18.950 Handout 4. Inverse and Implicit Function Theorems.

Theorem 1 (Inverse Function Theorem). *Suppose $U \subset \mathbf{R}^n$ is open, $f : U \to \mathbf{R}^n$ is C^1, $x_0 \in U$ and df_{x_0} is invertible. Then there exists a neighborhood V of x_0 in U and a neighborhood W of $f(x_0)$ in $\mathbf{R}^n$ such that f has a C^1 inverse $g = f^{-1} : W \to V$. (Thus $f(g(y)) = y$ for all $y \in W$ and $g(f(x)) = x$ for all $x \in V$.) Moreover,*

$$dg_y = (df_{g(y)})^{-1} \quad \text{for all } y \in W$$

and g is smooth whenever f is smooth.

Remark. The theorem says that a *continuously differentiable* function f between regions in $\mathbf{R}^n$ is *locally* invertible near points where its differential is invertible.

Proof. Without loss of generality, we may assume that $x_0 = 0$, $f(x_0) = 0$ and $df_{x_0} = I$. (Otherwise, replace f with $\widetilde{f}(x) = df_{x_0}^{-1}(f(x + x_0) - f(x_0))$. Note that if the theorem holds with $\widetilde{f}$, 0, 0, I and a function $\widetilde{g}$ in place of f x_0, $f(x_0)$, df_{x_0} and g respectively, then it is easily verified that the theorem as stated holds with $g(y) = x_0 + \widetilde{g}(df_{x_0}^{-1}(y - f(x_0)))$.)

Since df_x is continuous in x at x_0 (see Exercise 1), there exists a number $r > 0$ such that

$$x \in \overline{B}_r(0) \implies \|df_x - I\| \leq \frac{1}{2}.$$

(Recall that for a linear transformation $A : \mathbf{R}^n \to \mathbf{R}^m$ we define the norm of A by $\|A\| = \sup_{\{|v| \leq 1\}} |A(v)|$.) Fix $y \in B_{r/2}(0)$. Define a function ϕ by

$$\phi(x) = x - f(x) + y.$$

Note that $d\phi_x = I - df_x$ and hence

$$\|d\phi_x\| \leq 1/2 \quad \text{if } x \in \overline{B}_r(0).$$

Thus

$$\begin{aligned} |\phi(x)| &\leq |\phi(x) - y| + |y| = |\int_0^1 \frac{d}{dt}\phi(tx)dt| + |y| \\ &= |\int_0^1 d\phi_{tx} \cdot x dt| + |y| \leq \int_0^1 \|d\phi_{tx}\||x|dt + |y| \\ &\leq r/2 + r/2 = r \end{aligned} \qquad (1)$$

whenever $x \in \overline{B}_r(0)$. i.e. ϕ is a map from $\overline{B}_r(0)$ into itself. For any $x, z \in \overline{B}_r(0)$,

$$
\begin{aligned}
|\phi(z) - \phi(x)| &= \left| \int_0^1 \frac{d}{dt}\phi(x + t(z-x))dt \right| \\
&\leq \int_0^1 |d\phi_{x+t(z-x)} \cdot (z-x)|dt \\
&\leq \int_0^1 \|d\phi_{x+t(z-x)}\||z-x|dt \\
&\leq \frac{1}{2}|z-x|.
\end{aligned}
$$

Thus $\phi : \overline{B}_r(0) \to \overline{B}_r(0)$ is a contraction, and hence ϕ has a unique fixed point $x_y \in \overline{B}_r(0)$. i.e. there is a unique point $x_y \in \overline{B}_r(0)$ with $f(x_y) = y$. In fact $x_y \in B_r(0)$ since $\frac{r}{2} > |y| = |f(x_y)| \geq |x_y| - |x_y - f(x_y)| \geq |x_y| - \frac{1}{2}|x_y| = \frac{1}{2}|x_y|$. Set $W = B_{r/2}(0)$ and $V = f^{-1}(W) \cap B_r(0)$. Note then that V is open. Define $g : W \to V$ by $g(y) = x_y$. Then $f(g(y)) = y$ for all $y \in W$ and $g(f(x)) = x$ for all $x \in V$.

