HOMEWORK, DUE OCTOBER 8, 2007

Problem 1: Surfaces of Revolution

The set $S \subset \mathbb{R}^3$ obtained by rotating a regular plane curve C about an axis in the plane containing C which does not meet C is called a *Surface of Revolution*.

To be more concrete, let $I = (a, b) \subset \mathbb{R}$ be an open interval and

$$\alpha : I \to \mathbb{R}^3, \quad \alpha(t) = (f(t), 0, g(t))$$

be a parametrization of the curve $C = \alpha(I)$ in the xz-plane, where f, g are differentiable functions and $f(t) > 0$ for all $t \in I$. Let 's rotate C around the z-axis and denote the rotation angle about the z-axis by ϕ. Fix an angle ϕ_0. Thus, we obtain a map from the open set

$$U = \{(\phi, t) \in \mathbb{R}^2 \,|\, \phi_0 < \phi < \phi_0 + 2\pi,\ a < t < b\}$$

given by

$$\mathbf{x}_{\phi_0} : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad \mathbf{x}_{\phi_0}(\phi, t) = (f(t)\cos(\phi), f(t)\sin(\phi), g(t))\,.$$

The curve C is called the *generating curve* of S, the z-axis is called the *rotation axis* of S. The circles described by points on C are called the *parallels* of S, the various positions of C are called the *meridians* of S.

(1) Prove that $S \subset \mathbb{R}^3$ is a regular surface and that the two local parametrizations $\mathbf{x}_0$ and $\mathbf{x}_\pi$ suffice to cover S.
(2) Give an example of a surface of revolution.
(3) Prove that the rotation of a surface of revolution S around its axis is a diffeomorphism of S.
(4) Compute the first fundamental form of a surface of revolution in the local parametrization x_{ϕ_0}.

Problem 2: Differentiable Maps

(1) *(Distance Function)* Let $S \subset \mathbb{R}^3$ be a regular surface. Fix a point $p_0 \in \mathbb{R}^3$. The square of the distance function with respect to p_0 is given by

$$f : S \to \mathbb{R}, \quad f(p) = |p - p_0|^2 =< p - p_0, p - p_0 >\,.$$

Prove that $f : S \to \mathbb{R}$ is differentiable and determine its differential $df_p : T_pS \to \mathbb{R}$ at a point $p \in S$.
(2) Problem 2 on page 80.
(3) Problem 3 on page 80.

Problem 3: Graphs

Let $U \subset \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}$ a differentiable function. Consider the regular surface $S = \mathrm{graph}(f) \subset \mathbb{R}^3$, that is

$$S = \mathrm{graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in U,\ z = f(x, y)\}$$

(1) Show that the tangent plane of S at a point $p = (x_0, y_0, z_0)$ is given by

$$T_pS = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)\}.$$

(2) Show that the tangent plane of S at a point p is the graph of the differentiable function $df_p : \mathbb{R}^2 \to \mathbb{R}$.

(3) Compute the first fundamental form of S in the local parametrization $\mathbf{x}(x, y) = (x, y, f(x, y))$.

Problem 4

Problem 24 on page 91.

Problem 5

Problems 1 and 3 on page 99.

Not required

Prove the *Inverse Function Theorem* for regular surfaces:
Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces and $\phi : S_1 \to S_2$ a differentiable map. Suppose that for $p \in S_1$ the differential $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$ is an isomorphism. Then ϕ is a local diffeomorphism at $p \in S_1$.