\omega_x \in \Lambda_k(T_{xc}(M))$ for all $x \in M$. ω is said to be *real* if ω_x is real-valued on $T_x(M) \times \cdots \times T_x(M)$ for all $x \in M$. Let U be a coordinate patch and let $x_1, \ldots, x_m$ be a system of coordinates on it. For $y \in U$, let $\{(dx_1)_y, \ldots, (dx_m)_y\}$ be the basis of $T_y(M)^*$ dual to $\{(\partial/\partial x_1)_y, \ldots, (\partial/\partial x_m)_y\}$. Then there are unique functions $a_{i_1,\ldots,i_k}$ $(1 \le i_1 < i_2 < \cdots < i_k \le m)$ defined on U such that

$$\omega_y = \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,\ldots,i_k}(y)(dx_{i_1})_y \wedge \cdots \wedge (dx_{i_k})_y \quad (y \in U).$$

ω is said to be a k-form if all the $a_{i_1,\ldots,i_k}$ are C^∞ functions on U (for all possible choices of U).

Suppose ω $(x \mapsto \omega_x)$ is an assignment such that $\omega_x \in \Lambda_k(T_{xc}(M))$ for all $x \in M$. Let $Z_1, \ldots, Z_k$ be vector fields. Then the function

$$\omega(Z_1, \ldots, Z_k) : x \mapsto \omega_x((Z_1)_x, \ldots, (Z_k)_x)$$

is well defined on M. It is easy to show that ω is a k-form if and only if this function is C^∞ on M for all choices of $Z_1, \ldots, Z_k$. The map

$$(Z_1, \ldots, Z_k) \mapsto \omega(Z_1, \ldots, Z_k)$$

of $\mathfrak{J}(M) \times \cdots \times \mathfrak{J}(M)$ into $C^\infty(M)$ is skew-symmetric and $C^\infty(M)$-multilinear (i.e., C-multilinear and respects the module actions of $C^\infty(M)$); the correspondence between such maps and k-forms is a bijection. If ω is a k-form and $f \in C^\infty(M), f\omega : x \mapsto f(x)\omega_x$ is also a k-form. So the vector space of k-forms is also a module over $C^\infty(M)$. If ω is a k-form and ω' is a k'-form, then $x \mapsto \omega_x \wedge \omega'_x$ is usually denoted by $\omega \wedge \omega'$. It is a $(k + k')$-form, and $\omega \wedge \omega' = (-1)^{kk'} \omega' \wedge \omega$.

We write $\mathfrak{a}_0(M) = C^\infty(M)$ and $\mathfrak{a}_k(M)$ for the $C^\infty(M)$-module of all k-forms. Let $\mathfrak{a}(M)$ be the direct sum of all the $\mathfrak{a}_k(M)$ $(0 \le k \le m)$. Under $\wedge$, $\mathfrak{a}(M)$ is an algebra over $C^\infty(M)$.

Suppose $f \in C^\infty(M)$. Then for any vector field Z, $Zf \in C^\infty(M)$, and so there is a unique 1-form, denoted by df, such that

$$(df)(Z) = Zf \quad (Z \in \mathfrak{J}(M)). \tag{1.1.14}$$

If U is a coordinate patch with coordinates $x_1, \ldots, x_m$, then

$$(df)_y = \sum_{1 \le j \le m} \left(\frac{\partial f}{\partial x_j}\right)(y)(dx_j)_y \quad (y \in U).$$

In particular, on U, dx_j is the 1-form $y \mapsto (dx_j)_y$. More generally, there is a *unique* endomorphism d $(\omega \mapsto d\omega)$ of the vector space $\mathfrak{a}(M)$ with the following properties:

$$(1.1.15)\quad \begin{cases} \text{(i)} & d(d\omega) = 0 \text{ for all } \omega \in \mathfrak{a}(M) \\ \text{(ii)} & \text{if } \omega \in \mathfrak{a}_r(M),\ \omega' \in \mathfrak{a}_{r'}(M), \text{ then } d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^r \omega \wedge d\omega' \\ \text{(iii)} & \text{if } f \in \mathfrak{a}_0(M),\ df \text{ is the 1-form } Z \mapsto Zf\ (Z \in \mathfrak{J}(M)) \end{cases}$$

Let U be a coordinate patch, let $x_1, \ldots, x_m$ coordinates on it, and let

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

on U. Then on U

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} da_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{1.1.16}$$

The elements of $\mathfrak{a}(M)$ are called *exterior differential forms* on M. The endomorphism d $(\omega \mapsto d\omega)$ is the operator of *exterior differentiation* on $\mathfrak{a}(M)$.

We now discuss briefly some aspects of the theory of integration on manifolds. We confine ourselves to the integration of m-forms on m-dimensional manifolds.

We begin with unoriented or Lebesgue integration. Let M be, as usual, a C^∞ manifold of dimension m, and ω any m-form on M. It is then possible to associate with ω a nonnegative Borel measure on M. To see how this is done, consider a coordinate patch U with coordinates $x_1, \ldots, x_m$, and let $\tilde{U} = \{(x_1(y), \ldots, x_m(y)) : y \in U\}$; for any C^r function f on U, let $\tilde{f} \in C^r(\tilde{U})$ be such that $\tilde{f} \circ (x_1, \ldots, x_m) = f$. Now, we can find a real C^∞ function w_U on U such that $\omega = w_U dx_1 \wedge \cdots \wedge dx_m$ on U. The standard transformation formula for multiple integrals then shows that for any $f \in C_c(U)$, the integral

$$\int_{\tilde{U}} \tilde{f}(t_1, \ldots, t_m) | \tilde{w}_U(t_1, \ldots, t_m) | \, dt_1 \ldots dt_m$$

does not depend on the choice of coordinates $x_1, \ldots, x_m$. In other words, there is a nonnegative Borel measure μ_U on U such that for all $f \in C_c(U)$ and any system $(x_1, \ldots, x_m)$ of coordinates on U

$$\int_U f d\mu_U = \int_{\tilde{U}} \tilde{f} \, | \tilde{w}_U | \, dt_1 \cdots dt_m.$$

The measures μ_U are uniquely determined, and this uniqueness implies the existence of a unique nonnegative Borel measure μ on M such that μ_U is the restriction of μ to U for any U. Thus, for any coordinate patch U and any system $(x_1, \ldots, x_m)$ of coordinates on U we have, for all $f \in C_c(U)$,

$$\int_U f \, d\mu = \int_{\tilde{U}} \tilde{f}(t_1, \ldots, t_m) | \tilde{w}_U(t_1, \ldots, t_m) | \, dt_1 \cdots dt_m. \tag{1.1.17}$$

We write $\omega \sim \mu$ and say that μ *corresponds* to ω.

Let M be as above. M is said to be *orientable* if there exists an m-form on M which does not vanish anywhere on M. Two such m-forms, ω_1 and ω_2, are said to be *equivalent* if there exists a positive function g (necessarily C^∞) such that $\omega_2 = g\omega_1$. An *orientation* on M is an equivalence class of nowhere-vanishing m-forms on M. By M being *oriented* we mean that we are given M together with a distinguished orientation; the members of this class are then said to be positive (in symbols, >0).

Suppose now that M is oriented. Let η be any m-form on M with compact support. Select an m-form $\omega > 0$ and write $\eta = g\omega$, where $g \in C_c^\infty(M)$; let μ_ω be the measure corresponding to ω. We then define

$$\int_M \eta = \int_M g \, d\mu_\omega. \tag{1.1.18}$$

It is not difficult to show that this definition is dependent only on η and the orientation of M, and not on the particular choice of ω. Finally, if $\omega > 0$ is as above we often write $\int_M f\omega$ for $\int_M f \, d\mu_\omega$.

Theorem 1.1.3. *Let M be oriented and ω a positive m-form on M. Let μ be the nonnegative Borel measure on M which corresponds to ω. Then, given any differential operator D on M, there exists a unique differential operator $D^\dagger$ on M such that*

$$\int_M Df \cdot g \, d\mu = \int_M f \cdot D^\dagger g \, d\mu \tag{1.1.19}$$

for all $f, g \in C^\infty(M)$ with at least one of f and g having compact support. $D^\dagger$ has the same order as D and $D \mapsto D^\dagger$ is an involutive antiautomorphism of the algebra $Diff(M)$.

Proof. Given $D \in \mathrm{Diff}(M)$ and $g \in C^\infty(M)$, the validity of (1.1.19) for all $f \in C_c^\infty(M)$ determines $D^\dagger g$ uniquely. So if $D^\dagger$ exists, it is unique. It is also clear that if $D^\dagger$ is a differential operator such that (1.1.19) is satisfied whenever f and g are in $C_c^\infty(M)$, then (1.1.19) is satisfied whenever at least one of f and g lies in $C_c^\infty(M)$. The uniqueness implies quickly that the set $\mathfrak{D}_M$ of all $D \in \mathrm{Diff}(M)$ for which $D^\dagger$ exists is a subalgebra, that $\mathfrak{D}_M^\dagger = \mathfrak{D}_M$, and that $D \mapsto D^\dagger$ is an involutive antiautomorphism of $\mathfrak{D}_M$. It remains only to prove that $\mathfrak{D}_M = \mathrm{Diff}(M)$.

Let U be a coordinate patch, and let $(x_1, \ldots, x_m)$ be a coordinate system U with $\omega = w_U dx_1 \wedge \cdots \wedge dx_m$ on U, where $w_U > 0$ on U. Put $\tilde{U} = \{(x_1(y), \ldots, x_m(y)) : y \in U\}$ and for any $h \in C^\infty(U)$ denote by $\tilde{h}$ the element of $C^\infty(\tilde{U})$ such that $\tilde{h} \circ (x_1, \ldots, x_m) = h$. A simple partial integration shows that if $1 \leq j \leq m$, $f, g \in C_c^\infty(U)$,

$$\int_{\tilde{U}} \left(\frac{\partial \tilde{f}}{\partial t_j}\right) \tilde{g} \tilde{w}_U \, dt_1 \cdots dt_m = -\int_{\tilde{U}} \left(\frac{\partial \tilde{g}}{\partial t_j} + \frac{1}{\tilde{w}_U} \frac{\partial \tilde{w}_U}{\partial t_j} \tilde{g}\right) \tilde{f} \tilde{w}_U \, dt_1 \cdots dt_m.$$

If Z_j is the vector field $y \mapsto (\partial/\partial x_j)_y$ on U, and $\varphi_j \in C^\infty(U)$ is defined by $\varphi_j = w_U^{-1} \cdot (Z_j w_U)$, it is clear that $Z_j^\dagger$ exists and is the differential operator of order 1 given by $Z_j^\dagger = -(Z_j + \varphi_j)$. If $h \in C^\infty(U)$, $h^\dagger$ exists and coincides with h. But by Theorem 1.1.2, $\mathrm{Diff}(U)$ is algebraically generated by $C^\infty(U)$ and the vector fields Z_j, $1 \leq j \leq m$. Hence $\mathfrak{D}_U = \mathrm{Diff}(U)$. Moreover, the above argument shows that for any $E \in \mathrm{Diff}(U)$ the order of $E^\dagger$ is $\leq$ order of E.

