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ARITHMETIC SUBGROUPS OF ALGEBRAIC GROUPS

BY ARMAND BOREL AND HARISH-CHANDRA

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Introduction

A complex matric group $G \subset \mathbf{GL}(n, \mathbf{C})$ is algebraic, defined over $\mathbf{Q}$, if it consists of all invertible matrices whose coefficients annihilate some set of polynomials on $\mathbf{M}(n, \mathbf{C})$ with rational coefficients. In this case, let $G_{\mathbf{Z}}$ be the subgroup of elements of G which have integral coefficients, determinant ± 1, and $G_{\mathbf{R}} = G \cap \mathbf{GL}(n, \mathbf{R})$. Then $G_{\mathbf{Z}}$ is an arithmetically defined subgroup of $G_{\mathbf{R}}$, or more briefly, an *arithmetic subgroup* of $G_{\mathbf{R}}$. Typical examples are $\mathbf{SL}(n, \mathbf{Z}) \subset \mathbf{SL}(n, \mathbf{R})$, Siegel's modular group, or the group of units of a non-degenerate rational quadratic form; and the main purpose of this paper is to generalize facts known in these and other cases involving classical groups, chiefly from reduction theory. In particular, we shall prove that $G_{\mathbf{Z}}$ is finitely generated, and give necessary and sufficient conditions under which $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact, or of finite invariant measure. In analogy with the terminology used in the classical cases, we shall also call $G_{\mathbf{Z}}$ the *group of units* of $G_{\mathbf{R}}$.

In view of known facts about algebraic groups and algebraic tori, the main case to investigate is that of semi-simple groups, and this paper is mainly concerned with the latter. However, it turns out that the reductive groups (that is fully reducible groups, or groups whose identity component is isogenous to the product of an algebraic torus by a semi-simple group) form the natural domain of validity for some results, and part of the discussion will be carried out directly for reductive groups. The two following properties will be particularly useful for our purposes:

(A) Let $G_1 \supset \cdots \supset G_m$ be reductive real algebraic subgroups of $\mathbf{GL}(n, \mathbf{R})$. Then there exists $a \in \mathbf{SL}(n, \mathbf{R})$ such that the groups $a \cdot G_i \cdot a^{-1}$ are all stable under $x \to {}^t x$ (1.9).

(B) Let G be a connected complex algebraic reductive group defined over $\mathbf{Q}$, H a closed subgroup defined over $\mathbf{Q}$. Then H is reductive if and only if G/H can be realized as the closed orbit of a rational point in a rational representation defined over $\mathbf{Q}$ (3.8).

The statement (A) is known [21, § 7], and (B) is a slight sharpening of known facts. Proofs have been included for the sake of completeness. However, the techniques used in them will seldom occur elsewhere in the paper, so that the reader who wishes to proceed as directly as possible to the main part of the paper may skip 1.5 to 1.9, the proofs of 1.10, 1.11, and 3.4 to 3.8 without serious inconvenience. Section 2 is also preliminary, and collects some basic facts and notions about algebraic groups.

Section 4 introduces Siegel domains. Let G be a real algebraic semi-simple, or reductive group, and $G = K \cdot A \cdot N$ an Iwasawa decomposition of G (see 1.11). A Siegel domain $\mathfrak{S}_{t,\omega}(t > 0; \omega$ a compact set in $N)$ is the set of elements $K \cdot A_t \cdot \omega$, where A_t is the set of exponentials of elements on which the simple restricted roots have values smaller than t. Their chief properties are:

(i) the set of elements $a \cdot n \cdot a^{-1} (a \in A_t, n \in \omega)$ is relatively compact (4.2);

(ii) if G is semi-simple, $\mathfrak{S}_{t,\omega}$ has finite Haar measure (4.3).

When $G = \mathbf{GL}(n, \mathbf{R})$ and $K \cdot A \cdot N$ is the standard Iwasawa decomposition, it is classical that $\mathfrak{S}_{t,\omega}$ meets only a finite number of its right translates under $x \cdot G_{\mathbf{Z}} \cup G_{\mathbf{Z}} \cdot x$ $(x \in G_{\mathbf{Q}})$, and that $G = \mathfrak{S}_{t,\omega} \cdot \mathbf{SL}(n, \mathbf{Z})$ for t, ω big enough (see 2.5 for references). These facts will be used in the present paper.

Section 5 is devoted to a finiteness lemma (5.3, 5.4), which generalizes the finiteness of the number of integral reduced forms with a given non-zero determinant. This lemma, together with (A), (B), is used in 6.5 to show that if G is reductive, defined over $\mathbf{Q}$, there exist open sets U in $G_{\mathbf{R}}$ such that $G_{\mathbf{R}} = U \cdot G_{\mathbf{Z}}$, $K \cdot U = U$ for a suitable maximal compact subgroup K; and $U^{-1} \cdot U \cap (x \cdot G_{\mathbf{Z}} \cdot y)$ is finite if $x, y \in G_{\mathbf{Q}}$. The construction of U is analogous to Hermite's procedure to obtain a fundamental domain for the group of units of an indefinite rational form in the space of majorizing forms; however, we shall operate directly in the group, instead of using the symmetric space G/K. The finite generation of $G_{\mathbf{Z}}$ follows immediately. It is also shown (6.9) that in a rational representation of G defined over $\mathbf{Q}$, the integral points contained in a closed orbit form a finite number of orbits of $G_{\mathbf{Z}}$. Applied to the natural representation of $\mathbf{SL}(n, \mathbf{Z})$

in the space of quadratic forms, this yields the finiteness of the number of classes of integral forms with a given non-zero determinant. Theorem 6.5 is extended to general algebraic groups in 6.12. In 6.11 it is proved that if $\mu: G \to G'$ is an isogeny, defined over $\mathbf{Q}$, then $\mu(G_{\mathbf{Z}})$ and $G'_{\mathbf{Z}}$ are commensurable.

The finiteness of the measure of $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is proved in § 7 when G is semi-simple, in § 9 when the identity component G^0 of G has no non-trivial rational character defined over $\mathbf{Q}$ (see 7.8, 9.4). In § 7, the basic lemma is 7.5, which says roughly that if $\mathfrak{S}$ is a Siegel domain of a real algebraic semi-simple Lie group, and G_1 is a suitably embedded semi-simple subgroup of G, then $\mathfrak{S} \cdot x \cap G_1$ is contained in the union of a finite number of translates of a Siegel domain of G_1. Theorem 9.4 follows easily from 7.8, a result of Ono [22] on algebraic tori, and some remarks on rational characters made in § 8.

Section 10 is a preliminary to § 11, and discusses closed conjugacy classes. It is shown that if G is an algebraic group, the conjugacy class of an element x in G, (or in the Lie algebra of G), is closed if x is semi-simple, and not closed if G is reductive and x not semi-simple (10.1).

Section 11 gives a necessary and sufficient condition for $G_{\mathbf{R}}/G_{\mathbf{Z}}$ to be compact. When G is semi-simple, the condition is that $G_{\mathbf{Q}}$ consists of semi-simple elements. It generalizes the compactness of $G_{\mathbf{R}}/G_{\mathbf{Z}}$ when $G_{\mathbf{Q}}$ is the multiplicative group of elements of norm 1 in a division algebra over $\mathbf{Q}$, or when G is the orthogonal group of a rational form which does not represent zero. This condition had been conjectured by R. Godement. Its necessity follows easily from 10.1. Conversely, if $G_{\mathbf{Q}}$ consists of semi-simple elements, then 10.1 implies the existence of a locally faithful rational representation defined over $\mathbf{Q}$ (the adjoint representation) in which all rational points have closed orbits. The main part of the proof starts from that fact, and is a suitable adaptation of a known argument used in the classical case.

The definition of groups of units given above can be generalized by replacing $\mathbf{Q}$ and $\mathbf{Z}$ by a number field K and the ring of algebraic integers of K respectively. However this case is reduced to the previous one by the well-known operation of "restriction of the ground field", and the main results of the paper extend automatically to groups over number fields, as will be seen in § 12.

In § 13, we have relegated some remarks on algebraic groups, not used in the present paper, but which may be viewed as natural complements to some auxiliary results proved in §§ 1, 8.

The main results of this paper have been summarized in [3]. The applications to adele groups announced in [2] will be published elsewhere.

Notation. Linear representations will always be right representations. $H\backslash G$ (resp. G/H) is the space of cosets Hg (resp. gH) of a group G modulo a subgroup H. The identity component of a topological group G is denoted G^0. If a group G operates on a space M, the subgroup of G leaving a given point $m \in M$ fixed, (the *isotropy group* of m), is denoted by G_m. $\operatorname{diag}(a_1, \cdots, a_n)$ is a diagonal $n \times n$ matrix with diagonal coefficients $a_1, \cdots, a_n$. The Lie algebra of a Lie group $G, H, M, \cdots$ is denoted by the corresponding German letter.

1. Reductive real algebraic groups

1.1. Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ and $\theta: k + p \to k - p$ the corresponding Cartan involution. It is convenient to allow $\mathfrak{g}$ to be compact, then $\mathfrak{p} = 0$ and θ is the identity. The Cartan involutions and Cartan decompositions are conjugate under $\operatorname{Ad}\mathfrak{g}$ (see [20], for instance). The Cartan involutions of $\mathfrak{sl}(n, \mathbf{R})$ are the transformations $x \to -x^*$, where x^* is the adjoint of x with respect to some positive non-degenerate quadratic form on $\mathbf{R}^n$. They are also involutive automorphisms of $\mathfrak{gl}(n, \mathbf{R})$, to be called the Cartan involutions of $\mathfrak{gl}(n, \mathbf{R})$. Similarly, the automorphisms $x \to (x^{-1})^*$ of $\mathbf{SL}(n, \mathbf{R})$ or $\mathbf{GL}(n, \mathbf{R})$ will be called the Cartan involutions of these groups. The restriction of a Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$ to a semi-simple subalgebra $\mathfrak{g}$, which is stable under it, is a Cartan involution of $\mathfrak{g}$.

