18.950 Handout 3. Existence and Uniqueness of Solutions to O.D.E.'s

Let $l \in (0, \infty]$, $R \in (0, \infty)$, $y_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}$. Suppose

$$F : (t_0 - l, t_0 + l) \times \overline{B}_R(y_0) \to \mathbf{R}^n$$

is a continuous function. Consider the first order O.D.E. initial value problem:

$$\frac{du}{dt} = F(t, u) \quad \text{and} \quad u(t_0) = y_0. \tag{$\star$}$$

In general, we cannot expect to find a solution to the system $(\star)$ that exists for all t. e.g. for $n = 1$, if we take $F(t, u) = 1 + u^2$, $t_0 = 0$ and $y_0 = 0$, then the solution (which can be found by separation of variables) is $u(t) = \tan t$ which blows up as $t \to \pm\pi/2$. The theorem below gives sufficient conditions under which a unique solution to $(\star)$ exists *locally* in some neighborhood of t_0. Before stating the theorem, we need the following:

Definition. A map $f : \Omega \subseteq \mathbf{R}^m \to \mathbf{R}^n$ is said to be Lipschitz if there exists a number L such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in \Omega$. The infimum of the set of such L is called the Lipschitz constant of f, denoted $\operatorname{Lip} f$.

Theorem 1. *Suppose $F = F(t, y)$ as above is continuous. Suppose also that F is Lipschitz in the y variable uniformly for all $t \in (t_0 - l, t_0 + l)$. i.e. $\|F(t, y_1) - F(t, y_2)\| \leq L\|y_1 - y_2\|$ for some $L \in (0, \infty)$ and for all $y_1, y_2 \in B_R(y_0)$ and all $t \in (t_0 - l, t_0 + l)$. Then there exists $c > 0$ such that the initial value problem $(\star)$ has a unique solution $u(t)$ for $t \in (t_0 - c, t_0 + c)$.*

Proof. First note that by the fundametal theorem of calculus, u is a solution to $(\star)$ in an interval $(t_0 - c, t_0 + c)$ if and only if

$$u(t) = y_0 + \int_{t_0}^{t} F(s, u(s))ds \tag{1}$$

for $t \in (t_0 - c, t_0 + c)$. Let $M = \max_{I \times \overline{B}_R(y_0)} |F(t, y)|$ where $I = [t_0 - l_1, t_0 + l_1]$, $l_1 = \min\{l/2, 1\}$. Set $c = \min\{l_1, \frac{R}{M}, \frac{1}{2L}\}$. For $u \in \mathcal{C}((t_0 - c, t_0 + c), \overline{B}_R(y_0))$ (equipped with the *sup* metric, see handout 2), define a function $\phi(u)$ by

$$\phi(u)(t) = y_0 + \int_{t_0}^{t} F(s, u(s))ds. \tag{2}$$

Note that $\phi(u)$ is continuous (in fact differentiable), and for all $t \in (t_0 - c, t_0 + c)$,

$$|\phi(u)(t) - y_0| \leq M|t - t_0| \leq Mc \leq R. \tag{3}$$

Thus $\phi(u) \in \mathcal{C}\left((t_0 - c, t_0 + c), \overline{B}_R(y_0)\right)$. Moreover, ϕ is a contraction because

$$\begin{aligned} |\phi(u)(t) - \phi(v)(t)| &\leq \left| \int_{t_0}^{t} F(t, u(t)) - F(t, v(t)) dt \right| \\ &= \left| \int_{t_0}^{t} L|u(t) - v(t)| dt \right| \\ &\leq L|t - t_0| \|u - v\|_{\sup} \leq \frac{1}{2} \|u - v\|_{\sup} \end{aligned} \tag{4}$$

for all $t \in (t_0 - c, t_0 + c)$ and hence

$$\|\phi(u) - \phi(v)\|_{\sup} \leq \frac{1}{2} \|u - v\|_{\sup}.$$

Thus by the contraction mapping theorem, ϕ has a unique fixed point, which, by (1), is the desired unique solution to $(\star)$ in the interval $(t_0 - c, t_0 + c)$. $\square$

Corollary 1. *Suppose $F(t, y, x_1, \ldots, x_{m-1})$, where t, y, $x_i \in \mathbf{R}$, is a real valued function defined in some neighborhood of the point $(t_0, y_0, y_1, \ldots, y_{m-1}) \in \mathbf{R}^{m+1}$. If F continuous in all variables and Lipschitz in the variables $y, x_1, \ldots, x_{m-1}$, then the m^{th}-order O.D.E. initial value problem*

$$\begin{aligned} \frac{d^m u}{dt^m} &= F\left(t, u, \frac{du}{dt}, \ldots, \frac{d^{m-1}u}{dt^{m-1}}\right), \\ u(t_0) &= y_0, \\ \frac{du}{dt}(t_0) &= y_1, \\ &\vdots \\ \frac{d^{m-1}u}{dt^{m-1}}(t_0) &= y_{m-1} \end{aligned}$$

has a unique solution near $(t_0, y_0, y_1, \ldots, y_{m-1})$.