Let D be any differential operator on M. From what we have just proved it is clear that for each coordinate patch U one can find a differential operator $D_U^\dagger$ on U such that $\mathrm{ord}(D_U^\dagger) \leq \mathrm{ord}(D)$ and for all $f, g \in C_c^\infty(U)$

$$\int_M (Df) \cdot g \, d\mu = \int_M f \cdot (D_U^\dagger g) \, d\mu.$$

The uniqueness of $\dagger$ shows that the $D_U^\dagger$ match on overlapping coordinate patches. So there is a differential operator D' on M such that $C_U^\dagger$ is the restriction of D' to U for any arbitrary coordinate patch U. Moreover, if U is any coordinate patch, we have

$$\int_M (Df) g \, d\mu = \int_M f (D'g) \, d\mu$$

for all $f, g \in C_c^\infty(U)$. A simple argument based on partitions of unity shows that this equation is valid for all $f, g \in C_c^\infty(M)$. In other words, $D^\dagger$ exists and coincides with D'. Our construction makes it clear that $\mathrm{ord}(D^\dagger) \leq \mathrm{ord}(D)$

for all $D \in \mathrm{Diff}(M)$. Since $D^{\dagger\dagger} = D$, this shows that $\mathrm{ord}(D) \leq \mathrm{ord}(D^\dagger)$, so that necessarily $\mathrm{ord}(D) = \mathrm{ord}(D^\dagger)$ for all $D \in \mathrm{Diff}(M)$. The theorem is proved.

$D^\dagger$ is called the *formal adjoint of D relative to* ω.

Mappings. Let M, N be C^∞ manifolds. A continuous map

$$\pi : M \longrightarrow N$$

is said to be *differentiable* (C^∞) if for any open set $U \subseteq N$ and any $g \in C^\infty(U)$, $g \circ \pi \in C^\infty(\pi^{-1}(U))$. Suppose π is differentiable, $x \in M$, $y = \pi(x)$. Then with respect to coordinates $x_1, \ldots, x_m$ around x, and $y_1, \ldots, y_n$ around y, π is given by differentiable functions.

If g, g' are C^∞ around y and coincide in an open set containing y, then $g \circ \pi$ and $g' \circ \pi$ coincide in an open set containing x. Thus the map $g \mapsto g \circ \pi$ ($g \in C^\infty(N)$) induces an algebra homomorphism π^* ($\mathbf{u} \mapsto \mathbf{u} \circ \pi$) of $\mathbf{D}_y$ into $\mathbf{D}_x$. If $X_x \in T_{xc}(M)$, there is a unique $Y_y \in T_{yc}(N)$ such that $Y_y(\mathbf{u}) = X_x(\pi^*(\mathbf{u}))$; we write $Y_y = (d\pi)_x(X_x)$. Thus

$$(d\pi)_x(X_x)(\mathbf{u}) = X_x(\pi^*(\mathbf{u})) \quad (\mathbf{u} \in \mathbf{D}_y).$$

$(d\pi)_x$ is a linear map of $T_{xc}(M)$ into $T_{yc}(N)$, called the *differential of* π *at* x. It is clear that $(d\pi)_x$ maps the tangent space $T_x(M)$ into the tangent space $T_y(N)$. A special case of this arises when M is an open subset of the real line $\mathbf{R}$. In this case, for any $\tau \in M$, $D_\tau = (d/dt)_{t=\tau}$ is a basis for $T_\tau(M)$, and it is customary to write

$$\dot{\pi}(\tau) = \left(\frac{d}{dt}\right)_{t=\tau} \pi(t) = (d\pi)_\tau(D_\tau). \tag{1.1.20}$$

$\dot{\pi}(\tau)$ is thus an element of $T_{\pi(\tau)}(N)$.

If $p \geq 1$ is any integer, it is obvious that $\pi^*(\mathbf{J}_y^p) \subseteq \mathbf{J}_x^p$, so given any $v \in T_{xc}^{(\infty)}(M)$, there is a unique $v' \in T_{yc}^{(\infty)}(N)$ such that $v'(\mathbf{u}) = v(\pi^*(\mathbf{u}))$ for all $\mathbf{u} \in \mathbf{D}_y$. We write $v' = (d\pi)_x^{(\infty)}(v)$; thus

$$(d\pi)_x^{(\infty)}(v)(\mathbf{u}) = v(\pi^*(\mathbf{u})) \quad (\mathbf{u} \in \mathbf{D}_y). \tag{1.1.21}$$

It is obvious that $(d\pi)_x^{(\infty)}$ maps $T_x^{(r)}(M)$ into $T_y^{(r)}(N)$ for any integer $r \geq 0$ and that $(d\pi)_x^{(\infty)} | T_x(M) = (d\pi)_x$. We refer to $(d\pi)_x^{(\infty)}$ as the *complete differential of* π *at* x. If D is a differential operator on M, there need not in general exist a differential operator D' on N such that $(d\pi)_x^{(\infty)}(D_x) = D'_{\pi(x)}$ for all $x \in M$. If such a D' exists, we shall say (following Chevalley) that D and D' are π-*related*. Given $D \in \mathrm{Diff}(M)$ and $D' \in \mathrm{Diff}(N)$, it is easy to show that D and D' are π-related if and only if $D(u \circ \pi) = (D'u) \circ \pi$ for all $u \in C^\infty(N)$. If

$D_j \in \mathrm{Diff}(M)$ and $D'_j \in \mathrm{Diff}(N)$ are π-related ($j = 1, 2$), then $D_1 \circ D_2$ and $D'_1 \circ D'_2$ are π-related.

Let $\pi : M \longrightarrow N$ be a C^∞ map and ω any r-form on N. For $x \in M$ let $(\pi^*\omega)_x$ be the r-linear form defined by

$$(\pi^*\omega)_x(v_1, \ldots, v_r) = \omega_{\pi(x)}((d\pi)_x(v_1), \ldots, (d\pi)_x(v_r)), \tag{1.1.22}$$

for $v_1, \ldots, v_r \in T_{xc}(M)$. Then $(\pi^*\omega)_x \in \Lambda_r(T_{xc}(M))$, and $x \mapsto (\pi^*\omega)_x$ is an r-form on M. We denote this form by $\pi^*\omega$. $\pi^* : \omega \mapsto \pi^*\omega$ has the following properties:

$$\begin{cases} \text{(i)} \quad \pi^*(u\omega) = (u \circ \pi)\pi^*\omega \quad (u \in C^\infty(N)) \\ \text{(ii)} \quad d(\pi^*\omega) = \pi^*(d\omega) \\ \text{(iii)} \quad \pi^*(\omega_1 \wedge \omega_2) = (\pi^*\omega_1) \wedge (\pi^*\omega_2). \end{cases} \tag{1.1.23}$$

($\omega, \omega_1, \omega_2, \in \mathfrak{A}(N)$ are arbitrary).

We consider now the special case where the differentiable map π is a homeomorphism of M onto N and π^{-1} is also a differentiable map. π is then called a *diffeomorphism*. In this case π induces natural isomorphisms between the respective spaces of functions, differential operators, etc. For instance, let $N = M$ and $\alpha : x \mapsto \alpha(x)$ a diffeomorphism of M onto itself. Then α induces the automorphism $u \mapsto u^\alpha$ of $C^\infty(M)$ where $u^\alpha(x) = u(\alpha^{-1}(x))$ for all $x \in M$, $u \in C^\infty(M)$. This in turn induces the automorphism $D \mapsto D^\alpha$ of the algebra $\mathrm{Diff}(M)$; $D^\alpha(u) = (D(u^{\alpha^{-1}}))^\alpha$, for all $D \in \mathrm{Diff}(M)$ and $u \in C^\infty(M)$. The set of all diffeomorphisms of M is a group under composition. If α, β are diffeomorphisms of M onto itself, the $D^{\alpha\beta} = (D^\beta)^\alpha$ for $D \in \mathrm{Diff}(M)$. Similarly we have the automorphism $\omega \mapsto \omega^\alpha$ of $\mathfrak{A}(M)$.

Let π ($M \mapsto N$) be a C^∞ map ($m = \dim(M)$, $n = \dim(N)$), $x \in M$, and let $(d\pi)_x$ be surjective. Let $y = \pi(x)$. Then $m \geq n$, and it is well known that in suitable coordinates around x and y, π looks like the projection $(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_n)$ around the origin in $\mathbf{R}^m$. In fact, let $x_1, \ldots, x_m$ be coordinates around x, and $y_1, \ldots, y_n$ coordinates around y with $x_i(x) = y_j(y) = 0$, $1 \leq i \leq m$, $1 \leq j \leq n$. There are C^∞ functions $F_1, \ldots, F_n$ defined around $\mathbf{0}_m = (0, \ldots, 0) \in \mathbf{R}^m$ such that $y_j \circ \pi = F_j(x_1, \ldots, x_m)$ ($1 \leq j \leq n$) around x. Since $(d\pi)_x$ is surjective, a standard argument shows that the matrix $(\partial F_j/\partial t_k)_{1 \leq j \leq n, 1 \leq k \leq m}$ has rank n at $\mathbf{0}_m$. By permuting the x_i if necessary, we assume that the $n \times n$ matrix $(\partial F_j/\partial t_k)_{1 \leq j, k \leq n}$ is non-singular at $\mathbf{0}_m$. It is then clear that the functions $y_1 \circ \pi, \ldots, y_n \circ \pi, x_{n+1}, \ldots, x_m$ form a system of coordinates around x; and with respect to these and the y_j, π looks like the projection $(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_n)$. It follows from this that the set $M_1 = \{z : z \in M, (d\pi)_z \text{ is surjective}\}$ is open in M, that $\pi[M_1]$ is open in N, and that π is an open map of M_1 onto $\pi[M_1]$. π is called a *submersion* if $(d\pi)_x$ is surjective for all $x \in M$. If π is a submersion and $\pi[M] = N$, N is called a *quo-*

tient of M relative to π. It follows from the local description of π given above that if N is a quotient of M relative to π, then for any open set $U \subseteq N$, a function g on U is C^∞ if and only if $g \circ \pi$ is C^∞ on $\pi^{-1}(U)$. In other words, in this case, the differentiable structure of N is completely determined by π and the differentiable structure on M.

We now consider maps with injective differentials. Here it is necessary to exercise somewhat greater care than in the case of a submersion. Let M and N be C^∞ manifolds of dimensions m and n respectively, and $\pi\,(M \longrightarrow N)$ a C^∞ map. Let $x \in M$, $y = \pi(x)$ and suppose that $(d\pi)_x$ is injective. Then $m \leq n$, and in suitable coordinates around x and y, π looks like the injection $(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_m, 0, \ldots, 0)$ around the origin. More precisely, we can find all of the following: a coordinate patch U containing x with coordinates $x_1, \ldots, x_m$; a coordinate patch V containing y with coordinates $y_1, \ldots, y_n$; and a number $a > 0$ with the following properties:

(1.1.24)
- (i) $\xi\,(z \mapsto (x_1(z), \ldots, x_m(z)))$ is a diffeomorphism of U onto I_a^m, with $\xi(x) = \mathbf{0}_m$; $\eta\,(z' \mapsto (y_1(z'), \ldots, y_n(z')))$ is a diffeomorphism of V onto I_a^n, with $\eta(y) = \mathbf{0}_n$.[3]
- (ii) $\eta \circ \pi \circ \xi^{-1}$ is the map
$$(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_m, 0, \ldots, 0)$$
of I_a^m into I_a^n.