Given a subspace $\mathfrak{m}$ of $\mathfrak{g}$ or of $\mathfrak{gl}(n, \mathbf{R})$, and a Cartan involution θ of $\mathfrak{g}$ or $\mathfrak{gl}(n, \mathbf{R})$, we put

$$\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m} \cap \mathfrak{k}\,, \qquad \mathfrak{m}_{\mathfrak{p}} = \mathfrak{m} \cap \mathfrak{p}\,.$$

If $\mathfrak{m} = \theta(\mathfrak{m})$, then $\mathfrak{m} = \mathfrak{m}_{\mathfrak{k}} + \mathfrak{m}_{\mathfrak{p}}$ (and conversely), and $\mathfrak{m}$ is spanned by semi-simple elements.

It is standard that, given a real or complex representation of $\mathfrak{g}$ in a finite dimensional vector space V, there exists a Hilbert space structure on V with respect to which the elements of $\rho(\mathfrak{k})$ (resp. $\rho(\mathfrak{p})$) are skew-hermitian (resp. hermitian). The algebra $\rho(\mathfrak{g})$ and the analytic group it generates are then self-adjoint; this also implies that $\rho(x)$ is a semi-simple endomorphism of V with purely imaginary (resp. real) eigenvalues whenever $x \in \mathfrak{k}$ (resp. $x \in \mathfrak{p}$). If ρ is the adjoint representation, such a scalar product is given by $B^*(x, y) = -B(x, \theta(y)) (x, y \in \mathfrak{g})$, where B is the Killing form of $\mathfrak{g}$.

If $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R})$, and ρ is the identical representation, the preceding remarks show that $\mathfrak{g}$ is stable under some Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$. By the conjugacy of Cartan involutions, it follows that the Cartan involutions of $\mathfrak{g}$ are the restrictions to $\mathfrak{g}$ of the Cartan involutions of $\mathfrak{gl}(n, \mathbf{R})$ leaving $\mathfrak{g}$ invariant.

1.2. A subalgebra $\mathfrak{g}$ of a Lie algebra $\mathfrak{m}$ is *reductive in* $\mathfrak{m}$ if $\mathrm{ad}_{\mathfrak{m}}\mathfrak{g}$, (the image of $\mathfrak{g}$ in the adjoint representation of $\mathfrak{m}$), is completely reducible. A Lie algebra $\mathfrak{g}$ is *reductive* if it is so in itself. This is the case if and only if $\mathfrak{g}$ is the direct product of its center by a semi-simple ideal, which is then necessarily equal to the derived algebra $\mathscr{D}\mathfrak{g}$ of $\mathfrak{g}$ [5, § 6, No. 4].

LEMMA. *Let G be a closed subgroup of* $\mathbf{GL}(n, \mathbf{R})$ *with a finite number of connected components. Then the following conditions are equivalent:*

(i) *G is completely reducible;*

(ii) *$\mathfrak{g}$ is completely reducible;*

(iii) *$\mathfrak{g}$ is reductive in $\mathfrak{gl}(n, R)$;*

(iv) *$\mathfrak{g}$ is reductive, and its center consists of semi-simple endomorphisms.*

The identity component G^0 of G is invariant in G, of finite index. By an elementary fact (see e.g. G. D. Mostow, Amer. J. Math. 78 (1956), 200–221, Lemma 3.1), G is completely reducible if and only if G^0 is, hence (i) is equivalent to (ii). For the other equivalences, see [5, § 6, Nos. 3–6].

1.3. *A real algebraic group* is a subgroup of $\mathbf{GL}(n, \mathbf{R})$ which consists of all invertible matrices whose coefficients annihilate some set of polynomials with real coefficients, in n^2 indeterminates. A subalgebra of $\mathfrak{gl}(n, \mathbf{R})$ is algebraic if it is the Lie algebra of a real algebraic group. A real algebraic group is *reductive* if it is a completely reducible linear group.

1.4. LEMMA. *Let $\mathfrak{m}$ be an algebraic commutative fully reducible subalgebra of $\mathfrak{gl}(n, \mathbf{R})$. Then $\mathfrak{m} = \mathfrak{m}' + \mathfrak{m}''$, where $\mathfrak{m}'$ (resp. $\mathfrak{m}''$) consists of all elements of $\mathfrak{m}$ with purely imaginary (resp. real) eigenvalues, and is an algebraic subalgebra. $\mathfrak{m}$ is invariant under a Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$. If θ is a Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$ leaving $\mathfrak{m}$ invriant, then $\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m}'$, $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{m}''$, and θ leaves invariant every algebraic subalgebra of $\mathfrak{m}$.*

The algebra $\mathfrak{m}$ consists of semi-simple elements, and is therefore contained in a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{gl}(n, \mathbf{R})$ [4, 2.7]. It is known (see e.g. [11, p. 107]) that a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ is invariant under a Cartan involution of $\mathfrak{g}$. If we apply this to $\mathfrak{sl}(n, \mathbf{R})$, and recall that $\mathfrak{c}$ is the product of the center of $\mathfrak{gl}(n, \mathbf{R})$ by a Cartan subalgebra of $\mathfrak{sl}(n, \mathbf{R})$, we see that $\mathfrak{c}$ is invariant under a Cartan involution θ' of $\mathfrak{gl}(n, \mathbf{R})$. Using 1.1, we have then $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} + \mathfrak{c}_{\mathfrak{p}}$ where $\mathfrak{c}_{\mathfrak{k}}$ (resp. $\mathfrak{c}_{\mathfrak{p}}$) is the set of elements of $\mathfrak{c}$ with purely imaginary (resp. real) eigenvalues.

Let $x \in \mathfrak{m}$. We have $x = k + p (k \in \mathfrak{c}_{\mathfrak{k}}, p \in \mathfrak{c}_{\mathfrak{p}})$, and in order to prove our first contention, it is enough to show that $k, p \in \mathfrak{m}$ or also, since k and p are real matrices, that $k, p \in \mathfrak{m}_c$. After a suitable complex change of coordinates, we may assume $\mathfrak{c}$ to be diagonal, and write

$$k = \operatorname{diag}(i\mu_1, \cdots, i\mu_n) \qquad p = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$$
$$(\lambda_i, \mu_i \in \mathbf{R}, i = 1, \cdots, n) .$$

But [7a, p. 160] the smallest algebraic algebra in $\mathfrak{gl}(n, \mathbf{C})$ containing x is the set of matrices $\operatorname{diag}(a_1, \cdots, a_n)$, where $(a_1, \cdots, a_n)$ annihilates all linear forms with integral coefficients which are zero on $(\lambda_1 + i\mu_1, \cdots, \lambda_n + i\mu_n)$. It contains therefore k and p; *a fortiori* $k, p \in \mathfrak{m}_c$.

If now θ is a Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$ leaving $\mathfrak{m}$ invariant, and $\mathfrak{a}$ is an algebraic subalgebra of $\mathfrak{m}$, then we have $\mathfrak{m}' = \mathfrak{m}_{\mathfrak{k}}$, $\mathfrak{m}'' = \mathfrak{m}_{\mathfrak{p}}$ by 1.1, hence also $\mathfrak{a}' \subset \mathfrak{m}_{\mathfrak{k}}$, $\mathfrak{a}'' \subset \mathfrak{m}_{\mathfrak{p}}$, which shows that $\theta(\mathfrak{a}) = \mathfrak{a}$.

1.5. LEMMA. *Let* $\mathfrak{g}$ *be either semi-simple or equal to* $\mathfrak{gl}(n, \mathbf{R})$ *and* $\mathfrak{m}$ *a subalgebra of* $\mathfrak{g}$. *Then if* $\mathfrak{m}$ *is stable under a Cartan involution* θ *of* $\mathfrak{g}$, *it is reductive in* $\mathfrak{g}$, *and the restriction of* θ *to* $\mathcal{D}\mathfrak{m}$ *is a Cartan involution of* $\mathcal{D}\mathfrak{m}$. *Conversely, if* $\mathfrak{m}$ *is reductive and algebraic, it is stable under some Cartan involution of* $\mathfrak{g}$.

Using the last assertion of 1.1, and identifying $\mathfrak{g}$ with an algebraic subalgebra of $\mathfrak{gl}(n, \mathbf{R})$ $(n = \dim \mathfrak{g})$ by means of the adjoint representation, we see first that it is enough to prove the first part when $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{R})$.

Let $\mathfrak{m}$ be stable under a Cartan involution θ of $\mathfrak{g}$. The ideal $\mathfrak{n}$ formed by the nilpotent matrices of the radical of $\mathfrak{m}$ is then also stable under θ, hence (1.1) spanned by semi-simple matrices; thus $\mathfrak{n} = 0$ and $\mathfrak{m}$ is fully reducible [5; § 6, Théorème 4].