Next we show that g is differentiable, with $dg_y = (df_{g(y)})^{-1}$. First note that with $\psi : B_r(0) \to \mathbf{R}^n$ defined by $\psi(x) = x - f(x)$, we have that for $x_1, x_2 \in B_r(0)$,

$$
\begin{aligned}
|x_1 - x_2| - |f(x_1) - f(x_2)| &\leq |(x_1 - x_2) - (f(x_1) - f(x_2))| \\
&\leq |\psi(x_1) - \psi(x_2)| \\
&\leq \frac{1}{2}|x_1 - x_2|
\end{aligned}
$$

where the last inequality follows by estimating as in (1), using $d\psi_x = I - df_x$. Hence

$$
\frac{1}{2}|x_1 - x_2| \leq |f(x_1) - f(x_2)|
$$

for any $x_1, x_2 \in B_r(0)$, which implies

$$
|g(y_1) - g(y_2)| \leq 2|y_1 - y_2| \tag{2}
$$

for any $y_1, y_2 \in W = B_{r/2}(0)$. In particular, g is continuous.

Now fix $y \in W$, and let $A = df_{g(y)}$. Since W is open, there exists $\delta > 0$ such that $y + k \in W$ if $k \in B_\delta(0)$. Let $h = g(y+k) - g(y)$. Then $k = y + k - y = f(g(y+k)) - f(g(y)) = f(g(y)+h) - f(g(y))$ and hence, for $k \in B_\delta(0) \setminus \{0\}$,

$$\begin{aligned} \frac{|g(y+k) - g(y) - A^{-1}k|}{|k|} &= \frac{|A^{-1}(Ah - k)|}{|h|}\frac{|h|}{|k|} \\ &\leq \frac{\|A^{-1}\||k - Ah|}{|h|}\frac{|h|}{|k|} \\ &\leq 2\frac{\|A^{-1}\||f(g(y)+h) - f(g(y)) - Ah|}{|h|} \end{aligned} \tag{3}$$

where the last estimate follows from (2). Note that since $g(y+k) = g(y) \implies f(g(y+k)) = f(g(y)) \implies y + k = y \implies k = 0$, we have that $h \neq 0$ if $k \neq 0$. Sice $A = df_{g(y)}$, it follows from the definition of differentiability of f that the right hand side of (3) tends to 0 as $h \to 0$, and hence, since $|h| \leq 2|k|$ by (2), it follows that

$$\lim_{k \to 0} \frac{|g(y+k) - g(y) - A^{-1}k|}{|k|} = 0.$$

i.e. g is differentiable at y and

$$dg_y = (df_{g(y)})^{-1}. \tag{4}$$

Finally, note that the function $y \mapsto dg_y$ is the composition of the function $y \mapsto df_{g(y)}$ and matrix inversion $A \mapsto A^{-1}$. Matrix inversion is a smooth map of the entries, and the function $y \mapsto df_{g(y)}$ is continuous since g is continuous and f is C^1. Hence we conclude that $y \mapsto dg_y$ is continuous; i.e. that g is C^1. Repeatedly differentiating (4) shows that g is smooth if f is smooth. □

Exercise 1. Let $L(\mathbf{R}^n; \mathbf{R}^n)$ be the set of linear transformations from $\mathbf{R}^n$ into itself with the metric $d(A, B) = \|A - B\|$. (cf. Exercise 10 of handout 1.) Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}^n$ be a C^1 function. Show that the map $x \mapsto df_x$ is continuous as a map from U into $L(\mathbf{R}^n; \mathbf{R}^n)$.

Exercise 2. Suppose $g : [a, b] \to \mathbf{R}^n$ is continuous. Show that

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|dt$$

where $|\cdot|$ denotes the Euclidean norm. You may use without proof that $\left|\int_a^b h(t)dt\right| \leq \int_a^b |h(t)|dt$ for a scalar valued function h.

Exercise 3. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. Compute $f'(x)$ for all $x \in \mathbf{R}$. Show that $f'(0) > 0$, yet f is not one-to-one in any neighborhood of 0. This example shows that in the Inverse Function Theorem, the hypothesis that f is C^1 cannot be weakened to the hypothesis that f is differentiable.

Exercise 4. Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that f is C^1 and that $df_{(x,y)}$ is invertible for all $(x, y) \in \mathbf{R}^2$ and yet f is not a one-to-one function globally. Why doesn't this contradict the Inverse Function Theorem?