To see this, let $x'_1, \ldots, x'_m$ be coordinates around x and let $y'_1, \ldots, y'_n$ be coordinates around $y = \pi(x)$ with $x'_i(x) = y'_j(y) = 0$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $F_j (1 \leq j \leq n)$ be C^∞ functions around $\mathbf{0}_m$ such that $y'_j \circ \pi = F_j(x'_1, \ldots, x'_m)$ around $x (1 \leq j \leq n)$. Since $(d\pi)_x$ is injective, the matrix $(\partial F_j/\partial t_k)_{1 \leq j \leq n, 1 \leq k \leq m}$ has rank m at $\mathbf{0}_m$. By permuting the y'_j if necessary, we may assume that the $m \times m$ matrix $(\partial F_j/\partial t_k)_{1 \leq j,k \leq m}$ is nonsingular at $\mathbf{0}_m$. It is then clear that the functions $y'_1 \circ \pi, \ldots, y'_m \circ \pi$ form a system of coordinates around x. Let $x_i = y'_i \circ \pi (1 \leq i \leq m)$. Let G_p be C^∞ functions around $\mathbf{0}_m$ such that $y'_p \circ \pi = G_p(x_1, \ldots, x_m)$ around $x\,(m < p \leq n)$. Define $y_i = y'_i\ (i \leq m)$, $y_p = y'_p - G_p(y'_1, \ldots, y'_m)\ (m < p \leq n)$. Then we have (1.1.24) for suitable U, V, $a > 0$. It follows from (1.1.24) that there is a sufficiently small open set U around x such that π is a homeomorphism of U onto $\pi[U]$.

π is called an *immersion* if $(d\pi)_x$ is injective for all $x \in M$; an *imbedding* if it is an one to-one immersion; and a *regular imbedding* if it is an imbedding and if π is a homeomorphism of M onto $\pi[M]$, the latter being given the topology inherited from N. The properties (1.1.24) are not in general strong

[3]For any integer $k \geq 1$ and any $b > 0$, we write I_b^k for the cube in $\mathbf{R}^k$ defined by
$$I_b^k = \{(t_1, \ldots, t_k)\colon -b < t_j < b \quad \text{for} \quad 1 \leq j \leq k\}.$$
The origin of $\mathbf{R}^k$ is denoted by $\mathbf{0}_k$.

enough to ensure that a given imbedding is regular or has other nice properties. Note, however, that if π is an imbedding the equations (1.1.24) completely determine the differentiable structure of M in terms of π and the differentiable structure of N: if $W \subseteq M$ is open and f is a complex-valued function on W, f is C^∞ if and only if for each $x \in W$ one can find an open set U with $x \in U \subseteq W$, an open set V containing $y = \pi(x)$ with $\pi[U] \subseteq V$, and $g \in C^\infty(V)$ such that $f(z) = g(\pi(z))$, $z \in U$.

The next theorem describes some of the nice properties of regular imbeddings. Recall that a subset A of a topological space E is said to be *locally closed* (in E) if it is a relatively closed subset of some open subset of E, or equivalently, if it is open in its closure.

Theorem 1.1.4. *Let π be a regular imbedding of M into N. Then $\pi[M]$ is locally closed in N. For each $x \in M$, we can choose U, V, $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that, in addition to* (1.1.24), *we have*

$$\pi[U] = \pi[M] \cap V. \tag{1.1.25}$$

If P is any C^∞ manifold, and u is any map of P into M, u is C^∞ if and only if $\pi \circ u$ is a C^∞ map of P into N.

Proof. Let U', V', $x_1, \ldots, x_m$, $y_1, \ldots, y_n$, and $a' > 0$ be such that the relations (1.1.24) are satisfied (with U', V', and a' replacing U, V, and a, respectively). Since π is a homeomorphism onto $\pi[M]$, $\pi[U']$ is open in $\pi[M]$, so there is an open set V'' in N such that $\pi[U'] = V'' \cap \pi[M]$. Let $V_1 = V' \cap V''$. Then V_1 is an open subset of N containing $y = \pi(x)$ and $\pi[U'] = V_1 \cap \pi[M]$. Choose a with $0 < a < a'$ such that $\eta^{-1}(I_a^n) \subseteq V_1$ and $\xi^{-1}(I_a^m) \subseteq U'$. Then if we set $U = \xi^{-1}(I_a^m)$ and $V = \eta^{-1}(I_a^n)$, we have (1.1.25). Note that $\pi[U] = \pi[M] \cap V$ is closed in V by (1.1.24). Now select open sets $V_i (i \in I)$ in N such that $\pi[M] \subseteq \bigcup_{i \in I} V_i$ and $\pi[M] \cap V_i$ is closed in V_i for each $i \in I$. Then it is clear that $\pi[M]$ is closed in $\bigcup_{i \in I} V_i$; thus $\pi[M]$ is locally closed. For the last assertion, let P be a C^∞ manifold, and let u be a map of P into M such that $\pi \circ u$ is a C^∞ map of P into N. Let $p \in P$, $x = u(p)$, $y = \pi(x)$. There is an open set W in P containing p such that $(\pi \circ u)[W] \subseteq V$; then $u[W] \subseteq U$. It follows at once from a consideration of coordinates that u is a C^∞ map of W into U.

The universal property contained in the last assertion of Theorem 1.1.4 is an important consequence of the regularity of an imbedding. However, even some irregular imbeddings possess this property. Let π be an imbedding of M into N. We shall call π *quasi-regular* if the following property is satisfied: if P is any C^∞ manifold and u any map of P into M, u is C^∞ if and only if $\pi \circ u$ is C^∞ from P to N. There are imbeddings which are quasi-regular but not regular (Exercise 1).

Submanifolds. Let M, N be C^∞ manifolds. Then M is called a *submanifold* of N if

$$(1.1.26)\quad \begin{cases} \text{(i)} & M \subseteq N \text{ (set-theoretically)} \\ \text{(ii)} & \text{the identity map of } M \text{ into } N \text{ is an imbedding.} \end{cases}$$

M is said to be a *regular* (resp. *quasi-regular*) submanifold if the identity map of M into N is regular (resp. quasi-regular). If M is a submanifold of N and $x \in M$, we shall identify $T_{xc}^{(\infty)}(M)$ with its image in $T_{xc}^{(\infty)}(N)$ under the complete differential of the identity map of M into N.

As we have observed already, the relations (1.1.24) have the following consequence: given a subset $M \subseteq N$ and a topology on M under which M is a Hausdorff second countable space and which is finer than the one induced from N, there is *at most one* differentiable structure on M so that M becomes a submanifold of N. If such a structure exists, we shall equip M with it and refer to M as a submanifold of N. If the topology on M is the one induced by N, then the differentiable structure described above, if it exists, will convert M into a regular submanifold of N.

Theorem 1.1.5. *Let N be a C^∞ manifold and let $M \subseteq N$. In order that M, equipped with the relative topology, be a (regular) submanifold of N, it is necessary and sufficient that the following be satisfied. There exists an integer m with $1 \leq m \leq n$ such that given any $x \in M$, one can find an open set V of N containing x and $n - m$ real differentiable functions $f_1, \ldots, f_{n-m}$ on V such that*

$$(1.1.27)\quad \begin{cases} (i) & V \cap M = \{z : z \in V,\ f_1(z) = \cdots = f_{n-m}(z) = 0\} \\ (ii) & (df_1)_x, \ldots, (df_{n-m})_x \text{ are linearly independent elements of } T_x(N)^*. \end{cases}$$

If this is the case, $\dim(M) = m$, and M is a locally closed subset of N.

Proof. The only thing that needs to be proved is that if M satisfies the conditions described above, then it becomes a regular submanifold of N; Theorem 1.1.4 implies the remaining assertions. Also if $m = n$, (1.1.27) reduces to the condition $V \subseteq M$, so that in this case M is an open submanifold of N. We may thus assume $1 \leq m < n$.

Fix $x \in M$ and let $V, f_1, \ldots, f_{n-m}$ be as in (1.1.27). It is then clear that we can find a system of coordinates $x_1, \ldots, x_n$ in a neighborhood of x such that $x_j(x) = 0 (1 \leq j \leq n)$ and $x_{m+j} = f_j (1 \leq j \leq n - m)$. By replacing V by a smaller open set, we may assume that the homeomorphism ξ $(y \mapsto (x_1(y), \ldots, x_n(y)))$ maps V onto I_a^n for some $a > 0$. Then ξ maps $M \cap V$ onto $I_a^m \times \mathbf{0}_{n-m}$. In other words, $U = M \cap V$ is a regularly imbedded submanifold of V, hence of N. Since $x \in M$ is arbitrary, it follows that we can write $M = \bigcup_{i \in I} U_i$, where each U_i is open in M and is a regular submanifold of N. If $i, j \in I$ are

such that $U_{ij} = U_i \cap U_j \neq \phi$, then U_{ij} is open in both U_i and U_j and is a regular submanifold of N under each of the C^∞ structures induced by U_i and U_j. These two structures must be the same, so U_{ij} is an open submanifold of both U_i and U_j. It then follows that there is a unique C^∞ structure for M such that each $U_i (i \in I)$ becomes an open submanifold of M. This structure converts M into a regular submanifold of N.

Product manifolds. Let $M_j (j = 1, 2)$ be a C^∞ manifold of dimension m_j, and let $M = M_1 \times M_2$. Equip M with the product topology; it is then Hausdorff and second countable. Let $U \subseteq M$ be an open subset and f a complex function defined on U. We say f is C^∞ if the following condition is satisfied: for any $(a_1, a_2) \in U$ there are coordinate patches V_j around a_j and coordinates $x_{j1}, \ldots, x_{jm_j}$ on $V_j (j = 1, 2)$ such that (i) $V_1 \times V_2 \subseteq U$, and (ii) if $\tilde{V}_j$ is the image of V_j under the map $z \mapsto (x_{j1}(z), \ldots, x_{jm_j}(z))$, there is a C^∞ function φ on $\tilde{V}_1 \times \tilde{V}_2$ such that

$$f(b_1, b_2) = \varphi(x_{11}(b_1), \ldots, x_{1m_1}(b_1), x_{21}(b_2), \ldots, x_{2m_2}(b_2))$$

for all $(b_1, b_2) \in V_1 \times V_2$. $U \mapsto C^\infty(U)$ is a differentiable structure for M; it is called the *product* of the structures on M_1 and M_2. M is called the *product* of the C^∞ manifolds M_1 and M_2. If π_j is the natural projection of M on M_j, π_j is a submersion. If N is a C^∞ manifold and $u : y \mapsto (u_1(y), u_2(y))$ is a map of N into M, then u is C^∞ if and only if u_1 and u_2 are C^∞.

Suppose $x = (x_1, x_2) \in M$. Given functions $f_j \in C^\infty(U_j)$, where $x_j \in U_j$ $(j = 1, 2)$, we write $f_1 \otimes f_2$ for the element of $C^\infty(U_1 \times U_2)$ given by $(f_1 \otimes f_2)(a_1, a_2) = f_1(a_1) f_2(a_2) (a_j \in U_j)$. The map $f_1, f_2 \mapsto f_1 \otimes f_2$ induces a natural injection of $\mathbf{D}_{x_1} \otimes \mathbf{D}_{x_2}$ into $\mathbf{D}_x$. If X_j is a tangent vector to M_j at $x_j (j = 1, 2)$, there is exactly one tangent vector X to M at x such that for $\mathbf{u}_j \in \mathbf{D}_{x_j}$ $(j = 1, 2)$,

$$(1.1.28)\quad X(\mathbf{u}_1 \otimes \mathbf{u}_2) = \mathbf{u}_1(x_1) X_2(\mathbf{u}_2) + \mathbf{u}_2(x_2) X_1(\mathbf{u}_1);$$

$X_1, X_2 \mapsto X$ is a linear isomorphism of $T_{x_1}(M_1) \times T_{x_2}(M_2)$ with $T_x(M)$. More generally, if $v_j \in T_{x_j}^{(r_j)}(M_j)$ $(j = 1, 2)$ there is exactly one $v \in T_x^{(\infty)}(M)$ such that

$$(1.1.29)\quad v(\mathbf{f}_1 \otimes \mathbf{f}_2) = v_1(\mathbf{f}_1) v_2(\mathbf{f}_2) \quad (\mathbf{f}_j \in \mathbf{D}_{x_j});$$

$v \in T_x^{(r_1 + r_2)}(M)$ and the map $v_1 \otimes v_2 \mapsto v$ extends uniquely to a linear isomorphism of $T_{x_1c}^{(\infty)}(M_1) \otimes T_{x_2c}^{(\infty)}(M_2)$ onto $T_{xc}^{(\infty)}(M)$. We shall often identify these two spaces and write $v_1 \otimes v_2$ for the element v defined by (1.1.29); in particular, the tangent vector X defined by (1.1.28) is nothing but $X_1 \otimes 1_{x_2} + 1_{x_1} \otimes X_2$.