Let now $\mathfrak{m}$ be fully reducible, and algebraic. The existence of a Cartan involution θ of $\mathfrak{g}$ leaving $\mathfrak{m}$ invariant is known if $\mathfrak{m}$ is semi-simple [20], or if $\mathfrak{m}$ is commutative (1.4). Let now $\mathfrak{m}$ be neither semi-simple nor commutative. Its center $\mathfrak{c}$ is fully reducible, algebraic, therefore invariant under a Cartan involution θ of $\mathfrak{g}$. The centralizer $\mathfrak{z}(\mathfrak{c})$ of $\mathfrak{c}$ in $\mathfrak{g}$ is reductive in $\mathfrak{g}$ [4, §§ 3, 4], stable under θ, and θ induces a Cartan involution of $\mathcal{D}\mathfrak{z}(\mathfrak{c})$. By [20] there exists a Cartan involution θ' of $\mathfrak{g}$ which leaves $\mathcal{D}\mathfrak{z}(\mathfrak{c})$ and $\mathcal{D}\mathfrak{m}$ invariant. The restriction of θ and θ' to $\mathcal{D}\mathfrak{z}(\mathfrak{c})$ are Cartan involutions of $\mathcal{D}\mathfrak{z}(\mathfrak{c})$, and the analytic group generated by $\mathcal{D}\mathfrak{z}(\mathfrak{c})$ contains an element g such that $\theta'' = \operatorname{Ad} g \circ \theta \circ \operatorname{Ad} g^{-1}$ has the same restriction to $\mathcal{D}\mathfrak{z}(\mathfrak{c})$ as θ' (1.1). The Cartan involution θ'' leaves then invariant $\mathfrak{c}$, $\mathcal{D}\mathfrak{m}$, hence also $\mathfrak{m}$.

1.6. LEMMA. *Let* $\mathfrak{g} \supset \mathfrak{g}'$ *be algebraic subalgebras reductive in* $\mathfrak{gl}(n, \mathbf{R})$. *Then there exists a Cartan involution* θ *of* $\mathfrak{gl}(n, \mathbf{R})$ *leaving* $\mathfrak{g}$ *invariant. Any such Cartan involution is conjugate under an element of the analytic* *group G generated by* $\mathfrak{g}$ *to a Cartan involution leaving also* $\mathfrak{g}'$ *invariant.*

Let $\mathfrak{c}$ and $\mathfrak{c}'$ be the centers of $\mathfrak{g}$ and $\mathfrak{g}'$, and θ a Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$ leaving $\mathfrak{g}$ invariant. The algebra $\mathfrak{c} + \mathfrak{c}'$ is commutative, algebraic, reductive in $\mathfrak{gl}(n, \mathbf{R})$, hence so is $\mathfrak{c}'' = \mathscr{D}\mathfrak{g} \cap (\mathfrak{c} + \mathfrak{c}')$, and $\mathfrak{c}'' + \mathscr{D}\mathfrak{g}'$ is algebraic, reductive in $\mathscr{D}\mathfrak{g}$. By 1.5, $\mathfrak{c}'' + \mathscr{D}\mathfrak{g}'$ is stable under a Cartan involution of $\mathscr{D}\mathfrak{g}$. Using 1.1, this shows the existence of $g \in G$ such that $\theta' = \mathrm{Ad}g \circ \theta \circ \mathrm{Ad}g^{-1}$ leaves $\mathfrak{c}'' + \mathscr{D}\mathfrak{g}'$ invariant; since $\mathfrak{g}$ centralizes $\mathfrak{c}$, we also have $\theta'(\mathfrak{c}) = \mathfrak{c}$, hence $\theta'(\mathfrak{c} + \mathfrak{c}'') = \mathfrak{c} + \mathfrak{c}''$, and, by 1.4, $\theta'(\mathfrak{c}') = \mathfrak{c}'$; therefore $\theta'(\mathfrak{g}') = \mathfrak{g}'$.

This proves the second assertion of 1.6. The first one is the special case where $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{R})$ and $\mathfrak{g}' = \mathfrak{g}$.

1.7. LEMMA. *Let M be reductive algebraic subgroup of* $\mathbf{GL}(n, \mathbf{R})$, *$\theta$ a Cartan involution of* $\mathbf{GL}(n, \mathbf{R})$ *leaving the Lie algebra $\mathfrak{m}$ of M invariant,* $\mathfrak{gl}(n, \mathbf{R}) = \mathfrak{k} + \mathfrak{p}$ *the corresponding Cartan decomposition. Then $M = L \cdot \exp(\mathfrak{m}_{\mathfrak{p}})$ where L is a compact subgroup with Lie algebra $\mathfrak{m}_{\mathfrak{k}}$, which is connected if M is so.*

Let $\mathfrak{c}$ be the center of $\mathfrak{m}$, and Q the normalizer of $(\mathscr{D}\mathfrak{m})_{\mathfrak{k}}$ in M. The group Q normalizes $\mathfrak{c}$, hence also $\mathfrak{c}_{\mathfrak{k}}$ and $\mathfrak{c}_{\mathfrak{p}}$ (1.4), and $(\mathscr{D}\mathfrak{m})_{\mathfrak{p}}$, which is the orthogonal complement of $(\mathscr{D}\mathfrak{m})_{\mathfrak{k}}$ in $\mathscr{D}\mathfrak{m}$ with respect to the Killing form. By a standard result, $(\mathscr{D}\mathfrak{m})_{\mathfrak{k}}$ is equal to its normalizer in $\mathscr{D}\mathfrak{m}$; therefore the Lie algebra $\mathfrak{q}$ of Q is equal to $\mathfrak{c} + (\mathscr{D}\mathfrak{m})_{\mathfrak{k}}$. The algebras $\mathfrak{c}_{\mathfrak{k}} = \mathfrak{c} \cap \mathfrak{k}$ and $(\mathscr{D}\mathfrak{m})_{\mathfrak{k}} = \mathfrak{k} \cap \mathscr{D}\mathfrak{m}$, being algebraic, generate closed, hence compact, subgroups A, B of Q^0, with A central. $\mathfrak{c}_{\mathfrak{p}}$ is the Lie algebra of a vector subgroup V, which is invariant in Q. It follows immediately that $Q^0 = A \cdot B \cdot V$, and hence Q^0/V is compact. Since Q^0 has finite index in Q, the quotient Q/V is also compact. By Iwasawa's theorem (see e.g., [27, Exp. 22, Théorème 1]), there exists a compact subgroup L of Q, containing $A \cdot B$, such that Q is the semi-direct product of L and V. The Lie algebra of L is therefore equal to $\mathfrak{m}_{\mathfrak{k}}$. By the conjugacy of Cartan decompositions of $\mathscr{D}\mathfrak{m}$ under $\mathrm{Ad}\mathscr{D}\mathfrak{m}$, the group Q meets every connected component of M, therefore $M = Q \cdot M^0$. By a classical result of E. Cartan (see [20], [12, Lemma 31] for instance), the analytic subgroup generated by $\mathscr{D}\mathfrak{m}$ is equal to $B \cdot \exp((\mathscr{D}\mathfrak{m})_{\mathfrak{p}})$, therefore $M^0 = A \cdot B \cdot V \cdot \exp((\mathscr{D}\mathfrak{m})_{\mathfrak{p}}) = A \cdot B \cdot \exp \mathfrak{m}_{\mathfrak{p}}$, and $M = L \cdot V \cdot A \cdot B \cdot \exp(\mathfrak{m}_{\mathfrak{p}}) = L \cdot \exp(\mathfrak{m}_{\mathfrak{p}})$. Moreover $L = A \cdot B$ if $M = M^0$.

1.8. LEMMA. *Let $G \supset G'$ be reductive algebraic subgroups of* $\mathbf{GL}(n, \mathbf{R})$. *Then there exists a Cartan involution of* $\mathbf{GL}(n, \mathbf{R})$ *leaving G invariant. Every such Cartan involution is conjugate by an element* $\mathrm{Ad}g (g \in G^0)$ *to a Cartan involution leaving also G' invariant.*

As in 1.6, the first assertion is a special case of the second one. By 1.6, a Cartan involution leaving G invariant is conjugate by an automorphism

$\mathrm{Ad}x(x \in G^0)$ to a Cartan involution θ leaving also the Lie algebra $\mathfrak{g}'$ of G' invariant. Let $\mathfrak{gl}(n, \mathbf{R}) = \mathfrak{k} + \mathfrak{p}$ and $\mathbf{GL}(n, \mathbf{R}) = K \cdot P$ be the corresponding decompositions of $\mathfrak{gl}(n, \mathbf{R})$ and $\mathbf{GL}(n, \mathbf{R})$. By 1.7, $G' = L \cdot \exp(\mathfrak{g}'_{\mathfrak{p}})$, where L is a compact subgroup with Lie algebra $\mathfrak{g}'_{\mathfrak{k}}$. The involution θ clearly leaves invariant G^0, L^0, but not necessarily L. Assume that we have found $y \in G^0$ such that

$$(1) \qquad K' = y \cdot K \cdot y^{-1} \supset L \qquad y \cdot \exp(\mathfrak{g}'_{\mathfrak{k}}) \cdot y^{-1} = \exp(\mathfrak{g}'_{\mathfrak{p}}) \, .$$

Then $\theta' = \mathrm{Ad}y \circ \theta \circ \mathrm{Ad}y^{-1}$ will do. In fact it is conjugate under $y \cdot x$ to the initial Cartan involution, acts trivially on L, and acts by $p \to p^{-1}$ on $\exp(\mathfrak{g}'_{\mathfrak{p}})$, hence it leaves G' invariant.

