18.950 Handout 1. Metric Spaces.

Definition 1. *A set X together with a function $d : X \times X \to \mathbf{R}$ is called a* metric space*, and the function d a* metric *or* distance function*, if the following holds:*
For all $x, y, z \in X$,

(1) $d(x, y) \geq 0$.

(2) $d(x, y) = 0$ *iff* $x = y$.

(3) $d(x, y) = d(y, x)$.

(4) The triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

Exercise 1. Let (X, d) be a metric space, and $E \subset X$. Then (E, d_E) is a metric space where d_E is the restriction of d to $E \times E$.

Exercise 2. The Euclidean space $\mathbf{R}^n$ equipped with the metric $d(x, y) = |x - y| \equiv \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is a metric space.

Definition 2. *Let (X, d) be a metric space, $x_0 \in X$ and ρ a positive real number. The set $B_\rho(x_0) \equiv \{x \in X : d(x, x_0) < \rho\}$ is called the* open ball centered at x_0 with radius ρ.

Definition 3. *Let (X, d) be a metric space. A subset $U \subseteq X$ is said to be* open *if for each $y \in U$, there exists ρ such that $B_\rho(y) \subset U$.*

Exercise 3. Let (X, d) be a metric space. Then

(a) X and the empty set $\emptyset$ are open sets.

(b) The union of any collection of open sets is an open set.

(c) The intersection of any finite collection of open sets is an open set.

Exercise 4. The open ball $B_\rho(x_0)$ is an open set.

Definition 4. *Let* (X, d) *be a metric space. A subset* $C \subseteq X$ *is said to be* closed *if the complement* $X \setminus C$ *of* C *is open.*

Exercise 5. Let (X, d) be a metric space. Then

(a) X and $\emptyset$ are closed sets.

(b) The intersection of any collection of closed sets is closed.

(c) The union of any finite collection of closed sets is closed.

Definition 5. *Let* (X, d) *be a metric space and* $E \subseteq X$. *Let* $\mathcal{F}$ *be the collection of closed subsets* C *of* X *such that* C *contains* E. *The* closure *of* E, *denoted* $\overline{E}$ *is defined by* $\overline{E} = \cap_{C \in \mathcal{F}} C$.

Definition 6. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces,* $f : X \to Y$ *be a map and* $a \in X$. *The map* f *is said to be* continuous at $a \in X$ *provided that for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that whenever* $x \in X$, $d_X(x, a) < \delta$, *we have that* $d_Y(f(x), f(a)) < \epsilon$. *A map* $f : X \to Y$ *is* continuous *if it is continuous at each point of* X.

Exercise 6. Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is continous if and only if $f^{-1}(U)$ is an open subset of X whenever U is an open subset of Y.

Definition 7. *Let* a_n, $n \in \mathbf{N}$, ($\mathbf{N}$ *is the set of natural numbers* $1, 2, 3, \ldots$) *be a sequence of points in a metric space* (X, d) *and let* $a \in X$. *The sequence* a_n *is said to converge to* a, *written* $a_n \to a$ *or* $\lim_{n\to\infty} a_n = a$, *if for each* $\epsilon > 0$, *there exists* $N \in \mathbf{N}$ *such that* $d(a_n, a) < \epsilon$ *whenever* $n \geq N$.

Exercise 7. A map $f : X \to Y$ from the metric space (X, d_X) to the metric space (Y, d_Y) is continuous at a point $a \in X$ if and only if $f(a_n) \to f(a)$ for every sequence $a_n \in X$ with $a_n \to a$.

Definition 8. *Let (X, d) be a metric space. A sequence a_n of X is called a* Cauchy sequence *if for each $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $d(a_n, a_m) < \epsilon$ for all $n, m > N$.*

Exercise 8. Convergent sequences are Cauchy.

Definition 9. *A metric space (X, d) is said to be* complete *if every Cauchy sequence in X converges to a point in X.*

Definition 10. *A subset E of a metric space X is said to be* bounded *if there exists a number $R > 0$ and a point $a \in X$ such that $E \subset B_R(a)$.*

Exercise 9. Let a_n be a Cauchy sequence of a metric space X. Then the set $\{a_n\}$ is bounded.

Definition 11. *Let (X, d) be a metric space and $E \subseteq X$. An* open cover *of E is a collection of open sets U_α, $\alpha \in A$ where A is an arbitrary index set, such that $E \subseteq \cup_{\alpha \in A} U_\alpha$. A* finite subcover *of the open cover U_α is a finite subcollection $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ such that $E \subseteq \cup_{j=1}^{n} U_{\alpha_j}$.*

Definition 12. *Let (X, d) be a metric space. A subset $K \subseteq X$ is said to be* compact *if every open cover of K has a finite subcover.*

Theorem 1. *Any compact subset of a metric space is closed and bounded.*

Theorem 2. *A compact metric space is complete.*

Theorem 3. *A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 4. *A continuous function on a compact metric space attains its supremum and infimum. That is to say, if (X, d) is a compact metric space and $f : X \to \mathbf{R}$ a continuous function, then there exist points x_m, $x_M \in X$ such that $f(x_m) = m = \inf_{x \in X} f(x)$ and $f(x_M) = M = \sup_{x \in X} f(x)$. In particular, m and M are finite.*

Exercise 10. Let $L(\mathbf{R}^n; \mathbf{R}^m)$ denote the set of linear transformations $A : \mathbf{R}^n \to \mathbf{R}^m$. For $A \in L(\mathbf{R}^n; \mathbf{R}^m)$, define $\|A\|$ (the *norm* of A) by

$$\|A\| = \sup_{\{|v| \leq 1\}} |A(v)|.$$

For $A, B \in L(\mathbf{R}^n, \mathbf{R}^m)$, define $d(A, B) = \|A - B\|$. Then $(L(\mathbf{R}^n; \mathbf{R}^m), d)$ is a metric space.