18.950 Handout 2. The Contraction Mapping Theorem.

Here we prove a very useful fixed point theorem called the contraction mapping theorem. Later we will apply this theorem to prove existence and uniqueness of solutions to ODE's, and also to prove inverse and implicit function theorems.

Definition 1. *Let $f : X \to X$ be a map of a metric space to itself. A point $a \in X$ is called a* fixed point *of f if $f(a) = a$.*

Recall that a metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Exercise. For any positive integers n, m, a set $\Omega \subseteq \mathbf{R}^n$, a positive number R and a point $y_0 \in \mathbf{R}^m$, let $\mathcal{C}\left(\Omega; \overline{B}_R(y_0)\right)$ denote the collection of continuous maps $f : \Omega \to \overline{B}_R(y_0)$. Then $\mathcal{C}\left(\Omega, \overline{B}_R(y_0)\right)$ is a complete metric space with the *uniform* or *sup* metric defined by

$$d(f, g) = \|f - g\|_{\sup} = \sup_{x \in \Omega} |f(x) - g(x)|$$

for $f, g \in \mathcal{C}\left(\Omega; \overline{B}_R(y_0)\right)$.

Definition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\phi : X \to Y$ is called a* contraction *if there exists a positive number $c < 1$ such that*

$$d_Y\left(\phi(x), \phi(y)\right) \le c d_X\left(x, y\right)$$

for all $x, y \in X$.

A trivial example is the map $\phi(x) = \frac{1}{2}x$ for $x \in X = \mathbf{R}^n$.

Theorem 1 (Contraction mapping theorem). *Let (X, d) be a complete metric space. If $\phi : X \to X$ is a contraction, then ϕ has a unique fixed point.*

Proof. By definition of contraction, there exists a number $c \in (0,1)$ such that

$$d(\phi(x), \phi(y)) \leq cd(x, y). \tag{1}$$

Let $a_0 \in X$ be an arbitrary point, and define a sequence a_n inductively by setting $a_{n+1} = \phi(a_n)$. We claim that a_n is Cauchy. To see this, first note that for any $n \geq 1$, we have by (1) that $d(a_{n+1}, a_n) = d(\phi(a_n), \phi(a_{n-1})) \leq cd(a_n, a_{n-1})$, and so we can check easily by induction that

$$d(a_{n+1}, a_n) \leq c^n d(a_1, a_0) \tag{2}$$

for all $n \geq 1$. This and the triangle inequality then gives that for $m > n \geq 1$,

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, a_{m-1}) + d(a_{m-1}, a_{m-2}) + \ldots + d(a_{n+1}, a_n) \\ &\leq (c^{m-1} + c^{m-2} + \ldots + c^n) d(a_1, a_0) \\ &\leq \frac{c^n}{1-c} d(a_1, a_0). \end{aligned} \tag{3}$$

This shows that $d(a_m, a_n) \to 0$ as $n, m \to \infty$, i.e. that $\{a_n\}$ is Cauchy as claimed.

Since (X, d) is complete, there exists $a \in X$ such that $a_n \to a$. Being a contraction, ϕ is continuous, and hence

$$\phi(a) = \lim_{n \to \infty} \phi(a_n) = \lim_{n \to \infty} a_{n+1} = a.$$

Thus a is a fixed point of ϕ.

If $b \in X$ is also a fixed point of ϕ, then

$$d(a, b) = d(\phi(a), \phi(b)) \leq cd(a, b)$$

which implies, since $c < 1$, that $d(a, b) = 0$ and hence that $a = b$. Thus the fixed point is unique.

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