There remains to find $y \in G^0$ satisfying (1). Let M be the normalizer of $\mathfrak{g}'_{\mathfrak{p}}$ in G. It contains L, and is stable under θ. We view P as usual as a Riemannian symmetric space of $\mathbf{GL}(n, \mathbf{R})$ under the operations $p \to x \cdot p \cdot x^*$, where $x^* = \theta(x^{-1})$. Then $M_{\mathfrak{p}} = \exp \mathfrak{m}_{\mathfrak{p}}$ is a totally geodesic submanifold, and, by E. Cartan's fixed point argument, every compact subgroup of $\mathbf{GL}(n, \mathbf{R})$ leaving $M_{\mathfrak{p}}$ invariant has a fixed point on $M_{\mathfrak{p}}$ (for all this, see for instance [20]). In particular L has a fixed point a on $M_{\mathfrak{p}}$. Let y be the square root of a contained in $M_{\mathfrak{p}}$. Then, for any $x \in L$,

$$\begin{aligned}(y^{-1} \cdot x \cdot y)(y^{-1} \cdot x \cdot y)^* &= y^{-1} \cdot x \cdot y \cdot y \cdot x^* \cdot y^{-1} \\ &= y^{-1}x \cdot a \cdot x^* \cdot y^{-1} = y^{-1} \cdot y^2 \cdot y^{-1} = e \, ,\end{aligned}$$

and therefore $y^{-1} \cdot L \cdot y \subset K$, which is the first part of (1). Since M normalizes $\mathfrak{g}'_{\mathfrak{p}}$, we have $[\mathfrak{m}_{\mathfrak{p}}, \mathfrak{g}'_{\mathfrak{p}}] \subset \mathfrak{g}'_{\mathfrak{p}}$; but $[\mathfrak{m}_{\mathfrak{p}}, \mathfrak{g}'_{\mathfrak{k}}] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, hence $[\mathfrak{m}_{\mathfrak{p}}, \mathfrak{g}'_{\mathfrak{p}}] = 0$, and y commutes elementwise with $\exp \mathfrak{g}'_{\mathfrak{p}}$. Thus y also verifies the second equality of (1).

1.9. THEOREM (Mostow [21]). *Let $G_1 \supset \cdots \supset G_m$ be reductive real algebraic subgroups of* $\mathbf{GL}(n, \mathbf{R})$. *Then there exists $a \in \mathbf{SL}(n, \mathbf{R})$ such that the groups $a \cdot G_i \cdot a^{-1} (i = 1, \cdots, m)$ are self-adjoint.*

By 1.1, this theorem is equivalent to the existence of a Cartan involution of $\mathbf{GL}(n, \mathbf{R})$ leaving the G_i's invariant. By 1.8, there exists a Cartan involution leaving G_1 invariant. Assume $\theta(G_i) = G_i$ for $1 \leqq i \leqq k < m$. By 1.8, θ is conjugate under an element $\mathrm{Ad}x(x \in G_k^0)$ to a Cartan involution θ' leaving G_{k+1} stable. Since the G_i's $(i \leqq k)$ are still invariant under θ', the theorem follows by induction.

The following proposition strengthens 1.7, and extends to reductive real algebraic groups well-known properties of semi-simple Lie groups.

1.10. PROPOSITION. *Let G be a reductive algebraic subgroup of* $\mathbf{GL}(n, \mathbf{R})$, *$\theta$ a Cartan involution of* $\mathbf{GL}(n, \mathbf{R})$ *leaving G invariant* (1.6), $\mathfrak{gl}(n, \mathbf{R}) = \mathfrak{k} + \mathfrak{p}$, *and* $\mathbf{GL}(n, \mathbf{R}) = K \cdot P$ *the corresponding decompositions* *of* $\mathfrak{gl}(n, \mathbf{R})$ *and* $\mathbf{GL}(n, \mathbf{R})$. *Then the set L of fixed points of θ in G is a maximal compact subgroup with Lie algebra* $\mathfrak{g}_{\mathfrak{k}}$. *Every compact subgroup of G is conjugate by an element* $\mathrm{Ad}\,x(x \in \exp(\mathfrak{g}_{\mathfrak{p}}))$ *to a subgroup of L. The map $(x, y) \to x \cdot \exp(y)$ of $L \times \mathfrak{g}_{\mathfrak{p}}$ into G is an analytic homeomorphism.*

Let $g = k \cdot p(k \in K, p \in P)$ be an element of G. Since $\theta(g^{-1}) \cdot g = p^2$, we have $p^2 \in G$, hence also $p^{2m} \in G, (m \in \mathbf{Z})$. In a suitable coordinate system, $\log p = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, ($\lambda_i$ real), and $p^{2m} = \mathrm{diag}(\exp 2m\lambda_1, \cdots, \exp 2m\lambda_n)$. It is then an elementary fact that every polynomial with real coefficients over the space of $n \times n$ matrices which is annihilated by the elements $p^{2m}(m \in \mathbf{Z})$ is also annihilated by the diagonal matrices $\mathrm{diag}(\exp t\lambda_1, \cdots, \exp t\lambda_n)$ (t real). Since G is algebraic, this shows that G contains the 1-parameter group generated by $\log p$. Therefore $\log p \in \mathfrak{g}_{\mathfrak{p}}$, $p \in \exp \mathfrak{g}_{\mathfrak{p}}$, and $k \in G \cap K = L$. Thus $G = L \cdot \exp \mathfrak{g}_{\mathfrak{p}}$. That every compact subgroup of G is conjugate under $\exp \mathfrak{g}_{\mathfrak{p}}$ to a subgroup of L is proved by the fixed point argument of E. Cartan used in 1.8. The proof of the last statement for semi-simple Lie groups given in [12, Lemma 31] is valid without change here.

1.11. We now extend Iwasawa's decomposition to reductive algebraic groups. Let $G \subset \mathbf{GL}(n, \mathbf{R})$ be reductive, algebraic, θ a Cartan involution of $\mathbf{GL}(n, \mathbf{R})$ leaving G invariant, K its fixed point set, $\mathfrak{a}$ a maximal subalgebra of $\mathfrak{g}_{\mathfrak{p}}$. For $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda = \{x \in \mathfrak{g}, [a, x] = \lambda(a)x, a \in \mathfrak{a}\}$. Choose some order on $\mathfrak{a}^*$ and let $\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda$.

PROPOSITION. *We keep the previous notation. Then $\mathfrak{n}$ generates a closed unipotent subgroup N invariant under $A = \exp \mathfrak{a}$, and $G = K \cdot A \cdot N$. The map $(k, a, n) \to k \cdot a \cdot n$ of $K \times A \times N$ onto G is an analytic homeomorphism.*

Let $\mathfrak{c}$ be the center of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{c} \times \mathscr{D}\mathfrak{g}$, and $\mathfrak{a} = \mathfrak{c}_{\mathfrak{p}} \times \mathfrak{a}_1$, where $\mathfrak{a}_1$ is a maximal subalgebra of $(\mathscr{D}\mathfrak{g})_{\mathfrak{p}}$. Clearly, for $\lambda \neq 0, \mathfrak{g}_\lambda = \{x \in \mathscr{D}\mathfrak{g}, [a_1, x] = \lambda(a_1)x, a_1 \in \mathfrak{a}_1\}$. By the usual Iwasawa decomposition, the analytic group M generated by $\mathscr{D}\mathfrak{g}$ is of the form $M = K_1 \cdot A_1 \cdot N$ where $K_1 = K \cap M$, $A_1 = \exp \mathfrak{a}_1$, and N is closed, unipotent, with Lie algebra $\mathfrak{n}$. As was remarked in 1.7, $\mathfrak{c}_{\mathfrak{k}}$ and $\mathfrak{c}_{\mathfrak{p}}$ generate respectively a torus T and a vector subgroup V, therefore

$$G^0 = T \cdot K_1 \cdot V \cdot A_1 \cdot N = T \cdot K_1 \cdot A \cdot N .$$

But $G = K \cdot G^0$, with $K \supset T \cdot K_1$, hence $G = K \cdot A \cdot N$. Since V is central, and N is stable under conjugation by elements of A_1, the group N is also invariant under A. The second assertion is proved by standard arguments, as in the usual case [27; Exp. 11].

An Iwasawa decomposition $G = K \cdot A \cdot N$ and a Cartan involution θ

whose fixed point set is K, and which induces $x \to x^{-1}$ on A, will be called *compatible.*

Let G' be an open subgroup of G. Since $G = K \cdot G^\circ$ and θ leaves K pointwise fixed, we have $\theta(G') = G'$. The automorphism of G' induced by θ and the decomposition $G' = (K \cap G') \cdot A \cdot N$ will also be called a Cartan involution and an Iwasawa decomposition of G' respectively. Clearly, 1.8 to 1.11 are also valid for open subgroups of reductive real algebraic groups.

2. Algebraic groups

In this paragraph and the next we assume some familiarity with the elementary theory of affine algebraic sets and algebraic groups, and shall recall only some of the relevant definitions and facts. For more details, see e.g. [1, 8, 17].

2.1. A *complex algebraic group*, or simply, an *algebraic group*, is a subgroup G of $\mathbf{GL}(n, \mathbf{C})$, which consists of all invertible matrices $g = (g_{ij})$ whose coefficients annihilate some set of polynomials $\{P_\alpha[X_{11}, \cdots, X_{nn}]\}$ with complex coefficients. In other words, G is the intersection of $\mathbf{GL}(n, \mathbf{C})$ with an affine algebraic set in the space $\mathbf{M}(n, \mathbf{C})$ of $n \times n$ complex matrices. The group is said to be defined over a subfield k of $\mathbf{C}$ if the P_α may be chosen so as to have coefficients in k.[1] The intersection of all the fields of definition is also a field of definition. An algebraic group is also a complex Lie group; it is connected as a manifold if and only if it is irreducible as an algebraic set, or if and only if it is connected in the Zariski topology.

Let G be an algebraic group, and A a subring of $\mathbf{C}$. Then G_A will denote the subgroup of elements of G whose coefficients are in A and whose determinant is a unit of A. If A is a field of definition, then G_A is an algebraic A-group in the terminology of [2], an algebraic group over A in [7], and if G is connected, G_A is Zariski dense in G [26, p. 44]. Conversely, if H is an algebraic group over A, in the sense of [7], then $H = G_A$, where G is the smallest complex algebraic group containing H.