If D_j is a differential operator on $M_j (j = 1, 2)$, then $D: (x_1, x_2) \mapsto (D_1)_{x_1} \otimes (D_2)_{x_2}$ is a differential operator $M_1 \times M_2$; we write $D_1 \otimes D_2$ for D.

These considerations can be extended easily to products of more than two manifolds.

1.2. Analytic Manifolds

We begin by recalling the definition of an analytic function of m variables, real or complex. Let $U \subseteq \mathbf{R}^m$ be any open set and let f be a function defined on U with values in $\mathbf{C}$. f is said to be *analytic* on U if, given any $(x_1^0, \ldots, x_m^0) \in U$, we can find an $\eta > 0$ and a power series

$$\sum_{r_1,\ldots,r_m \geq 0} c_{r_1,\ldots,r_m}(x_1 - x_1^0)^{r_1} \cdots (x_m - x_m^0)^{r_m} \quad (c_{r_1,\ldots,r_m} \in \mathbf{C})$$

around $(x_1^0, \ldots, x_m^0)$ such that the series converges absolutely and uniformly for all $(x_1, \ldots, x_m)$ with $\max_{1 \leq j \leq m} |x_j - x_j^0| < \eta$, to the sum $f(x_1, \ldots, x_m)$. For an open set $U \subseteq \mathbf{C}^m$, a similar definition of a *complex analytic* or *holomorphic* function on U can be given. The functions which are analytic on U form an algebra under the usual operations. Analytic functions of analytic functions are analytic.

The definition of a *real* analytic manifold is similar to that of a C^∞ manifold. Let M be a Hausdorff space satisfying the second axiom of countability. A *real analytic structure* for M is an assignment

$$\mathfrak{A} : U \mapsto \mathfrak{A}(U) \quad (U \text{ open}, \subseteq M)$$

such that

(i) $\mathfrak{A}$ possesses properties (i) and (ii) of a differentiable structure (cf. §1.1).

(ii) There exists an integer $m > 0$ with the following property: for each $x \in M$, can find an open set U containing x and m *real* functions $x_1, \ldots, x_m$ from $\mathfrak{A}(U)$ such that (a) the map $\xi : y \mapsto (x_1(y), \ldots, x_m(y))$ is a homeomorphism of U with an open subset of $\mathbf{R}^m$, and (b) if W is any open subset of U, $\mathfrak{A}(W)$ is precisely the set of all functions of the form $F \circ \xi$, with F analytic on $\xi[W]$.

The pair $(M, \mathfrak{A})$ (and, by abuse of language, M itself) is said be a *real analytic manifold of dimension m*. For an open $U \subseteq M$, the elements of $\mathfrak{A}(U)$ are called the *analytic functions* on U. As before, any open set such as U in (ii) above is called a *coordinate patch;* and $x_1, \ldots, x_m$ are called *analytic coordinates* on U.

Let $U \subseteq M$ be open and let f be a complex-valued function defined on U. We define f to be C^∞ if for each $x \in U$, f is a C^∞ function of the local analytic coordinates around x. The assignment $U \mapsto C^\infty(U)$ is easily seen to be a differentiable structure for M. We shall call this the C^∞ structure underlying the analytic structure. Note that $\mathfrak{A}(U) \subseteq C^\infty(U)$ for all open U. The entire theory of differentiable manifolds now becomes available to M.

Let M and N be analytic manifolds and π a map of M into N. The definition of the analyticity of π is analogous to the C^∞ case. π is called an *analytic isomorphism* or an *analytic diffeomorphism* if it is bijective and if both π and π^{-1} are analytic. It is a consequence of the classical theorem on implicit and inverse functions that if $\pi\ (M \longrightarrow N)$ is analytic and bijective and if $(d\pi)_x$ is bijective for all $x \in M$, then π^{-1} is analytic, so that π is an analytic diffeomorphism.

Let M be an analytic manifold and D a differential operator on M. For any open set $U \subseteq M$, let D_U denote the restriction of D to U. D is called *analytic* if for each open U, $D_U : f \mapsto D_U f$ leaves $\mathfrak{A}(U)$ invariant. Let U be a coordinate patch, $x_1, \ldots, x_m$ analytic coordinates on U, and let $D_U = \sum_{|\alpha| \leq r} a_{(\alpha)} \partial^{(\alpha)}$. Then if D is analytic, $a_{(\alpha)} \in \mathfrak{A}(U)$; conversely, if for each $x \in M$ we can find analytic coordinates $x_1, \ldots, x_m$ around x such that $D = \sum_{|\alpha| \leq r} a_{(\alpha)} \partial^{(\alpha)}$ on an open set around x with analytic $a_{(\alpha)}$, then D is an analytic differential operator. Similarly, a definition of analyticity can be given for differential forms. The analytic differential operators form a subalgebra of Diff(M). If ω is an analytic m-form which is real and vanishes nowhere, D an analytic differential operator, and $D^\dagger$ the formal adjoint of D with respect to ω, then it is easy to verify that $D^\dagger$ is analytic. If ω, ω' are analytic r-forms, then $d\omega$ and $\omega \wedge \omega'$ are analytic; if $\pi\ (M \longrightarrow N)$ is analytic and ω is an analytic r-form on N, so is $\pi^*\omega$ on M.

The concepts of products and quotients of analytic manifolds as well as submanifolds of analytic manifolds are defined exactly as in the C^∞ case, with analytic functions and coordinate systems replacing the C^∞ ones. The definitions and results of §1.1 concerning maps with surjective and injective differentials remain valid with this modification. In particular, Theorems 1.1.4 and 1.1.5 remain true in the analytic case: if N is an analytic manifold and M a subset of N equipped with the relative topology, then M is a regular analytic submanifold of N of dimension m $(1 \leq m \leq n)$ if and only if for each $x \in M$ we can find an open subset V of N containing x and $n - m$ real-valued analytic functions $f_1, \ldots, f_{n-m}$ on V such that (i) $V \cap M$ is precisely the set of common zeros of $f_1, \ldots, f_{n-m}$ in V, and (ii) $(df_1)_x, \ldots, (df_{n-m})_x$ are linearly independent elements of $T_x(N)^*$.

A *complex analytic* or *holomorphic* manifold of complex dimension m is defined in the same way as a real analytic manifold, with holomorphic functions replacing real analytic functions. Given a complex analytic manifold M of dimension m, there is an underlying real analytic structure for M in which M is a real analytic manifold of dimension $2m$; if $U \subseteq M$ is open and f is a real-valued function on U, f will be analytic in this real analytic structure if and only if the following is satisfied: for each $x \in U$, we can find holomorphic coordinates $z_1, \ldots, z_m$ around x such that f is a real analytic function of the $2m$ real functions $\mathrm{Re}(z_1), \ldots, \mathrm{Re}(z_1)$, $\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_m)$ in a sufficiently small open neighborhood of x.

Let M be a complex analytic manifold and let $x \in M$. Two functions defined and holomorphic in an open set containing x are called *equivalent* if they coincide in some open neighborhood of x. The equivalence classes are called the *germs of holomorphic functions at* x. In the usual way, they form an algebra over $\mathbf{C}$, denoted by $\mathbf{H}_x$; for any $\mathbf{f} \in \mathbf{H}_x$, write $\mathbf{f}(x)$ for the common value at x of the elements of $\mathbf{f}$. The *holomorphic tangent vectors* to M at x are then the linear functions v on $\mathbf{H}_x$ such that $v(\mathbf{fg}) = \mathbf{f}(x)v(\mathbf{g}) + \mathbf{g}(x)v(\mathbf{f})$ for all $\mathbf{f}, \mathbf{g} \in \mathbf{H}_x$. They form a complex vector space, the so called *holomorphic tangent space to M at x;* this vector space is denoted by $T_x(M)$. More generally, let $\mathbf{J}_x$ be the ideal in $\mathbf{H}_x$ of all $\mathbf{u}$ with $\mathbf{u}(x) = 0$; then for any integer $r \geq 0$, a *holomorphic differential expression* at x is a linear function v on $\mathbf{H}_x$ which vanishes on $\mathbf{J}_x^{r+1}$. The set of all such is a vector space denoted by $T_x^{(r)}(M)$. As before, we put $T_x^{(\infty)}(M) = U_{r \geq 0} T_x^{(r)}(M)$. Holomorphic vector fields, differential forms, and differential operators can now be defined as in the real analytic case; no changes are needed.

The same situation provails with respect to the concepts of quotient and submanifolds of complex analytic manifolds. In particular, the analogues of Theorems 1.1.4 and 1.1.5 are true in the complex analytic case also.

Algebraic sets. The version of Theorem 1.1.5 for analytic manifolds is very useful in showing that certain subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are regular analytic submanifolds. The simplest examples are obtained when we take M to be the set of zeros of a collection of *polynomials*. For example, let $p \geq 1, q \geq 1$ be integers and let F be the polynomial on $\mathbf{R}^{p+q}$ defined by

$$F(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

Let M be the set of zeros of F and $M_0 = M \backslash \{\mathbf{0}\}$ ($\mathbf{0}$ is the origin in $\mathbf{R}^{p+q}$). Then, for $x \in M$, $(dF)_x \neq 0$ if and only if $x \in M_0$. So M_0 is a regular analytic submanifold of dimension $p + q - 1$. It is not difficult to show that M does not look like a manifold around $\mathbf{0}$. $\mathbf{0}$ is called a singular point, and points of M_0 are called regular; the set of regular points is thus open in M and forms a regular submanifold of $\mathbf{R}^{p+q}$. We now prove a theorem of H. Whitney [1] which asserts that the above example is somewhat typical. We work in $\mathbf{R}^n$; the case of sets of zeros in $\mathbf{C}^n$ of complex polynomials can be handled similarly.

Let $U \subseteq \mathbf{R}^n$ be an open set, fixed throughout this discussion; let $\mathcal{P}$ be the algebra of all polynomial functions on $\mathbf{R}^n$ with real coefficients. For any subset $\mathfrak{F} \subseteq \mathcal{P}$ let

$$Z(\mathfrak{F}) = \{u : u \in U, P(u) = 0 \ \forall \ P \in \mathfrak{F}\}. \tag{1.2.1}$$

Any subset of U which is $Z(\mathfrak{F})$ for some $\mathfrak{F} \subseteq \mathcal{P}$ is called an *algebraic subset of* U. For any subset M of U, let

$$\mathcal{J}(M) = \{P : P \in \mathcal{P}, P(u) = 0 \ \forall \ u \in M\}; \tag{1.2.2}$$

$\mathcal{J}(M)$ is an ideal in $\mathcal{P}$. Note that $Z(\mathfrak{F})$ is also the set of common zeros of the elements of $\mathcal{J}(Z(\mathfrak{F}))$, so that any algebraic subset of U is of the form $Z(\mathcal{J})$ for some ideal $\mathcal{J} \subseteq \mathcal{P}$. Now, if $\mathcal{J}$ is an ideal in $\mathcal{P}$, we can find $P_1, \ldots, P_r \in \mathcal{J}$ such that $\mathcal{J} = \mathcal{P}P_1 + \cdots + \mathcal{P}P_r$ (Hilbert basis theorem); $\{P_1, \ldots, P_r\}$ is called an *ideal basis* for $\mathcal{J}$. So any algebraic set is of the form $Z(\mathfrak{F})$ for a finite subset $\mathfrak{F}$ of $\mathcal{P}$.