Next we prove the *Implicit Function Theorem.* This theorem gives conditions under which one can solve, locally, a system of equations

$$f_i(x, y) = 0, \qquad i = 1, 2, \ldots n$$

where $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$, for y in terms of x. (Thus, $y = (y_1, \ldots, y_n)$ where $y_1, \ldots, y_n$ are regarded as n unknowns, satisfying the n equations $f_i(x, y) = 0$, $i = 1, \ldots, n$.) Geometrically, the set of solutions (x, y) to the system of equations is the graph of a function $y = g(x)$. Note that we have from linear algebra that if for each i, the function f_i is linear with constant coefficients in the variables y_j, then whenever the (constant) $n \times n$ matrix $\left(\frac{\partial f_i}{\partial y_j}\right)_{1\leq i,j\leq n}$ is invertible, the system of equations is solvable for y in terms of x. Implicit function theorem says that whenever f_i are C^1 and this matrix is invertible at a point (a, b), then the system is solvable for y in terms of x locally in a neighborhood of (a, b).

We shall use the following notation: For an $\mathbf{R}^n$ valued function $f(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_n(x, y))$ in a domain $U \subset \mathbf{R}^{m+n} \equiv \mathbf{R}^m \times \mathbf{R}^n$, where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, we shall denote by $d_x f$ the partial differential represented by the $n \times m$ matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i\leq n, 1\leq j\leq m}$ and by $d_y f$ the partial differential represented by the $n \times n$ matrix $\left(\frac{\partial f_i}{\partial y_j}\right)_{1\leq i,j\leq n}$.

Theorem 2 (Implicit Function Theorem). *Let $U \subset \mathbf{R}^{m+n} \equiv \mathbf{R}^m \times \mathbf{R}^n$ be an open set, $f : U \to \mathbf{R}^n$ a C^1 function, $(a, b) \in U$ a point such that $f(a, b) = 0$ and $d_y f|_{(a,b)}$ invertible. Then there exists a neighborhood V of*

(a,b) in U, a neighborhood W of a in $\mathbf{R}^m$ and a C^1 function $g : W \to \mathbf{R}^n$ such that

$$\{(x,y) \in V \,:\, f(x,y) = 0\} = \{(x, g(x)) \,:\, x \in W\}\,.$$

Moreover,

$$dg_x = -\left.(d_y\, f)^{-1}\right|_{(x,g(x))} \, d_x\, f|_{(x,g(x))}$$

and g is smooth if f is smooth.

Proof. Define $F : U \to \mathbf{R}^{m+n}$ by $F(x,y) = (x, f(x,y))$. Then F is C^1 in U, $F(a,b) = (a,0)$ and $\det dF_{(a,b)} = \det \left. d_y\, f\right|_{(a,b)} \neq 0$. Hence by the Inverse Function Theorem, F has a C^1 inverse $F^{-1} : \widetilde{W} \to V$ for neighborhoods V of (a,b) and $\widetilde{W}$ of $(a,0)$ in $\mathbf{R}^m \times \mathbf{R}^n$. Set $W = \{x \in \mathbf{R}^m : (x,0) \in \widetilde{W}\}$. Then W is open in $\mathbf{R}^m$. Note then that if $x \in W$, then $(x,0) \in \widetilde{W}$ so that $(x,0) = F(x_1, y_1)$ where $(x_1, y_1) \in V$ is uniquely determined by x. (In fact, by the definition of F, $x_1 = x$.) Define $g : W \to \mathbf{R}^n$ by setting $y_1 = g(x)$. Thus $g(x)$ is defined by $F^{-1}(x,0) = (x, g(x))$; i.e. by $g(x) = \pi \circ F^{-1}(x,0)$ where $\pi : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection map $\pi(x,y) = y$. Then $\{(x,y) \in V : f(x,y) = 0\} = \{(x,y) \in V : F(x,y) = (x,0)\} = \{(x, g(x)) : x \in W\}$. Since π is a smooth map and F^{-1} is C^1, it follows that g is C^1. The formula for dg_x follows by differentiating the identity

$$f(x, g(x)) \equiv 0 \quad \text{on } W$$

using the chain rule. By repeatedly differentiating this identity, it follows that g is smooth if f is smooth. □