An algebraic group G is not necessarily a closed subset of $\mathbf{M}(n, \mathbf{C})$; however, if we add one coordinate and put $g_{n+1,n+1} = (\det g)^{-1}$, $g_{n+1,i} = g_{i,n+1} = 0$ $(i = 1, \cdots, n)$, then G becomes an algebraic subgroup of $\mathbf{SL}(n+1, \mathbf{C})$, which is closed in $\mathbf{M}(n+1, \mathbf{C})$. This operation clearly does not change G_A.

[1] This is the definition of [1], with the universal field specialized to C. Being in characteristic zero, we need not distinguish between Zariski k-closed and defined over k (or between defined and quasi-defined over k [1]).

2.2. A *rational representation* $\pi: G \to \mathbf{GL}(V)$ is a homomorphism of G into $\mathbf{GL}(V)$ (V finite dimensional complex vector space) whose restriction to each connected component of G is a rational map of G into the space of endomorphisms $E(V)$ of V. The coefficients of $\pi(g)$ with respect to a basis of V are regular functions on each connected component of G. If G is a closed set in $\mathbf{M}(n, \mathrm{C})$, they are therefore polynomials in the coefficients of g, whose coefficients are constant on each connected component of G. From this and the end remark in 2.1, it follows that in general the coefficients of $\pi(g)$ are polynomials in those of g and in $(\det g)^{-1}$.

The rational representation π will be said to be defined over k if each component of G is, and if there exists a basis (v_i) of v such that the coefficients of $\pi(g)$ with respect to that basis are regular functions defined over k on each connected component of G. In that case, we shall denote by V_A the set of linear combinations of the v_i's with coefficients in $A \supset k$.

Given a rational representation $\pi: G \to \mathbf{GL}(V)$, we shall most often write $v \cdot g$ for $v \cdot \pi(g)$ $(v \in V, g \in G)$. An orbit $v \cdot G$ is always open and everywhere dense in its Zariski closure (smallest affine algebraic set containing it), therefore the latter coincides with the closure in the ordinary topology. If G is connected, and π is defined over k, then $v \cdot G$, its closure and the isotropy group G_v of v are defined over $k(v)$.

Let G be a connected algebraic group, H an algebraic subgroup, both defined over k. Then $H\backslash G$ is in a canonical way an algebraic variety defined over k. Given a rational representation $\pi: G \to \mathbf{GL}(V)$ and a point $v \in V$ for which $G_v = H$, the map $g \to v \cdot \pi(g)$ induces a regular bijective map of $H\backslash G$ onto $v \cdot G$, defined over k if π is. Since we are in characteristic zero and these varieties are non-singular, this map is in fact birational and biregular.

2.3. PROPOSITION. *Let G be a connected algebraic group defined over* $\mathbf{R}$, $\pi: G \to \mathbf{GL}(V)$ *a rational representation of G defined over* $\mathbf{R}$, *and X an orbit of G in V. Then $X_{\mathrm{R}} = X \cap V_{\mathrm{R}}$ is the union of a finite number of orbits of $(G_{\mathrm{R}})^0$, which are closed if X is so.*

$\dim_{\mathrm{R}}$ and $\dim_{\mathrm{C}}$ will denote the topological and the complex dimension, respectively. Let $s = \dim_{\mathrm{C}} G$, $t = \dim_{\mathrm{C}} X$. The $s - t = \dim_{\mathrm{C}} G_x (x \in X)$. We assume X_{R} to be non-empty, hence X and its closure $\bar{X}$ are defined over $\mathbf{R}$. The latter is an irreducible algebraic set of complex dimension t. The group G_{R} being Zariski-dense in G, the set $\bar{X}$ is the smallest algebraic set containing $(\bar{X})_{\mathrm{R}}$, hence $(\bar{X})_{\mathrm{R}}$ is a real algebraic irreducible set of dimension t, and $(\bar{X})_{\mathrm{R}} = A \cup B$, where A is a manifold of dimension t, the set of real simple points of $\bar{X}$, and B a real algebraic set of dimension $<t$ [31, §§ 10, 11]. In view of their characterizations, A and B are both in-

variant under $G_{\mathbf{R}}$. Let now $x \in X_{\mathbf{R}}$. It is a simple point of $\bar{X}$, hence $x \in A$, $x \cdot G_{\mathbf{R}}$ is an open submanifold of A, and $x \cdot G^0_{\mathbf{R}}$ is a connected component of A. But A, being the complement of a real algebraic set in a real algebraic set, has only a finite number of components [31, Theorem 4], which proves the first part of our assertion. If moreover $X = \bar{X}$, then $(\bar{X})_{\mathbf{R}} = X_{\mathbf{R}} \subset A$, hence B is empty, A is a closed submanifold with a finite number of components, which must then also be closed.

2.4. PROPOSITION. *Let G be a connected algebraic group, H an algebraic subgroup, k a field of definition for G and H, and assume $H \backslash G$ to be an affine algebraic set. Then there exists a rational representation π: $G \to \mathbf{GL}(V)$ defined over k and a point $v \in V_k$ such that $G_v = H$, $v \cdot G$ is closed, and $g \to v \cdot g$ induces a biregular birational map of $H \backslash G$ onto $v \cdot G$.*[2]

Let A and B be rings of regular (i.e. rational, everywhere defined) functions, defined over k, of G and $H \backslash G$ respectively. The variety $H \backslash G$ is defined over k, non-singular and affinely imbeddable. It is therefore also biregularly homeomorphic over k to an affine algebraic set defined over k (see A. Weil, Amer. J. Math. 78 (1950), p. 509–524, Theorem 7). Therefore

(i) B is a finitely generated k-algebra and separates the points of $H \backslash G$.[3]

We let G operate on $A \otimes \mathbf{C}$ on the right by $(f \cdot g)(x) = f(g \cdot x)(f \in A \otimes \mathbf{C}; g, x \in G)$. The transpose of the canonical projection $G \to H \backslash G$ identifies then $B \otimes \mathbf{C}$ with the ring of invariants of H.

Let $b_1, \cdots, b_s$ be a finite system of generators of B, P_i a finite dimensional vector subspace of A such that $P_i \otimes \mathbf{C}$ is invariant under G and $b_i \in P_i$ [25, Theorem 12], P the direct sum of the P_i, $v = (b_1, \cdots, b_s)$, and π the natural representation of G in $V = P \otimes \mathbf{C}$. From (i) it is clear that $G_v = H$. Let $X = v \cdot G$, and μ be the map of $H \backslash G$ into the Zariski closure $\bar{X}$ of X defined by $g \to v \cdot g$. It is rational, defined over k, everywhere regular. Let B' be the ring of regular functions on $\bar{X}$. Since X is Zariski dense in $\bar{X}$, the elements of B' are determined by their restriction to X, and μ induces an injective homomorphism ${}^t\mu$ of B' into $B \otimes \mathbf{C}$. Let us prove now that ${}^t\mu$ is surjective. For this, it is enough to show that $b_i \in {}^t\mu(B')$ $(i = 1, \cdots, s)$. In P_i, we may always find a basis $f_1, \cdots, f_t$ such that $f_j(e) = 0 (j \geqq 2)$. We have $b_i \cdot g = \sum_j a_j(g) \cdot f_j$, where the a_j's are regular functions on G, therefore

[2] Propositions 2.4, 2.5 and their proofs are also valid if $\mathbf{C}$ is replaced by a universal field of arbitrary characteristic.

[3] In our case, this also follows directly from the fact that G_k is Zariski dense in G [26].

$$b_i(g) = (b_i \cdot g)(e) = \sum_j a_j(g) f_j(e) = a_1(g) \cdot f_1(e) .$$

This implies that $b_i = {}^t\mu(x_1 \cdot f_1(e))$, where x_1 is the first coordinate function with respect to the basis (f_j). Thus $b_i \in {}^t\mu(B')$.

Thus, v is a regular map of $H\backslash G$ into $\bar{X}$, defined over k, whose transpose is an isomorphism of the rings of regular functions. Since $H\backslash G$ and $\bar{X}$ are affine, this implies that v is a biregular, bijective, hence also that $X = \bar{X}$. q.e.d.

The following proposition is a slight strengthening of Proposition 5 in [8, Exp. 10] and is proved in the same way. It will be used only in 3.7.

2.5. PROPOSITION. *Let G be a connected algebraic group, H an algebraic subgroup, and k a field of definition for G and H. Then there exists a rational representation π: $G \to \mathbf{GL}(V)$, defined over k, a point $v \in V_k$, $(v \neq 0)$, such that H is the set of elements of G which leave the 1-dimensional subspace $[v]$ spanned by v invariant.*[2]

Again let A be as in 2.4, and I the ideal of H in $A \otimes \mathbf{C}$. It is invariant under H, and has a finite system of generators belonging to A. By [25, Theorem 12] there exists a finite dimensional subspace P of A such that $P \otimes \mathbf{C}$ is invariant under G, and that $P \cap I$ contains a system of generators of I. Let $d = \dim P \cap I$ and π the natural representation of G in $V = \bigwedge^d (P \otimes \mathbf{C})$. Then $v = P \cap I$ fulfills our condition.