Suppose now that M is an algebraic subset of U. For any $u \in M$, let $r_M(u)$ be the dimension of the linear space spanned by the differentials $(dP)_u$, $P \in \mathcal{J}(M)$. $r_M(u)$ is called the *rank* of M at u. The relation

$$d(PQ)_u = P(u)(dQ)_u + Q(u)(dP)_u$$

shows that if $\{Q_1, \ldots, Q_p\}$ is an ideal basis for $\mathcal{J}(M)$, $r_M(u)$ is also the dimension of the linear space spanned by $(dQ_1)_u, \ldots, (dQ_p)_u$. Put

$$r = \max_{u \in M} r_M(u) \tag{1.2.3}$$

$$M_0 = \{u : u \in M, r_M(u) = r\} \tag{1.2.4}$$

The points of M_0 are called *regular;* those of $M \setminus M_0$ are called *singular*. Now, for any $P_1, \ldots, P_s \in \mathcal{J}(M)$, $(dP_1)_u, \ldots, (dP_s)_u$ are linearly independent if and only if the matrix $(\partial P_i/\partial t_j)_{1 \leq i \leq s, 1 \leq j \leq n}$ is of rank s at u. It follows easily from this that M_0 is a nonempty open subset of M, being the set of all $u \in M$ where the rank of the matrix $((\partial Q_i/\partial t_j)_u)$ is maximum.

Theorem 1.2.1. (*Whitney*) *Let notation be as above. Then M_0 is a nonempty open subset of M and is a regular analytic submanifold of $\mathbf{R}^n$ of dimension $n - r$.*

Proof. We follow Whitney's proof. It is enough to prove that each point of M_0 can be surrounded by a connected open subset of M_0 which is a regular analytic submanifold of $\mathbf{R}^n$ of dimension $n - r$. Fix $u_0 \in M_0$; we may assume that $u_0 = \mathbf{0}$. We can then select $P_1, \ldots, P_r \in \mathcal{J}(M)$ such that $(dP_1)_0, \ldots, (dP_r)_0$ are linearly independent. The matrix $(\partial P_i/\partial t_j)_{1 \leq i \leq r, 1 \leq j \leq n}$ therefore has rank r at $\mathbf{0}$. By permuting the coordinates if necessary, we may assume that

$$\left(\frac{\partial(P_1, \ldots, P_r)}{\partial(t_1, \ldots, t_r)}\right)_0 \neq 0.$$

It is then obvious that

$$\left(\frac{\partial(P_1,\ldots,P_r,t_{r+1},\ldots,t_n)}{\partial(t_1,\ldots,t_r,t_{r+1},\ldots,t_n)}\right)_0 \neq 0.$$

Write $y_i = P_i$, $1 \leq i \leq r$, $y_i = t_i$, $r+1 \leq i \leq n$. Clearly, we can choose an open set V and an $a > 0$ such that (i) $\mathbf{0} \in V \subseteq U$, and (ii) $v \mapsto (y_1(v), \ldots, y_n(v))$ is an analytic diffeomorphism of V with the cube I_a^n. Let

$$V_0 = \{v : v \in V, y_1(v) = \cdots = y_r(v) = 0\}; \tag{1.2.5}$$

then V_0 is a connected regular analytic submanifold of $\mathbf{R}^n$ of dimension $n-r$, $\mathbf{0} \in V_0$, and $V \cap M \subseteq V_0$. It is now enough to prove that $V_0 \subseteq M$. For suppose this proved: then $V_0 = V \cap M$, so V_0 is an open subset of M. Since $(dP_1)_v, \ldots, (dP_r)_v$ are linearly independent for all $v \in V_0$, $V_0 \subseteq M_0$. So V_0 would be an open subset of M_0 containing u_0 and imbedded as a regular analytic submanifold of dimension $n-r$ of $\mathbf{R}^n$.

We now prove that $V_0 \subseteq M$. Let A be the algebra of all real-valued analytic functions on V. Write $\mathfrak{s} = \mathfrak{s}(M)$ and $\bar{\mathfrak{s}} = A\mathfrak{s}$, the ideal in A generated by $\mathfrak{s}$. We claim that $\bar{\mathfrak{s}}$ is invariant under the derivations $\partial/\partial y_j$, $r+1 \leq j \leq n$. It is enough to prove that $\partial/\partial y_j\, \mathfrak{s} \subseteq \bar{\mathfrak{s}}$ for $r+1 \leq j \leq n$. Fix j with $r+1 \leq j \leq n$, $F \in \mathfrak{s}$. Write $P_l = t_l$ if $r+1 \leq l \leq n$ and $l \neq j$, and $P_j = F$. Then

$$\frac{\partial(P_1,\ldots,P_n)}{\partial(y_1,\ldots,y_n)} = \frac{\partial(P_1,\ldots,P_n)}{\partial(t_1,\ldots,t_n)} \cdot \frac{\partial(t_1,\ldots,t_n)}{\partial(y_1,\ldots,y_n)}. \tag{1.2.6}$$

Now

$$\frac{\partial(P_1,\ldots,P_n)}{\partial(y_1,\ldots,y_n)} = \frac{\partial F}{\partial y_j}.$$

Furthermore,

$$\varphi = \frac{\partial(t_1,\ldots,t_n)}{\partial(y_1,\ldots,y_n)} \in A.$$

On the other hand, consider

$$P = \frac{\partial(P_1,\ldots,P_n)}{\partial(t_1,\ldots,t_n)}.$$

We have

$$P = \frac{\partial(P_1,\ldots,P_r,F)}{\partial(t_1,\ldots,t_r,t_j)}.$$

Since $P_1, \ldots, P_r, F \in \mathfrak{s}$, P has to vanish at all points of M, as otherwise there would be points of M where $\mathfrak{s}$ has rank $\geq r+1$. So $P = 0$ on M. Since P is a *polynomial*, $P \in \mathfrak{s}$. (1.2.6) now shows that

$$\frac{\partial F}{\partial y_j} = \varphi P \in \bar{\mathfrak{s}}.$$

It follows from the above result that for any $F \in \bar{\mathfrak{s}}$ and any multiindex $(\alpha) = (\alpha_{r+1}, \ldots, \alpha_n)$, $(\partial/\partial y_{r+1})^{\alpha_{r+1}} \cdots (\partial/\partial y_n)^{\alpha_n} F \in \bar{\mathfrak{s}}$. Now, it is trivial to see that any element of $\mathfrak{s}$ vanishes at $\mathbf{0}$. So if $F \in \bar{\mathfrak{s}}$, *all* the derivatives $(\partial/\partial y_{r+1})^{\alpha_{r+1}} \cdots (\partial/\partial y_n)^{\alpha_n} F$ vanish at $\mathbf{0}$. Since F is analytic and V_0 is connected, this implies that F vanishes on V_0. In particular, all elements of $\mathfrak{s}$ vanish on V_0. So $V_0 \subseteq M$. As mentioned earlier, this is sufficient to prove the theorem.

1.3. The Frobenius Theorem

The aim of this section is to introduce the concept of involutive systems of tangent spaces on an analytic manifold and to prove that such systems are integrable. At the local level this is just the classical Frobenius theorem. However, for applications to the theory of Lie groups, the local form of the theorem is not adequate, and it becomes necessary to construct global integral manifolds. We shall follow Chevalley's elegant method of doing this. We restrict ourselves to the analytic case; the C^∞ versions of our theorems can be proved by means of analogous arguments.

Let M be an analytic manifold of dimension m. An assignment $\mathfrak{L} : x \mapsto \mathfrak{L}_x (x \in M)$ is called a *system of tangent spaces* (*of rank* p) if $\mathfrak{L}_x$ is a linear subspace of dimension p contained in $T_x(M)$ for all $x \in M$. The system $\mathfrak{L}$ is said to be *nontrivial* if $1 \leq p \leq m-1$. *We shall consider only nontrivial systems in this section.* Given a system $\mathfrak{L}$ of tangent spaces of rank p, a vector field X is said to *belong to* $\mathfrak{L}$ on an open set U if $X_x \in \mathfrak{L}_x$ for all $x \in U$. $\mathfrak{L}$ is said to be an *analytic system* (*a.s.*) if for each $x \in M$ we can find an open set U containing x and p analytic vector fields ($p = \operatorname{rank} \mathfrak{L}$) $X_1, \ldots, X_p$ on U such that $(X_1)_y, \ldots, (X_p)_y$ span $\mathfrak{L}_y$ for all $y \in U$. $\mathfrak{L}$ is said to be an *involutive analytic system* (*i.a.s.*) if it has the additional property: let U be an open subset of M and let X, Y be two analytic vector fields which belong to $\mathfrak{L}$ on U; then $[X, Y]$ belongs to $\mathfrak{L}$ on U.

Given an a.s. $\mathfrak{L}$, an analytic submanifold S of M is said to be an *integral manifold* of $\mathfrak{L}$ if (*a*) S is connected, and (b) for each $y \in S$, $T_y(S) = \mathfrak{L}_y$. We do *not* require that S be a regular analytic submanifold, and so the topology of S could be strictly finer than the one induced from M. $\mathfrak{L}$ is said to be *integrable* if each point of M lies in some integral manifold of $\mathfrak{L}$.

An integrable a.s. $\mathfrak{L}$ is necessarily involutive. To prove this, we need only verify that if $x \in M$ and X and Y are analytic vector fields which belong to $\mathfrak{L}$ in some open neighborhood of x, then $[X, Y]_x \in \mathfrak{L}_x$. Now, there is an integral manifold S of $\mathfrak{L}$ through x. Replacing S by a sufficiently small open subset of it containing x, we may assume that S is a (connected) regular submanifold of M and that $S \subseteq U$, where U is open in M and where X and Y are defined on U and belong to $\mathfrak{L}$ on it. Then X' $(y \mapsto X_y)$ and Y' $(y \mapsto Y_y)$ $(y \in S)$ are analytic vector fields of S; if i is the identity map of S into

M, X', X and Y', Y are i-related. So $[X',Y']$ and $[X,Y]$ are i-related. This implies that $[X,Y]_x \in \mathfrak{L}_x$.

