Problem 1: Surfaces of Revolution

Let $S$ be a surface of revolution and $C$ its generating curve. (See the previous homework assignment for definition).

(1) Show that one can find local parametrizations $x(u, v)$ of $S$ such that in this local parametrizations the first fundamental form is given by $E = E(v)$, $F = 0$, $G = 1$, where $E(v)$ is a function depending on $v$, but not on $u$. (Note: in the previous homework assignment the variables $(u, v)$ where denoted by $(\phi, t)$.)

(2) Let $s$ be the arc length of the regular curve $C$ and $\rho = \rho(s)$ the distance to the rotation axis of the point of $C$ corresponding to $s$. Show Pappus' Theorem: The area of $S$ is

$$2\pi \int_0^l \rho(s) ds,$$

where $l$ is the length of $C$.

(3) Compute the area of the torus which is obtained by rotating a circle $\alpha(v) = (R + r \cos(v), 0, r \sin(v))$ with $0 \leq v < 2\pi$ and $0 < r < R$, around the $z$-axis.

Problem 2: Tubular Surfaces

Let $I \subset \mathbb{R}$ be an open interval and $\alpha : I \rightarrow \mathbb{R}^3$ a regular parametrized curve, parametrized by arc length, with nonzero curvature $\kappa(s)$ everywhere. Let $K = \sup_{s \in I} \kappa(s)$. Let $n(s)$ be the normal vector to $\alpha$ in $s$ and $b(s)$ the bi-normal vector of $\alpha$ at $s$. Fix some constant $\frac{1}{K} > r > 0$. The parametrized surface

$$x(s, v) = \alpha(s) + r(n(s) \cos(v) + b(s) \sin(v))$$

with $s \in I$, $0 < v < 2\pi$ is called the tube of radius $r$ around $\alpha$.

(1) Show that the parametrized surface $x$ is regular.

(2) Show that a unit normal vector to the surface at $p = x(s, v)$ is given by $N(s, v) = -(n(s) \cos(v) + b(s) \sin(v))$.

(3) Show that the area of the tube is $2\pi r$ times the length of the curve $\alpha(I)$.

Problem 3: Generalized Helicoids

Read Example 3 on page 94. Solve Problem 13 on page 101.
Problem 4

Problem 15 on page 90.

Problem 5

Problem 3 on page 99

Problem 6

Problem 2 on page 99.

Not required

I will probably not cover this by Wednesday - so keep it in mind for the next homework!

For every $k > 0$ let

$$P_k = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + ky^2\}$$

be the paraboloid in $\mathbb{R}^3$. This is a regular surface. Study the Gauss map and its differential at the point $p = (0, 0, 0) \in P_k$. 