3. Reductive algebraic groups. Affine homogeneous spaces

3.1. An *algebraic torus* (a torus, in the terminology of [1]) is an algebraic group which is birationally isomorphic to a direct product of groups $\mathbf{C}^*$ (multiplicative group of non-zero complex numbers), or equivalently [1], a connected group which is diagonalizable. A connected subgroup of $\mathbf{GL}(n, \mathbf{C})$ is an algebraic torus if and only if its Lie algebra is algebraic, commutative, and consists of semi-simple elements (or is reductive in $\mathfrak{gl}(n, \mathbf{C})$). An algebraic group G will be called *reductive* if $G^0 = T \cdot G'$ where T is an algebraic torus, central in G^0, and G' is semi-simple. For an algebraic group $G \subset \mathbf{GL}(n, \mathbf{C})$ the following conditions are equivalent:

(i) G is reductive;

(ii) G is fully reducible;

(iii) $\mathfrak{g}$ is reductive in $\mathfrak{gl}(n, \mathbf{C})$.

In fact, G is reductive, or fully reducible, if and only if G^0 is, and this equivalence follows from the previous remarks and the characterization of fully reducible subalgebras of $\mathfrak{gl}(n, \mathbf{C})$ [5, § 6, No. 5]. Let π be a rational representation of G. If G is a torus, then so is $\pi(G)$, [1; 9.4], therefore, $\pi(G)$ is reductive if G is. The reductive algebraic groups are the algebraic groups, all of whose rational representations are completely reducible.

3.2. PROPOSITION. *Let G be an algebraic group defined over* **R**. *Then the following conditions are equivalent:*

(i) *G is reductive;*

(ii) *$G_{\mathbf{R}}$ is reductive;*

(iii) *$\mathfrak{g}$ is the complexification of the (real) Lie algebra of a maximal compact subgroup of G^0.*

Let $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{R}}$ be the Lie algebras of G and $G_{\mathbf{R}}$, over **C** and **R** respectively. Then $\mathfrak{g} = \mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$. Thus $\mathfrak{g}_{\mathbf{R}}$ is fully reducible if and only if $\mathfrak{g}$ is, and the equivalence of (i) and (ii) follows from 3.1 and 1.2. Let now $G^0 = T \cdot G'$ be reductive, A and B be maximal compact subgroups of T and G' respectively. Then $A \times B$ is a maximal compact subgroup of $T \times G'$, which moreover contains the (finite) kernel of the natural homomorphism of $T \times G'$ onto G^0, therefore $A \cdot B$ is a maximal compact subgroup of G^0. Since the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ of A and B are real forms of the Lie algebras of T and G', this shows that (i) $\Rightarrow$ (iii). If now $\mathfrak{g}$ is the complexification of the Lie algebra of a compact group, it is certainly fully reducible, hence (iii) $\Rightarrow$ (i).

3.3. PROPOSITION. *Let G be a reductive connected algebraic group, $\pi: G \to \mathbf{GL}(V)$ a rational representation. Then the invariant polynomials separate the closed orbits of G.*

Let X, Y be two distinct closed orbits. These are two affine subvarieties with empty intersection. There exists therefore a polynomial P vanishing on X, and a polynomial Q vanishing on Y such that $P + Q = 1$. We have then $P \cdot g + Q \cdot g = 1$ for any $g \in G$. Let K be a maximal compact subgroup of G, dk the Haar measure on K with total measure 1, and

$$P^* = \int_K P \cdot k\, dk \qquad Q^* = \int_K Q \cdot k\, dk\ .$$

We still have $P^* + Q^* = 1$, and $P^*(x) = Q^*(y) = 0 (x \in X,\ y \in Y)$, hence $P^* = 1$ on Y. Moreover P^* and Q^* are invariant under K. Since the natural representation of G on the space of polynomials on V of a given degree is rational, and G is the smallest algebraic subgroup of G containing K (3.2), P^* and Q^* are invariant under G, whence our contention.

3.4. REMARK. The property (iii) also characterizes reductive groups among connected complex Lie groups. In fact, let G be complex, connected, K a maximal compact subgroup, and assume that $\mathfrak{g} = \mathfrak{k} \otimes \mathbf{C}$. Of course $\mathfrak{k} = \mathfrak{c} \times \mathscr{D}\mathfrak{k}$ with $\mathfrak{c}$ the Lie algebra of a torus (usual sense) and $\mathscr{D}\mathfrak{k}$ a compact semi-simple Lie algebra, therefore $\mathfrak{g} = (\mathfrak{c} \otimes \mathbf{C}) \times (\mathscr{D}\mathfrak{k} \otimes \mathbf{C})$, $G = T \cdot G'$, with G' complex semi-simple, T isomorphic to a product of $\mathbf{C}^*$'s, and $T \cap G'$ central, hence finite. $T \cap G'$ is also the intersection of any two maximal compact subgroups of T and G', therefore is uniquely characterized by the maximal compact subgroup K. This implies that if G and G^* are complex, connected, reductive, and have isomorphic maximal compact subgroups, then they are isomorphic (as complex Lie groups at first). Moreover, it is known that a reductive group has only one structure of algebraic group compatible with the given complex analytic structure. (For a more general theorem, see G. Hochschild and G. D. Mostow, Amer. J. Math. 88 (1961), p.p. 111–136, § 8.) Therefore a complex analytic isomorphism between two reductive groups is necessarily birational, biregular.

3.5. THEOREM. *Let G be a connected reductive algebraic group, and H an algebraic subgroup. Then G/H is an affine algebraic variety if and only if H is reductive.*

This result is not new. Since a non-singular affine variety is a Stein manifold, it is a consequence of the two following assertions, where G and H are as in the theorem:

(a) If G/H is a Stein manifold, then H is reductive.

(b) If H is reductive, G/H is affine.

The assertion (a) is due to Matsushima [18]. Assertion (b) is stated in [18] without proof, and attributed to Iwahori-Sugiura. For the sake of completeness, we insert here a proof of (b), and also one for (a), which is shorter than that of [18], although based on a similar idea. A consequence of (a) and (b) is that a quotient G/H, with G connected, reductive, is Stein if and only if it is an affine variety. We also remark that the statement 3.5 makes sense in arbitrary characteristic, but we do not know whether it holds true.

3.6. PROOF OF (a). G/H^0 is a finite Galois covering of G/H, hence is a Stein manifold [6], and we may assume H to be connected. We denote by $H_i(X)$ the i^{th} singular homology group of the space X, with complex coefficients.

Let $n = \dim_{\mathbf{C}} G$, $m = \dim_{\mathbf{C}} H$. We claim first that H reductive is equivalent to $H_m(H) \neq 0$. As a manifold H is the topological product of a euclidean space by a maximal compact subgroup, hence H reductive implies $H_m(H) \neq 0$. A complex analytic group being the quotient by a finite group of the semi-direct product of a semi-simple Lie by a solvable group, it is enough to prove the converse for H solvable. A maximal compact subgroup K of H is then commutative, of real dimension $\geqq m$. Since H is a complex linear group, $\mathfrak{k}$ and $i \cdot \mathfrak{k}$ are linearly independent over $\mathbf{R}$, hence $\mathfrak{h} = \mathfrak{k} + i\mathfrak{k}$ (direct), and $\mathfrak{h} \cong \mathfrak{k} \otimes \mathbf{C}$.

G is a locally trivial fibering, with typical fibre H, base G/H, and structural group H, therefore the Betti numbers of G are majorized by those of $(G/H) \times H$ (as follows from the existence of a spectral sequence). We have $H_n(G) \neq 0$; since G/H is Stein, $H_i(G/H) = 0$ $(i > n - m = \dim_{\mathbf{C}} G/H)$ [6], hence by the Künneth rule, $H_i(H) \neq 0$ for some $i \geqq m$. But G is Stein (it is algebraic), hence so is H (as a closed submanifold of G), and $H_i(H) = 0$ for $i > m$; we must then have $H_m(H) \neq 0$. As was already proved, this implies that H is reductive.

3.7. PROOF OF (b). Let $K \supset L$ be maximal compact subgroups of G and H respectively and ρ a faithful representation of K in $\mathbf{GL}(n, \mathbf{R})$. It follows from 3.4 that ρ extends to a biregular birational isomorphism of G onto the smallest complex algebraic group containing $\rho(K)$. We identify G with the latter. Thus G is defined over $\mathbf{R}$, and H, the smallest complex algebraic group containing L, is also defined over $\mathbf{R}$. The group $G_{\mathbf{R}}$, being real algebraic, with identity component K, is compact and consequently $G_{\mathbf{R}} = K$. By 2.5, there exists a rational representation $\pi: G \rightarrow \mathbf{GL}(V)$ defined over $\mathbf{R}$, and a non-zero element $v \in V_{\mathbf{R}}$ such that H is the subgroup of G leaving $\mathbf{C} \cdot v$ invariant. This yields a real 1-dimensional representation of L. Since L is compact, its image has at most 2 elements. By 1,10, applied to H, viewed as real reductive algebraic group, we have $H/H^0 \cong L/L^0$, the subgroup H' of H leaving v fixed has index $\leqq 2$, and G/H' is either equal to G/H or a two-fold covering of G/H. But the quotient of an affine variety by a finite group is an affine variety (see Serre, Symposium de Topologia Algebrica, Mexico, 1958, 24–53, § 13), therefore it is enough to prove that G/H' is affine. Changing slightly the notation, we may assume H to be the isotropy group of v. Let now $X = v \cdot G$, and $\bar{X}$ its closure. We want to prove that $X = \bar{X}$. The set $Y = \bar{X} - X$ is algebraic, defined over $\mathbf{R}$. Let $S_1, \cdots, S_q$ be polynomials with real coefficients which generate the ideal of Y and $S = S_1^2 + \cdots + S_q^2$. On $V_{\mathbf{R}}$, this polynomial vanishes on $Y_{\mathbf{R}}$ only. Therefore, for $x \in X_{\mathbf{R}}$, S has a constant sign on $x \cdot K$. We let G operate in the usual fashion on the ring A of polynomials over V, and consider as in 3.3 the average S^* of S over K:

$$S^* = \int_K S \cdot k \, dk \ ,$$

where dk is a Haar measure on K. It is invariant under K, hence also under G; consequently, S^* is constant on X, and therefore on $\bar{X}$. Since S does not vanish on the connected set $x \cdot K (x \in X_{\mathbf{R}})$, S^* is not zero on $\bar{X}$; on the other hand, Y is invariant under K, and S vanishes on Y, therefore S^* vanishes on Y. This is a contradiction, unless $Y = \emptyset$, $X = \bar{X}$. Thus X is an affine algebraic set. The map $g \rightarrow v \cdot g$ induces an isomorphism of $H \backslash G$ onto X and allows one to identify $H \backslash G$ with X (2.2).