Let M (resp. N) be an analytic manifold and $\mathfrak{M}$ (resp. $\mathfrak{N}$) an a.s. on M (resp. N). $\mathfrak{M}$ and $\mathfrak{N}$ are called *isomorphic* if there is an analytic diffeomorphism π $(M \longrightarrow N)$ such that $(d\pi)_x(\mathfrak{M}_x) = \mathfrak{N}_{\pi(x)}$ for all $x \in M$. If $\mathfrak{L}(x \mapsto \mathfrak{L}_x)$ is an a.s. on M and $U \subseteq M$ an open set, $\mathfrak{L}$ induces on U an a.s. $\mathfrak{L} \mid U$, by restriction. Let $a > 0$ and let us consider the cube I_a^m in $\mathbf{R}^m$. Let $t_1, \ldots, t_m$ be the usual coordinates in $\mathbf{R}^m$. For any $x \in I_a^m$ let $\mathfrak{L}_x^{p,m,a}$ be the linear span of $(\partial/\partial t_1)_x, \ldots, (\partial/\partial t_p)_x$. Then $\mathfrak{L}^{p,m,a}: x \mapsto \mathfrak{L}_x^{p,m,a}$ is an i.a.s. If $a_{p+1}, \ldots, a_m$ are fixed numbers between $-a$ and $+a$, the submanifold

$$\{(t_1, \ldots, t_m): t_{p+1} = a_{p+1}, \ldots, t_m = a_m\}$$

is an integral manifold of $\mathfrak{L}^{p,m,a}$. $\mathfrak{L}^{p,m,a}$ is called a *canonical* i.a.s. The classical Frobenius theorem asserts that, *locally*, every i.a.s. is isomorphic to a canonical one.

The proof of the local Frobenius theorem depends on the following two lemmas; the first lemma proves the theorem in question for the case $p = 1$.

Lemma 1.3.1. *Let M be an analytic manifold, X any real analytic vector field on M, and $x \in M$ a point such that $X_x \neq 0$. Then there are analytic coordinates $x_1, \ldots, x_m$ around x such that $X_y = (\partial/\partial x_1)_y$ for all y in an open neighborhood of x.*

Proof. Select analytic coordinates $z_1, \ldots, z_m$ around x such that $z_1(x) = \cdots = z_m(x) = 0$ and $X_x(z_1) \neq 0$. Then there are real analytic functions $G_1, \ldots, G_m$, defined on I_a^m (for some $a > 0$) such that $G_1(0, \ldots, 0) \neq 0$ and

$$X_y = \sum_{1 \leq i \leq m} G_i(z_1(y), \ldots, z_m(y)) \left(\frac{\partial}{\partial z_i}\right)_y$$

for all y in an open neighborhood of x. Consider the system of differential equations

$$\frac{du_i}{dt} = G_i(u_1(t), \ldots, u_m(t)) \quad (1 \leq i \leq m). \tag{1.3.1}$$

By the standard existence theorem (cf. Appendix, Theorem 1.4.1), we can select b with $0 < b < a$ and real analytic functions $u_1, \ldots, u_m$ on I_b^m such that

(a) $|u_j(t,y_2, \ldots, y_m)| < a$ for $1 \leq j \leq m$ and $(t,y_2, \ldots, y_m) \in I_b^m$

(b) for fixed $(y_2, \ldots, y_m) \in I_b^{m-1}$, the functions $u_1(\cdot, y_2, \ldots, y_m), \ldots, u_m(\cdot, y_2, \ldots, y_m)$ satisfy (1.3.1) on the open interval $(-b,b)$ with the initial conditions

$$u_1(0,y_2, \ldots, y_m) = 0, \quad u_2(0,y_2, \ldots, y_m) = y_2, \quad \ldots, \quad u_m(0,y_2, \ldots, y_m) = y_m.$$

Then the analytic map

$$\tau: (t,y_2, \ldots, y_m) \mapsto (u_1(t,y_2, \ldots, y_m), \ldots, u_m(t,y_2, \ldots, y_m))$$

has the nonvanishing Jacobian $G_1(0, \ldots, 0)$ at $\mathbf{0}$, and $\tau(\mathbf{0}) = \mathbf{0}$. So it is an analytic diffeomorphism on an open set containing $\mathbf{0}$. Therefore, there exist functions $F_1, \ldots, F_m$, defined and analytic around $\mathbf{0}$, vanishing at $\mathbf{0}$, such that the map $(v_1, \ldots, v_m) \mapsto (F_1(v_1, \ldots, v_m), \ldots, F_m(v_1, \ldots, v_m))$ inverts τ around $\mathbf{0}$. Let $x_j = F_j(z_1, \ldots, z_m)$. Then $x_1, \ldots, x_m$ form a system of analytic coordinates around x. It is easy to verify that $X_y = (\partial/\partial x_1)_y$ for all y in some open set containing the point x.

Lemma 1.3.2. *Let M be an analytic manifold, $x \in M$, and let $X_1, \ldots, X_p$ be real analytic vector fields defined on an open set U containing x such that (i) $(X_1)_y, \ldots, (X_p)_y$ are linearly independent for $y \in U$, and (ii) $[X_j,X_k] = 0$, $1 \leq j, k \leq p$. Then we can choose coordinates $x_1, \ldots, x_m$ around x such that, in an open set around x,*

$$X_j = \frac{\partial}{\partial x_j} + \sum_{1 \leq s < j} a_{js} \frac{\partial}{\partial x_s} \quad (1 \leq j \leq p), \tag{1.3.2}$$

where the a_{js} are defined and analytic around x.

Proof. We prove this by induction on p. For $p = 1$ this follows at once from Lemma 1.3.1. Let $1 < p \leq m$, and assume the result for $X_1, \ldots, X_{p-1}$. Then we can choose a connected open set V with $x \in V \subseteq U$ and coordinates $u_1, \ldots, u_m$ on V such that

$$X_j = \frac{\partial}{\partial u_j} + \sum_{1 \leq s < j} b_{js} \frac{\partial}{\partial u_s} \quad (1 \leq j \leq p-1), \tag{1.3.3}$$

where the b_{js} are analytic on V. Write $X_p = \sum_{1 \leq s \leq m} g_s \, \partial/\partial u_s$, where the g_s are analytic on V, and put $X'_p = \sum_{p \leq s \leq m} g_s \, \partial/\partial u_s$. From (1.3.3) and the condition (i) of the lemma we conclude easily that $(X'_p)_y \neq 0$ for all $y \in V$. On the other hand, the conditions $[X_p,X_j] = 0$ yield the relations

$$\sum_{1 \leq s \leq m} g_s \left[\frac{\partial}{\partial u_s}, X_j\right] - \sum_{1 \leq s \leq m} (X_j g_s) \frac{\partial}{\partial u_s} = 0 \tag{1.3.4}$$

on V, for $1 \leq j \leq p-1$. Now (1.3.3) shows that, for $1 \leq s \leq m$ and $1 \leq j \leq p-1$, $[\partial/\partial u_s, X_j]$ is a linear combination of only the $\partial/\partial u_t$ with $1 \leq t \leq p-1$. Hence (1.3.4) implies that $X_j g_s = 0$ on V for $1 \leq j \leq p-1$, $p \leq s \leq m$. A simple argument based on (1.3.3) now shows that $\partial/\partial u_j g_s = 0$ on V for $1 \leq j \leq p-1$, $p \leq s \leq m$. Since V is connected, this implies that, for each s with $p \le s \le m$, g_s is a function of $u_p, u_{p+1}, \ldots, u_m$ only. An application of Lemma 1.3.1 now shows that we can replace $u_p, \ldots, u_m$ by analytic functions $v_p, \ldots, v_m$ with the following properties: (a) $u_1, \ldots, u_{p-1}, v_p, \ldots, v_m$ form a system of coordinates around x, and (b) $X'_p = \partial/\partial v_p$ around x. Let $x_j = u_j$ for $1 \le j \le p-1$ and $x_j = v_j$ for $p \le j \le m$. Then (1.3.2) is satisfied in the coordinate system $(x_1, \ldots, x_m)$.

Theorem 1.3.3. (*Local Frobenius Theorem*) *Let $\mathfrak{L}$ $(x \mapsto \mathfrak{L}_x)$ be an involutive nontrivial analytic system of tangent spaces of rank p on an analytic manifold M of dimension m. Then, for any $x \in M$, we can find an open set U containing x and an $a > 0$ such that $\mathfrak{L} | U$ is isomorphic to the canonical i.a.s. $\mathfrak{L}^{p,m,a}$. In particular, $\mathfrak{L}$ is integrable.*

Proof. The theorem is equivalent to the following: given $x \in M$ we can choose analytic coordinates $x_1, \ldots, x_m$ around x such that $\mathfrak{L}_y$ is spanned by $(\partial/\partial x_1)_y, \ldots, (\partial/\partial x_p)_y$ for all y in an open set containing x. Since the canonical involutive analytic systems $\mathfrak{L}^{p,m,a}$ are integrable, this would imply that $\mathfrak{L}$ is integrable. Fix $x \in M$. Let $z_1, \ldots, z_m$ be analytic coordinates around x and let $Z_1, \ldots, Z_p$ be analytic vector fields such that (i) $Z_1, \ldots, Z_p$ are defined on an open set U containing x and the $z_1, \ldots, z_m$ are coordinates on U, and (ii) $(Z_1)_y, \ldots, (Z_p)_y$ span $\mathfrak{L}_y$ for all $y \in U$. We may then write $Z_j = \sum_{1 \le r \le m} a'_{jr} \partial/\partial z_r$, where the a'_{jr} are analytic functions on U. Clearly, some $p \times p$ submatrix of $(a'_{jr})_{1 \le j \le p, 1 \le r \le m}$ is invertible at x. We may assume without losing generality that $(a'_{jr})_{1 \le j, r \le p}$ is invertible at x and that U is so small that this matrix is invertible on U. Let b_{ij} $(1 \le i, j \le p)$ be the entries of the inverse matrix. Then the b_{ij} are analytic functions on U. Let $X_j = \sum_{1 \le s \le p} b_{js} Z_s$. Then: (i) $(X_1)_y, \ldots, (X_p)_y$ span $\mathfrak{L}_y$ for all $y \in U$, and (ii) $X_j = \partial/\partial z_j + \sum_{p+1 \le r \le m} c_{jr} \partial/\partial z_r$, $1 \le j \le p$, the c_{jr} being analytic functions on U.

We now claim that $[X_j, X_k] = 0$, $1 \le j, k \le p$. Fix such j, k. Since $\mathfrak{L}$ is involutive, $[X_j, X_k]$ belongs to $\mathfrak{L}$ on U. Therefore $[X_j, X_k] = \sum_{1 \le s \le p} f_s X_s$, where the f_s are analytic functions on U; in particular, f_s is the coefficient of $\partial/\partial z_s$ in $[X_j, X_k]$ for $1 \le s \le p$. On the other hand, the formula (ii) above for the X_r shows that $[X_j, X_k]$ is a linear combination of only the $\partial/\partial z_r$ with $p+1 \le r \le m$. This implies that the f_s are all zero, i.e., that $[X_j, X_k] = 0$.

Now use Lemma 1.3.2 to choose analytic coordinates $x_1, \ldots, x_m$ around x such that, for $1 \le j \le p$, $X_j = \partial/\partial x_j + \sum_{1 \le s < j} a_{js} \partial/\partial x_s$, the a_{js} being analytic around x. This representation shows that $(\partial/\partial x_1)_y, \ldots, (\partial/\partial x_p)_y$ span $\mathfrak{L}_y$ for all y in some open set containing x. This completes the proof of the theorem.

Let $U \subseteq M$ be an open set, $x_1, \ldots, x_m$ a system of coordinates on U, and $a > 0$. We say that $(U; x_1, \ldots, x_m; a)$ is *adapted to* $\mathfrak{L}$ if the map $u \mapsto (x_1(u), \ldots, x_m(u))$ is an analytic diffeomorphism of U with I_a^m and if $\mathfrak{L}_u$ is

spanned by $(\partial/\partial x_1)_u, \ldots, (\partial/\partial x_p)_u$ for all $u \in U$. In this case, for any $\mathbf{a} = (a_{p+1}, \ldots, a_m) \in I_a^{m-p}$, we define $U(\mathbf{a})$ by

$$U(\mathbf{a}) = \{u : u \in U, x_{p+1}(u) = a_{p+1}, \ldots, x_m(u) = a_m\}. \tag{1.3.5}$$

The $U(\mathbf{a})$ are regularly imbedded integral manifolds of $\mathfrak{L}$.

