HOMEWORK, DUE OCTOBER 22, 2007

Problem 1: Gauss map and its differential

For every $k > 0$ let

$$P_k = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = x^2 + ky^2\}$$

be the paraboloid in $\mathbb{R}^3$. This is a regular surface. Study the Gauss map and its differential at the point $p = (0, 0, 0) \in P_k$.

Problem 2: Gauss map

(1) Problem 8 on page 151.
(2) Find a regular surface $S \subset \mathbb{R}^3$ such that the image of the Gauss map $N : S \to \mathbb{S}^2$ in $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is
   (a) the point $p = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \in \mathbb{S}^2$ for a fixed $0 < \theta < \pi$ and $0 < \phi < 2\pi$.
   (b) the equator.
   (c) the set U, where $U \subset \mathbb{S}^2$ is any open set.

Problem 3: Curvature of Curves

Let $S \subset \mathbb{R}^3$ be a regular surface with Gaussian curvature $K > 0$. Let $C \subset S$ be a regular curve. Let κ_1, κ_2 be the principal curvatures of S at p. Let κ be the curvature of the curve C at p

(1) Show that the curvature κ of the curve C at p satisfies

$$\kappa \geq \min(|\kappa_1|, |\kappa_2|).$$

(2) Assume that S has the property that $|\kappa_1| \leq 1$ and $|\kappa_2| \leq 1$ for all $p \in S$. Is it true that the curvature κ of the curve C at p satisfies $|\kappa| \leq 1$? If yes, prove it, if no, give a counterexample.

Problem 4: The Pseudosphere

Problem 6 on page 168.

Problem 5: Mean curvature

Problem 5 on page 151.

Not required

Compute the Gauss curvature of the ellipsoid

$$E = \{(x, y, z) \in \mathbb{R}^3 \,|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$