For future reference, we state here a consequence of 2.4, 2.5, 3.4 to

3.7, the only one to be used in the sequel.

3.8. THEOREM. *Let G be a connected reductive algebraic group, defined over* $\mathbf{Q}$, *and H an algebraic subgroup. Then the two following conditions are equivalent:*

(i) *H is reductive, defined over* $\mathbf{Q}$*;*

(ii) *there exists a rational representation $\pi: G \to \mathbf{GL}(V)$ of G, defined over $\mathbf{Q}$, and an element $v \in V_{\mathbf{Q}}$, such that $v \cdot G$ is closed and $G_v = H$.*

In fact if (i) is true, then $H \backslash G$ is affine by 3.5, and (ii) follows from 2.4. If (ii) is true, then $H \backslash G$ may be identified with $v \cdot G$ (see end of the preceding proof) hence is affine, and H is reductive by 3.6. Finally, since $H = G_v$, $(v \in V_{\mathbf{Q}})$, H is defined over $\mathbf{Q}$.

4. Siegel domains

4.1. Let G be an open subgroup of a real algebraic, reductive group, $G = K \cdot A \cdot N$ an Iwasawa decomposition of G (1.11), and Σ the set of simple restricted roots, in the ordering for which the roots of $\mathrm{ad}_{\mathfrak{g}} \mathfrak{a}$ in $\mathfrak{n}$ are positive. Let $A_t = \{a \in A, \lambda(\log a) \leqq t; \lambda \in \Sigma\}$. A *Siegel domain* in G is a subset of the form $K \cdot A_t \cdot \omega$ where ω is compact in N. It will usually be denoted by $\mathfrak{S}_{t,\omega}$, or simply by $\mathfrak{S}$ if t, ω need not be specified. Clearly, $\mathfrak{S}_{t,\omega} \subset \mathfrak{S}_{t',\omega'}$ if $t \leqq t'$, $\omega \subset \omega'$.

It is understood once and for all that the choice of a Siegel domain presupposes an Iwasawa decomposition which, unless otherwise stated, will be written $K \cdot A \cdot N$ or equivalently, pre-supposes a Cartan involution θ, a maximal subalgebra on which $\theta = -\mathrm{Id}$, and an ordering of the restricted roots.

The union of the Siegel domains, with respect to a fixed Iwasawa decomposition, is G, and any finite union of such Siegel domains is contained in a Siegel domain.

Let us say that two families A, B of subsets of G are equivalent if every element of A (resp. B) is contained in an element of B (resp. A). It is clear that we get a family equivalent to the set of $\mathfrak{S}_{t,\omega}$ if we replace above A_t by $a \cdot A_0 (a \in A)$, or if we allow λ to run through all the positive roots in the definition of A_t.

4.2. PROPOSITION. *Let G be an open subgroup of a real algebraic, reductive group, $G = K \cdot A \cdot N$ an Iwasawa decomposition of G, and ω a compact set in N. Then the set of elements $a \cdot n \cdot a^{-1} (a \in A_t, n \in \omega)$ is relatively compact in N.*

Let $(x_i)(1 \leqq i \leqq m = \dim \mathfrak{n})$ be a basis of $\mathfrak{n}$, such that $[h, x_i] = \lambda_i(h) x_i$ $(h \in \mathfrak{a})$, where λ_i is a linear form on $\mathfrak{a}$ which is >0 for the given ordering $(1 \leqq i \leqq m)$, and that $\lambda_i \leqq \lambda_j$ if $i \leqq j$. For each j, the elements $x_i (i \geqq j)$ span then an ideal of $\mathfrak{n}$ and it follows from standard properties of simply connected nilpotent groups that $(t_1, \cdots, t_m) \to \exp(t_1 x_1) \cdots \exp(t_m x_m)$ is an analytic homeomorphism of $\mathbf{R}^m$ onto N. Further, if $n = \prod_i \exp(t_i x_i)$ and $a \in A$, then

$$a \cdot n \cdot a^{-1} = \prod_i \exp t_i \lambda_i(\log a) x_i \ .$$

The λ_i's are positive roots, therefore linear combinations with positive coefficients of the simple roots. Hence if $a \in A_t$ there exists a constant t' such that $\lambda_i(\log a) \leqq t'$ for all i. If further $n \in \omega$, then the t_i's are also bounded, and the proposition is proved.

4.3. PROPOSITION. *Let G be an open subgroup of a real algebraic, semisimple Lie group. Then any Siegel domain has finite Haar measure.*

Let $G = K \cdot A \cdot N$ be the underlying Iwasawa decomposition, and dk, da, dn Haar measures on K, A and N. Then, in the notation of 4.2,

$$dg = \exp[\sigma(\log a)]\, dk \cdot da \cdot dn \qquad (\sigma = \lambda_1 + \cdots + \lambda_m)$$

is a Haar measure on G [9, Lemma 35], hence

$$\int_{\mathfrak{S}_{t,\omega}} dg = c \cdot \int_{\lambda(\log a) \leqq t,\, \lambda \in \Sigma} \exp[\sigma(\log a)] da \ .$$

Let $\lambda_1, \cdots, \lambda_r$ be the simple roots. Then $\sigma = m_1\lambda_1 + \cdots + m_r\lambda_r$ with $m_i > 0 (1 \leqq i \leqq r)$. Since G is semi-simple, $r = \dim \mathfrak{a}$, the λ_i form a coordinate system on $\mathfrak{a}$, and the exponential $\mathfrak{a} \to A$ carries the euclidean measure on the Haar measure, we have, up to a constant factor

$$\int_{\lambda(\log a) \leqq t,\, \lambda \in \Sigma} \exp[\sigma(\log a)] da = \prod_i \int_{-\infty}^{t} \exp(m_i \lambda_i) d\lambda_i < \infty \ .$$

4.4. *Siegel domains in* $\mathbf{GL}(n, \mathbf{R})$. In $\mathbf{GL}(n, \mathbf{R})$ we consider the usual Iwasawa decomposition where $K = \mathbf{O}(n)$, A is the subgroup of diagonal matrices with positive coefficients, and N the group of upper triangular matrices with diagonal coefficients equal to 1. If we denote the diagonal coefficients $a_1 \cdots a_n$ of an element $a \in A$ by $\exp \mu_1, \cdots, \exp \mu_n$, the simple positive roots are the linear forms $\mu_i - \mu_{i+1} (i = 1, \cdots, n-1)$. Therefore the subset $\mathfrak{S}_{t,u} = \{k \cdot a \cdot \nu\}$ where $k \in \mathbf{O}(n)$, $a = \operatorname{diag}(a_1, \cdots, a_n)$, $a_i \leqq t \cdot a_{i+1} (i = 1, \cdots, n-1)$, $\nu = (n_{ij}) \in N$, $|n_{ij}| \leqq u (i < j)$ is a Siegel domain in the sense of 4.1. Clearly $\mathfrak{S}_{t,u} \subset \mathfrak{S}_{t',u'}$ if $t \leqq t'$, $u \leqq u'$ and $\mathbf{GL}(n, \mathbf{R})$ is the union of the $\mathfrak{S}_{t,u} (t, u > 0)$. If t and u are big enough (in fact $t > 2 \cdot 3^{-1/2}$, $u > 1/2$ will do), $\mathfrak{S}_{t,u}$, or its intersection with $\mathbf{SL}(n, \mathbf{R})$, or with the group of matrices of determinant ± 1, will be called a *standard Siegel domain* of the group in question. These domains will be important for us, because of the following result; for which references are given below.

4.5. PROPOSITION. *Let $\mathfrak{S}$ be a standard Siegel domain of* $\mathbf{GL}(n, \mathbf{R})$. *Then*

(a) $\mathbf{GL}(n, \mathbf{R}) = \mathfrak{S} \cdot \mathbf{SL}(n, \mathbf{Z})$.