The local Frobenius theorem is not adequate for applications, since the integral manifolds have been constructed only locally. For full effectiveness it is necessary to obtain them in the large. This was done by Chevalley [1]; we shall follow his method of "piecing together" the local integral manifolds to obtain the global ones. However, this has to be done with some care, because the global manifolds are not always regularly imbedded.

It is easy to see that an arbitrary integral manifold of $\mathfrak{L}$ is a union of open subsets of the form $U(\mathbf{a})$. In fact, let $(U; x_1, \ldots, x_m; a)$ be adapted to $\mathfrak{L}$ and let S be an integral manifold of $\mathfrak{L}$ with $S \cap U \ne \phi$; then $S \cap U$ is open in S. If $\bar{x}_j = x_j | S \cap U$, we have $d\bar{x}_j = 0$ $(p+1 \le j \le m)$, so that these $\bar{x}_j$ are *locally constant* on $S \cap U$. In other words, each connected component of $S \cap U$ (in the topology induced by S) is contained in some $U(\mathbf{a})$. Since these components are open in S, it follows that $S \cap U(\mathbf{a})$ is open in S for any $\mathbf{a} \in I_a^{m-p}$. But then, for any such $\mathbf{a}$, the identity map of $S \cap U(\mathbf{a})$ into $U(\mathbf{a})$ is analytic with a bijective differential. This shows that $S \cap U(\mathbf{a})$ is open in S, as well as in $U(\mathbf{a})$; both S and $U(\mathbf{a})$ induce the same topology on it.

Lemma 1.3.4. *If S_1 and S_2 are any two integral manifolds of $\mathfrak{L}$, then $S_1 \cap S_2$ is open in S_1 as well as in S_2; both S_1 and S_2 induce the same topology on it. The integral manifolds of $\mathfrak{L}$ are all quasi-regularly imbedded in M.*

Proof. Let $u \in S_1 \cap S_2$. Select an open set U containing u, coordinates $x_1, \ldots, x_m$ on U, and $a > 0$ such that $(U; x_1, \ldots, x_m; a)$ is adapted to $\mathfrak{L}$. Let $\mathbf{a} \in I_a^{m-p}$ be such that $u \in U(\mathbf{a})$. It is then clear from what we said above that $S_1 \cap S_2 \cap U(\mathbf{a})$ is open in S_1 as well as in S_2, both of which induce the same topology on it. This leads at once to the first assertion. For the second, let S be any integral manifold of $\mathfrak{L}$, N any analytic manifold, and π an analytic map of N into M such that $\pi[N] \subseteq S$. We shall prove that π is an analytic map of N into the analytic manifold S. Fix $y \in N$ and let $u = \pi(y)$. Choose an open set U containing u, coordinates $x_1, \ldots, x_m$ on U, and $a > 0$, such that $(U; x_1, \ldots, x_m; a)$ is adapted to $\mathfrak{L}$. Let $\mathbf{a} \in I_a^{m-p}$ be such that $u \in U(\mathbf{a})$, and let T be the connected component of $S \cap U(\mathbf{a})$ containing u(in the topology of S). We claim that T is also the connected component of $S \cap U$ in the topology of U, which contains u. Indeed, if T' is the component in question, then obviously $T \subseteq T'$. On the other hand, since S is second countable, $S \cap U$ has at most countably many connected components (in the topology of S), so that $S \cap U \subseteq \bigcup_{\mathbf{b} \in F} U(\mathbf{b})$ for some countable set $F \subseteq I_a^{m-p}$. But then the map $u \mapsto (x_{p+1}(u), \ldots, x_m(u))$, which is continuous on $S \cap U$ in the topology induced from U, takes at most countably many values. Therefore, it must be a constant on each connected component of $S \cap U$ in the topology of U, in particular on T'. Thus $T' \subseteq S \cap U(\mathbf{a})$; and, since we have already proved that both S and $U(\mathbf{a})$ induce the same topology on $S \cap U(\mathbf{a})$, we must have $T = T'$. This proves our claim. If W is the connected component of $\pi^{-1}(U)$ containing y, it is then clear that W is open and $\pi[W] \subseteq T$. Since T is open in the regularly imbedded $U(\mathbf{a})$, π is an analytic map of W into T. Hence π is an analytic map of W into S; this leads to the second assertion.

Lemma 1.3.5. *Let A be a connected Hausdorff space which is locally connected. Suppose $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is open in A and each connected component of A_n is second countable for each n. Then A is itself second countable.*

Proof. Let $\mathfrak{C}_n$ be the class of (open) sets which are connected components of A_n, and $\mathfrak{C} = \bigcup_{n=1}^{\infty} \mathfrak{C}_n$. Since there cannot exist an uncountable family of mutually disjoint nonempty open sets in a second countable space it follows that, given $F \in \mathfrak{C}$, there are only countably many $F' \in \mathfrak{C}$ such that $F \cap F' \neq \phi$. We now define the families $J_0, J_1, \ldots$ of open subsets of A as follows. We select $E \in \mathfrak{C}$ arbitrarily and define $J_0 = \{E\}$; for $s \geq 1$, $J_s = \{F: F \in \mathfrak{C}, F \cap F' \neq \phi$ for some $F' \in J_{s-1}\}$. The J_s $(s \geq 0)$ are well defined inductively. A simple induction on s shows that they are all countable. Let $B = \bigcup_{s=0}^{\infty} \bigcup_{F \in J_s} F$. Then B is open and second countable. If $v \in Cl(B)$, we can find $F \in \mathfrak{C}$ such that $v \in F$; and as $F \cap B \neq \phi$, there is an $s \geq 0$ and an $F' \in J_s$ such that $F \cap F' \neq \phi$. This shows that $F \in J_{s+1}$ and hence that $v \in B$. B is thus open and closed. Since A is connected, $A = B$. A is thus second countable.

Theorem 1.3.6. (*Global Frobenius Theorem*) *Let M be an analytic manifold $\mathfrak{L}$ $(x \mapsto \mathfrak{L}_x)$ an involutive analytic system of tangent spaces of rank p. Given any point of M, there is one and exactly one maximal integral manifold of $\mathfrak{L}$ containing that point. Any (nonempty) integral manifold of $\mathfrak{L}$ is quasi-regularly imbedded in M and is an open submanifold of precisely one maximal integral manifold of $\mathfrak{L}$.*

Proof. Let $\mathfrak{I}$ be the collection of all subsets of M which are unions of integral manifolds of $\mathfrak{L}$. It follows from Lemma 1.3.4 that $\mathfrak{I}$ is a topology for M finer than its original topology. It is clear that $(M,\mathfrak{I})$ is a Hausdorff locally connected space. Let $\{M_\xi : \xi \in J\}$ be the set of connected components of $(M,\mathfrak{I})$. Each M_ξ is an open subspace of $(M,\mathfrak{I})$ and if S is any integral manifold of $\mathfrak{L}$, the underlying topological space of S is an open subspace of exactly one M_ξ.

We now prove that the M_ξ are second countable. Fix $\xi \in J$. Let U be an

open set with coordinates $x_1, \ldots, x_m$ and let $a > 0$ be such that $(U; x_1, \ldots, x_m; a)$ is adapted to $\mathfrak{L}$. Since M_ξ as well as the $U(\mathbf{a})$ are open in $(M,\mathfrak{I})$, it follows that $M_\xi \cap U(\mathbf{a})$ is an open subspace of $U(\mathbf{a})$ for all $a \in I_a^{m-p}$. Now, $M_\xi \cap U$ is open in M_ξ and is the disjoint union of the $M_\xi \cap U(\mathbf{a})$, so each connected component of $M_\xi \cap U$ is an open subspace of some $U(\mathbf{a})$ and is therefore second countable. Since M (and hence M_ξ) can be covered by countably many open sets such as U, Lemma 1.3.5 can be used to conclude that M_ξ is second countable.

It is now obvious that there is a unique analytic structure on M_ξ such that each integral manifold of $\mathfrak{L}$ contained in M_ξ is an open submanifold of M_ξ. With this structure, M_ξ becomes a submanifold of M. It is also obvious that each M_ξ is a maximal integral manifold of $\mathfrak{L}$. Theorem 1.3.6 is completely proved.

It may be remarked that Theorems 1.3.3 and 1.3.6 are valid in the complex analytic case also. No change is necessary either in the formulations or in the proofs.

1.4. Appendix

In this appendix we discuss briefly some elementary results on analytic systems of ordinary differential equations. We work in $\mathbf{R}^m$ or $\mathbf{C}^m$. For any $a > 0$, let

$$I_a^m = \{(t_1, \ldots, t_m) : t_j \in \mathbf{R}, |t_j| < a \quad \text{for} \quad 1 \leq j \leq m\},$$
$$J_a^m = \{(z_1, \ldots, z_m) : z_j \in \mathbf{C}, |z_j| < a \quad \text{for} \quad 1 \leq j \leq m\}.$$

Let $a > 0$ and let $G_1, \ldots, G_m$ be m real functions defined and analytic on I_a^m. We consider the system of ordinary differential equations:

$$\frac{du_j}{dt} = G_j(u_1(t), \ldots, u_m(t)) \quad (1 \leq j \leq m). \tag{1.4.1}$$

If the G_j are defined and holomorphic on J_a^m, we consider the system

$$\frac{du_j}{dz} = G_j(u_1(z), \ldots, u_m(z)) \quad (1 \leq j \leq m). \tag{1.4.2}$$

Theorem 1.4.1. *Let $a > 0$ and let $G_1, \ldots, G_m$ be real functions defined and analytic on I_a^m. Then*

(*a*) *if u_j, v_j $(1 \leq j \leq m)$ are analytic functions defined on an open interval Δ containing 0 such that $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ are both solutions of* (1.4.1) *on Δ with $u_j(0) = v_j(0)$ $(1 \leq j \leq m)$, then $u_j = v_j$ on Δ for $1 \leq j \leq m$.*

(b) *there exists* b *with* $0 < b < a$ *and real analytic functions* u_j *on* I_b^{m+1} $(1 \leq j \leq m)$ *such that*

(i) $|u_j(t, y_1, \ldots, y_m)| < a$ *for* $(t, y_1, \ldots, y_m) \in I_b^{m+1}$

(ii) $$\frac{\partial u_j(t, y_1, \ldots, y_m)}{\partial t} = G_j(u_1(t, y_1, \ldots, y_m), \ldots, u_m(t, y_1, \ldots, y_m))$$

$$u_j(0, y_1, \ldots, y_m) = y_j$$

for $(t, y_1, \ldots, y_m) \in I_b^{m+1}$, $1 \leq j \leq m$.

Proof. (a) If $(\varphi_1, \ldots, \varphi_m)$ is solution of (1.4.1), we have, for $1 \leq j \leq m$,

$$\varphi_j'(0) = G_j(\varphi_1(0), \ldots, \varphi_m(0))$$

$$\varphi_j''(0) = \sum_{1 \leq r \leq m} \frac{\partial G_j}{\partial t_r}(\varphi_1(0), \ldots, \varphi_m(0))\varphi_r'(0),$$

and so on. A simple induction on $s \geq 1$ shows that the initial vector $(\varphi_1(0), \ldots, \varphi_m(0))$ completely determines the values of all the derivatives $\varphi_j^{(s)}(0)$ $(s \geq 1, 1 \leq j \leq m)$. So if the φ_j are analytic on an open interval Δ containing 0, they are completely determined on Δ by the initial vector $(\varphi_1(0), \ldots, \varphi_m(0))$. (a) follows at once from this.

