Lattices and Non-Lattices

We will discuss several constructions, basic properties and important theorems about lattices in semisimple Lie groups, focussing on matrix subgroups of $\mathrm{SL}(n, \mathbb{R})$.

Please let me (wienhard@math.princeton.edu) know soon, which talk you would like to give. Even if it says that one topic consists of 1 talk, a group of two people can joined up to prepare and present the topic. Talks are supposed to be around 60 minutes, some might be a bit longer, but should not exceed 90 minutes. If you have a question regarding one of the topics or would like to have further references, please contact me.

1. Basics on Lattices (1-2 talks) – Chi Li, Hajo Hein

Outline.

(1) Give the definition of a lattice, of a cocompact (uniform) lattice. Mention some examples, they will be discussed in more detail later, e.g. every lattice in $\mathbb{R}^n$ is isomorpic to $\mathbb{Z}^n$, $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{SL}(n, \mathbb{R})$, hyperbolic surfaces. We will mainly consider lattices in classical Lie groups $G \subset \mathrm{SL}(n, \mathbb{R})$, discuss some examples of such groups (see [16, Examples 3.22/3.23]) and give definition. State Borel's theorem: all linear semisimple Lie groups contain at least one cocompact and one non-cocompact lattice (original paper is [5]).

(2) Important notions for lattices:
  (a) commensurability and the commensurater
  (b) irreducibility
  (c) Zariski topology/Zariski closure (see [16, § 3.E])

(3) Borel density theorem (see [16, Corollary 4.44]), state the result and discuss the proof (assume that G is generated by unipotent elements).

(4) Important properties of lattices (sketch the proofs):
  (a) lattices are finitely presented
  (b) Selberg's lemma: every lattice has a torsionfree subgroup of finite index

References. Dave Morris,Introduction to Arithmetic Groups [16], mainly Chapter 4.

2. Construction of Lattices I - geometric (1-2 talks) – Vivek Shende

Outline. There are geometric ways to construct lattices by looking at tilings of the hyperbolic plane and their symmetry groups. This only works in low dimensions. Discuss first the example of hyperbolic surfaces, getting the fundamental domain in the hyperbolic plane. Then discuss the general construction of lattices in $\mathrm{SO}(n,1)$, $n \leq 9$ using Coxeter groups.

References. Yves Benoist, Five lectures on lattices in semisimple Lie groups, [3], Chapter 1.

3. Construction of Lattices II - arithmetic (2 talks) – Peter Varju, Ilya Vinogradov

Outline.

(1) Show that $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$ is a lattice. Discuss other examples in [3, Chapter2], cover the examples mention in [9, Section 2.1]. Cover Mahler's compactness criterium and Godement's compactness criterium.
(2) Give general definition of an arithmetic subgroup and general overview (see [3, Chapter2]).
(3) State Borel-HarishChandra's Theorem that arithmetic subgroups are lattices (the original articles are [7, 8])
(4) State Margulis's arithmeticity theorem.
(5) State Margulis's commensurater criterium for arithmeticity.

References.

(1) Yves Benoist, Five lectures on lattices in semisimple Lie groups, [3], Chapter 2,
(2) Dave Morris, Introduction to Arithmetic Groups [16], Chapter 6.
(3) M. Gromov and I. Piatetski-Shapiro,Nonarithmetic groups in Lobachevsky spaces [9], Section 2.1.

4. Congruence subgroup property (1 talk)

Outline. Describe the congruence subgroup problem for $\mathrm{SL}(n,\mathbb{Z})$. Describe the more general problem. Survey results and applications. This is more a survey talk, which should not go into too much detail and difficulty.

References. :

(1) Raghunathan, The congruence subgroup problem [13].
(2) Rapinchuk and Prasad, Developments on the Congruence Subgroup Problem after the work of Bass, Milnor and Serre [14].
(3) Bass, Milnor and Serre, Solution of the congruence subgroup problem for $\mathrm{SL}_n\,(n \geq 3)$ and $\mathrm{Sp}_{2n}\,(n \geq 2)$ [1].

5. Construction of non-arithmetic lattices (1 talk) – Will Cavendish

Outline. Gromov and Piatetski-Shapiro gave a beautiful construction of non-arithmetic lattices in $\mathrm{SO}(1, n)$, by interbreeding two arithmetic lattices. Describe the construction and explain the proof [9].

References.

(1) M. Gromov and I. Piatetski-Shapiro,Nonarithmetic groups in Lobachevsky spaces [9].
(2) Dave Morris, Introduction to Arithmetic Groups [16], Chapter § 6.D.

6. Rigidity (2 talks) – Kevin Wilson, Daniel Shenfeld

Outline.

(1) State Weil's local rigidity theorem.
(2) State Mostow's rigidity theorem and sketch the proof for $\mathrm{SO}(n, 1)$ $(n \geq 3)$
(3) State Margulis's superrigidity theorem, sketch some ideas of the proof.
(4) Prove Margulis's superrigidity theorem implies Margulis's arithmeticity theorem.
(5) State Margulis's normal subgroup thoerem.

References.

(1) Spatzier, An invitation to rigidity theory [15].
(2) Benedetti and Petronio, Lectures on hyperbolic geometry [2, Chapter C].
(3) Zimmer, Ergodic Theory and and semisimple groups, [17].
(4) Dave Morris, Introduction to Arithmetic Groups [16], Chapter 12.
(5) G. Margulis, Discrete subgroups of semisimple Lie groups, [11].

7. Manifolds of pinched negative curvature (1 talk) – Yi Wang, Zhiren Wang

Outline. Gromov and Thurston gave a nice construction of manifolds admitting a Riemannian metric of negative sectional curvature which is pinched arbitrarily close to -1, but which do not admit any metric of constant negative sectional curvature. Describe the construction and explain the proof.

References. M. Gromov and W. Thurston, Pinching constants for hyperbolic manifolds [10].

Other References

(1) M. Raghunathan, Discrete subgroups of Lie groups, [12].
(2) Y. Benoist, Réseaux des groupes de Lie, [4].
(3) Borel, Introduction aux groupes arithmétiques, [6].

References

1. H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. (1967), no. 33, 59–137.
2. Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.
3. Y. Benoist, *Five lectures on lattices in semisimple lie groups*, Lecture notes, available on the webpage, 2004.
4. ______, *Réseaux des groupes de lie*, Lecture notes, available on the webpage, 2008.
5. Armand Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.
6. ______, *Introduction aux groupes arithmétiques*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341, Hermann, Paris, 1969.
7. Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Bull. Amer. Math. Soc. **67** (1961), 579–583. MR MR0141670 (25 #5067)
8. ______, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535. MR MR0147566 (26 #5081)
9. M. Gromov and I. Piatetski-Shapiro, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 66, 93–103. MR MR932135 (89j:22019)
10. M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), no. 1, 1–12. MR MR892185 (88e:53058)
11. G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR MR1090825 (92h:22021)
12. M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

13. ______, *The congruence subgroup problem*, Proc. Indian Acad. Sci. Math. Sci. **114** (2004), no. 4, 299–308.
14. A. Rapinchuk and G. Prasad, *Developments on the congruence subgroup problem after the work of bass, milnor and serre*, Survey, available on the webpage, ????
15. R. J. Spatzier, *An invitation to rigidity theory*, Modern dynamical systems and applications, Cambridge Univ. Press, Cambridge, 2004, pp. 211–231.
16. D. Witte Morris, *Introduction to arithmetic groups*, Book in preparation, available on the webpage, 2010.
17. Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.