(b) Replacing a by a smaller positive number, we may assume that the power series expansions of the G_j around the origin converge absolutely and uniformly in I_a^m. Hence the G_j are restrictions to I_a^m of holomorphic functions on J_a^m. We also denote the latter by G_j. Let $0 < c < a$, and

$$\gamma = \max_{1 \leq j \leq m} \sup_{(z_1, \ldots, z_m) \in J_c^m} |G_j(z_1, \ldots, z_m)|.$$

Then γ is finite. Choose a constant $L \geq 1$ such that

$$\max_{1 \leq j \leq m} |G_j(z_1, \ldots, z_m) - G_j(z_1', \ldots, z_m')| \leq L \max_{1 \leq j \leq m} |z_j - z_j'| \tag{1.4.3}$$

for all $(z_1, \ldots, z_m), (z_1', \ldots, z_m') \in J_c^m$. Finally, select b with $0 < b < c(1 + \gamma)^{-1}$ and $Lb < 1$.

Now define a sequence

$$(u_{1,N}, \ldots, u_{m,N}) \quad (N = 0, 1, 2, \ldots)$$

of vector-valued functions as follows. Put

$$u_{j,0} = 0 \quad (1 \leq j \leq m); \tag{1.4.4}$$

for $N \geq 1$ and $(z, z_1, \ldots, z_m) \in J_b^{m+1}$, put

$$\begin{aligned} &u_{j,N}(z, z_1, \ldots, z_m) \\ &\quad = z_j + \int_0^z G_j(u_{1,N-1}(z', z_1, \ldots, z_m), \ldots, u_{m,N-1}(z', z_1, \ldots, z_m))dz', \end{aligned} \tag{1.4.5}$$

where the integral is taken along the line segment from 0 to z. We claim that for any $N \geq 0$, the $u_{j,N}(1 \leq j \leq m)$ are well defined and holomorphic on J_b^{m+1}, and that

$$u_{j,N}(z, z_1, \ldots, z_m)| < c$$

for $1 \leq j \leq m$ and $(z, z_1, \ldots, z_m) \in J_b^{m+1}$. We prove this claim by induction on N. For $N = 0$ there is nothing to prove. Let $N \geq 1$ and assume the result for $N - 1$. It is clear from (1.4.5) that $u_{j,N}$ is well defined and holomorphic on J_b^{m+1}. Further, if $(z, z_1, \ldots, z_m) \in J_b^{m+1}$, we have for $1 \leq j \leq m$

$$\begin{aligned} |u_{j,N}(z, z_1, \ldots, z_m)| &\leq b + \gamma \left| \int_0^z dz' \right| \\ &\leq b(1 + \gamma) \\ &< c, \end{aligned}$$

carrying forward the induction. Our claim is thus proved.

Now for $N \geq 1$ and $(z, z_1, \ldots, z_m) \in J_b^{m+1}$

$$\begin{aligned} &|u_{j,N+1}(z, z_1, \ldots, z_m) - u_{j,N}(z, z_1, \ldots, z_m)| \\ &\quad \leq Lb \max_{1 \leq j \leq m} \sup_{(z, z_1, \ldots, z_m) \in J_b^{m+1}} |u_{j,N}(z, z_1, \ldots, z_m) - u_{j,N-1}(z, z_1, \ldots, z_m)|, \end{aligned}$$

from (1.4.5) and (1.4.3). Applying this estimate in succession and noting that $|u_{j,1}(z, z_1, \ldots, z_m)| < c$ for $1 \leq j \leq m$ and that $(z, z_1, \ldots, z_m) \in J_b^{m+1}$, we get

$$\max_{1 \leq j \leq m} \sup_{(z, z_1, \ldots, z_m) \in J_b^{m+1}} |u_{j,N+1}(z, z_1, \ldots, z_m) - u_{j,N}(z, z_1, \ldots, z_m)| \leq c(Lb)^N.$$

Since $Lb < 1$, it follows that the series

$$\sum_{N \geq 0} \{u_{j,N+1}(z, z_1, \ldots, z_m) - u_{j,N}(z, z_1, \ldots, z_m)\}$$

converges uniformly in J_b^{m+1} for $1 \leq j \leq m$. Let $u_j(z, z_1, \ldots, z_m)$ be the sum. Then u_j is holomorphic on J_b^{m+1} and

$$u_j(z, z_1, \ldots, z_m) = \lim_{N \to \infty} u_{j,N}(z, z_1, \ldots, z_m) \quad (1 \leq j \leq m) \tag{1.4.6}$$

for $(z, z_1, \ldots, z_m) \in J_b^{m+1}$. (1.4.6) and (1.4.5) now yield

$$
\begin{aligned}
& u_j(z,z_1,\ldots,z_m) \\
(1.4.7)\qquad & = z_j + \int_0^z G_j(u_1(z',z_1,\ldots,z_m),\ldots,u_m(z',z_1,\ldots,z_m))\,dz'
\end{aligned}
$$

for $1 \leq j \leq m$ and $(z,z_1,\ldots,z_m) \in J_b^{m+1}$. Restricting to I_b^{m+1} and differentiating (1.4.7) with respect to z, we get

$$
\frac{\partial u_j(t,y_1,\ldots,y_m)}{\partial t} = G_j(u_1(t,y_1,\ldots,y_m),\ldots,u_m(t,y_1,\ldots,y_m))
$$
$$
u_j(0,y_1,\ldots,y_m) = y_j
$$

for $1 \leq j \leq m$ and $(t,y_1,\ldots,y_m) \in I_b^{m+1}$. The u_j being analytic on I_b^{m+1}, the theorem is proved.

The holomorphic version of Theorem 1.4.1 with the differential equations (1.4.2) instead of (1.4.1) is proved as above with minor variations. We leave its formulation and proof to the reader.

In applications it often happens that the G_j depend analytically on certain parameters. In this case, the solutions u_j also have the same analytic dependence on these parameters.

Theorem 1.4.2. *Let N be an analytic manifold, $a > 0$, and let the real functions G_j be defined and analytic on $I_a^m \times N$. Fix $x \in N$. Then we can find b with $0 < b < a$, an open subset N_x of N containing x, and real analytic functions $u_1,\ldots,u_m$ on $I_b^{m+1} \times N_x$ such that*

$$
\frac{\partial u_j(t,y_1,\ldots,y_m,x')}{\partial t} = G_j(u_1(t,y_1,\ldots,y_m,x'),\ldots,u_m(t,y_1,\ldots,y_m,x'))
$$
$$
u_j(0,y_1,\ldots,y_m,x') = y_j
$$

for $1 \leq j \leq m$, $(t,y_1,\ldots,y_m) \in I_b^{m+1}$ and $x' \in N_x$.

Proof. We may assume that for some $d > 0$, $N = I_d^n$, $x = (0,\ldots,0)$, and that the G_j are the restrictions to $I_a^m \times I_d^n$ of functions (denoted again by G_j) defined and holomorphic on $J_a^m \times J_d^n$. let $0 < c < a$, $0 < e < d$, and let $N' = J_e^n$. Define γ by

$$
\gamma = \max_{1\leq j\leq m} \sup_{(z_1,\ldots,z_m)\in J_c^m,\, x'\in N'} |G_j(z_1,\ldots,z_m,x')|
$$

and let $L \geq 1$ be a constant such that

$$
|G_j(z_1,\ldots,z_m,x') - G_j(z'_1,\ldots,z'_m,x')| \leq L \max_{1\leq j\leq m} |z_j - z'_j|
$$

for all $x' \in N'$ $(z_1,\ldots,z_m)$, $(z'_1,\ldots,z'_m) \in J_c^m$. Choose b such that $0 < b < c(1+\gamma)^{-1}$ and $Lb < 1$; we then define the sequence $u_{j,N}$ as follows. For $N = 0$, put $u_{j,0} = 0$ $(1 \leq j \leq m)$; for $N \geq 1$ write

$$
\begin{aligned}
& u_{j,N}(z,z_1,\ldots,z_m,x') \\
& = z_j + \int_0^z G_j(u_1(z',z_1,\ldots,z_m,x'),\ldots,u_m(z',z_1,\ldots,z_m,x'),x')\,dz'
\end{aligned}
$$

for $1 \leq j \leq m$, $(z,z_1,\ldots,z_m) \in J_b^{m+1}$, $x' \in N'$. Theorem 1.4.2 is now proved by arguing exactly as in the previous theorem. We leave the details to the reader.

The same proof also gives the holomorphic version of the above result.

EXERCISES

1. Consider $\mathbf{C}^2$ as a four-dimensional real analytic manifold, and let $\mathbf{T}^2 = \{(z_1,z_2): z_1,z_2 \in \mathbf{C}, |z_1| = |z_2| = 1\}$; show that $\mathbf{T}^2$ is a regularly imbedded compact submanifold. Prove that if $\alpha \in \mathbf{R}$ is irrational, the map $t \mapsto (e^{it},e^{i\alpha t})$ $(t \in \mathbf{R})$ is an imbedding of $\mathbf{R}$ into $\mathbf{T}^2$ which is quasi-regular but not regular.

2. Let $n \geq 2$ and let π be the map of $\mathbf{R}^n$ into $\mathbf{R}^1$ given by

$$
\pi(x_1,\ldots,x_n) = x_1^2 + \cdots + x_n^2.
$$

Let $M = \mathbf{R}^n \setminus \{0\}$, $N = \{t: t \in R, t > 0\}$. Let $D = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. Prove that there is a unique differential operator $\tilde{D}$ on N such that D and $\tilde{D}$ are π-related. Calculate $\tilde{D}$.

3. (a) Let F be a field of characteristic 0; V (resp. W) a vector space over F of finite dimension m (resp. n); and γ a linear map of V onto W with kernel U. Let λ (resp. μ) be a nonzero element of $\Lambda_m(V)$ (resp. $\Lambda_n(W)$). Prove that there is exactly one $\nu \in \Lambda_{m-n}(U)$ with the following property: let $u_1,\ldots,u_{m-n},v_1,\ldots,v_n$ be a basis for V such that $u_1,\ldots,u_{m-n}$ span U; then

$$
\nu(u_1,\ldots,u_{m-n}) = \frac{\lambda(u_1,\ldots,u_{m-n},v_1,\ldots,v_n)}{\mu(\gamma v_1,\ldots,\gamma v_n)}
$$

We write $\nu = (\lambda/\mu)_\gamma$.

(b) Let M and N be analytic manifolds of dimensions m and n respectively. Let $\omega^1 \in \mathfrak{A}_m(M)$ and $\omega^2 \in \mathfrak{A}_n(N)$, and suppose that ω^1 and ω^2 vanish nowhere. Let π be a submersion of M onto N, and for each $y \in N$ let $P_y = \pi^{-1}(\{y\})$. Prove that the P_y are closed regular submanifolds of M. For $y \in N$ and $x \in P_y$ let $\omega_x^y = (\omega_x^1/\omega_x^2)_{(d\pi)_x}$. Prove that $\omega^y: x \mapsto \omega_x^y$ is an element of $\mathfrak{A}_{m-n}(P_y)$ for each $y \in N$ and that $y \mapsto \omega^y$ is analytic in a natural sense.