(b) *For $x, y \in \mathbf{GL}(n, \mathbf{Q})$, the intersection $\mathfrak{S}^{-1} \cdot \mathfrak{S} \cap x \cdot \mathbf{SL}(n, \mathbf{Z}) \cdot y$ is finite.*

Let P_n be the space of real symmetric positive non-degenerate $n \times n$ matrices, and $\pi: X \to {}^tX \cdot X$ the usual projection of $\mathbf{GL}(n, \mathbf{R})$ onto P_n. We let as usual $\mathbf{GL}(n, \mathbf{R})$ act on the right on P_n by $F \to F[X] = {}^tX \cdot F \cdot X$ ($F \in P_n$, $X \in \mathbf{GL}(n, \mathbf{R})$). For $t, u > 0$, the *Siegel domain* $\mathfrak{S}'_{t,u}$ in P_n is the set of matrices $D[T]$ where

$$D = \operatorname{diag}(d_1, \cdots, d_n) \qquad (d_i \leqq t \cdot d_{i+1};\ i = 1, \cdots, n-1)$$
$$T = (t_{ij}) \in N \qquad (|\, t_{ij} \,| \leqq u,\ i < j)\ .$$

Clearly, $\mathfrak{S}_{t,u} = \pi^{-1}(\mathfrak{S}'_{t^2,u})$; the Siegel domains of $\mathbf{GL}(n, \mathbf{R})$ defined above are the inverse images of the Siegel domains in P_n. From the definitions, it is clear that $q \cdot \mathfrak{S}_{t,u} = \mathfrak{S}_{t,u}$, and $\mathfrak{S}'_{t,u}[q \cdot I] = \mathfrak{S}'_{t,u}$ for $q > 0$. Since $g \in \mathbf{GL}(n, \mathbf{Q})$ may be written as $g = q \cdot g'$, where g' has integral coefficients, and $\pi(X \cdot Y) = \pi(X)[Y]$, Proposition 4.5 follows from

(a′) *For $t \geqq 4/3$, $u \geqq 1/2$, $P_n = \bigcup_{S \in \mathbf{SL}(n,\mathbf{Z})} \mathfrak{S}'_{t,u}[S]$.*

(b′) *For each positive number q, the set of integral matrices S with determinant smaller than q in absolute value for which $\mathfrak{S}'_{t,u}[S] \cap \mathfrak{S}'_{t,u} \neq \emptyset$, is finite.*

The statement (a′), which is more or less implicit in Hermite's work [14], is proved in [16]; in fact S is allowed in [16] to have determinant -1, but it is obvious from the proof that it may be taken in $\mathbf{SL}(n, \mathbf{Z})$. The second assertion is a well known theorem of Siegel [28].

Clearly, the interior of a standard Siegel domain contains a standard Siegel domain, a finite union of standard Siegel domains is contained in a standard Siegel domain, and every element of $\mathbf{GL}(n, \mathbf{R})$ belongs to a standard Siegel domain.

5. A finiteness lemma

5.1. Let G be a locally compact group, acting continuously on the right on a locally compact space M, and G_x the isotropy group of a point $x \in M$. Then $\mu: g \to x \cdot g$ induces a continuous bijective map of the space $G_x \backslash G$ of cosets $G_x \cdot g$ onto the orbit $x \cdot G$ of x. It follows from a theorem of Arens [19, p. 65] that if G is countable at infinity, and $x \cdot G$ is *closed*, then μ is a homeomorphism of $G_x \backslash G$, endowed with the quotient topology, onto $x \cdot G$, endowed with the induced topology. Since a compact set of $G_x \backslash G$ is always the image of a compact set of G, this can also be expressed by saying that, under the assumptions made, the set of $g \in G$ for which $x \cdot g$ belongs to a fixed compact set of M is of the form $G_x \cdot \Omega$, with Ω compact in G. Of this we shall need the following special case.

5.2. PROPOSITION. *Let G be a Lie group with finitely many connected components, π a continuous complex or real linear representation of G in a finite dimensional vector space V, $v \in V$ a point whose orbit $v \cdot \pi(G)$ is closed, and Q a compact subset of V. Then $\{g \in G, v \cdot \pi(g) \in Q\} = G_x \cdot \Omega$, with Ω compact.*

By a *lattice* Γ in a finite dimensional vector space over $\mathbf{Q}$ or $\mathbf{R}$ we mean as usual a discrete additive subgroup which is generated by a vector space basis of V. A subspace W of V is said to be rational with respect to Γ if $W \cap \Gamma$ is a lattice in W, or, equivalently, if W is defined by linear equations with integral coefficients in a coordinate system in which Γ is the lattice of integral points.

5.3. LEMMA. *Let G be a subgroup of finite index in a semi-simple real algebraic group, θ a Cartan involution of G, $G = K \cdot A \cdot N$ an Iwasawa decomposition of G compatible with θ, and $\mathfrak{S}$ a Siegel domain of G (with respect to the decomposition $K \cdot A \cdot N$). Let π be a real representation of G in a finite dimensional vector space V, and Γ a lattice in V with respect to which the maximal eigenspaces V_i of $\pi(A)$ are rational. Let $v \in V$ be a point whose orbit is closed and whose isotropy group G_v is stable under θ. Then $v \cdot \pi(\mathfrak{S}) \cap \Gamma$ is finite.*

We denote by $|v|$ the norm of $v \in V$ for a euclidean norm with respect to which the elements of $\pi(A)$ are self-adjoint (1.1). The space V is the direct sum of the V_i's, which are mutually orthogonal. We denote by E_i the orthogonal projection of V onto V_i, and by μ_i the weight of α in V_i. Since V_i is rational with respect to Γ, the subgroup of Γ spanned by the intersections $\Gamma \cap V_i$ has finite index, and $E_i(\Gamma)$ is also a *lattice* in V_i. There exists therefore a constant $c > 0$ such that $w \in \Gamma$, $E_i(w) \neq 0$ implies $|E_i(w)| \geqq c$.

Let now $x = k_x \cdot a_x \cdot n_x \in \mathfrak{S}$, and $y_x = x \cdot a_x^{-1}$, $z_x = y_x \cdot a_x^{-1}$. In the sequel, we shall drop π, and write $w \cdot g$ instead of $w \cdot \pi(g)(w \in V, g \in G)$.

Let $w = v \cdot x$, and $w_i = E_i(w)$. Then

(1) $E_i(v \cdot y_x) = \exp[-\mu_i(\log a_x)] \cdot w_i\ , \quad E_i(v \cdot z_x) = \exp[-2\mu_i(\log a_x)]w_i\ .$

It follows from 4.2 that

$$\{y_x\}_{x \in \mathfrak{S}} = K \cdot \{a \cdot n \cdot a^{-1}\}_{a \cdot n \in \mathfrak{S}}$$

is relatively compact. Consequently, $\{v \cdot y_x)_{x \in \mathfrak{S}}$ is relatively compact and there exists a constant c' such that $|w \cdot a_x{}^{-1}| = |v \cdot y_x| \leqq c'$. Hence

(2) $$|E_i(v \cdot y_x)| = |w_i| \exp[-\mu_i(\log a_x)] \leqq c'\ .$$

Assume now $w \in \Gamma$, and fix an i. There are two possibilities:

(i) $w_i = 0$; then $E_i(v \cdot y_x) = E_i(v \cdot z_x) = 0$;

(ii) $w_i \neq 0$; then, by the above, $| w_i | \geqq c$, hence, using (1),

$$| E_i(v \cdot z_x) | = | E_i(v \cdot y_x) |^2 / | w_i | \leqq c'^2/c \ . \tag{3}$$

In both cases, $| E_i(v \cdot z_x) |$ has a bound which is independent of i and of $x \in \mathfrak{S}$. This proves therefore that if $v \cdot x \in \Gamma$, $x \in \mathfrak{S}$, then $v \cdot z_x$ belongs to some compact set of V. Since $v \cdot G$ is closed by assumption, 5.2 shows the existence of a compact subset Q of G such that

$$v \cdot x \in \Gamma, x \in \mathfrak{S} \Rightarrow z_x \in G_v \cdot Q \ . \tag{4}$$

We now fix such an x, and drop the index x. We have

$$z = k \cdot a \cdot n \cdot a^{-2} = k \cdot a^{-1} \cdot a^2 \cdot n \cdot a^{-2} \in G_v \cdot Q \ .$$

But $\theta(k) = k$, $\theta(a) = a^{-1}$, and, by assumption, $\theta(G_v) = G_v$, hence

$$\begin{aligned} \theta(z) = k \cdot a \cdot \theta(a^2 \cdot n \cdot a^{-2}) = x \cdot n^{-1} \cdot \theta(a^2 \cdot n \cdot a^{-2}) \in G_v \cdot \theta(Q) \\ x \in G_v \cdot \theta(Q) \cdot \theta(a^2 \cdot n \cdot a^{-2})^{-1} \cdot n \ . \end{aligned} \tag{5}$$

By definition of $\mathfrak{S}$, the element n lies in a fixed compact set. Further if we have $\mathfrak{S} = \mathfrak{S}_{t,\omega}$ in the notation of 4.1, then $a^2 \in A_{2t}$ and therefore, by 4.2, $a^2 \cdot n \cdot a^{-2}$ lies in a fixed compact set. Then so does $\theta(a^2 \cdot n \cdot a^{-2})^{-1}$, and (5) shows the existence of a compact set $Q' \subset G$ such that

$$v \cdot x \in \Gamma, x \in \mathfrak{S} \Rightarrow x \in G_v \cdot Q' \ .$$

But then $v \cdot x$ belongs to $\Gamma \cap v \cdot Q'$, which is finite.

REMARK. The lemma and its proof are valid if G is reductive, provided $\pi(A)$ is real diagonalizable, as will always happen if π is a rational representation. In fact we shall use the following form of the lemma (only for $G = \mathbf{GL}(n, \mathbf{C})$, and we could also limit ourselves to the case $G = \mathbf{SL}(n, \mathbf{C})$.

5.4. LEMMA. *Let G be a reductive complex algebraic group defined over $\mathbf{Q}$, θ a Cartan involution of $G_{\mathbf{R}}$, $G_{\mathbf{R}} = K \cdot A \cdot N$ an Iwasawa decomposition of $G_{\mathbf{R}}$ compatible with θ, where A is contained in an algebraic torus T of G which is defined over $\mathbf{Q}$ and is isomorphic over $\mathbf{Q}$ to a product of groups $\mathbf{C}^*$, and $\mathfrak{S}$ a Siegel domain. Let $\pi\colon G \to \mathbf{GL}(V)$ be a rational representation of G which is defined over $\mathbf{Q}$, and Γ a lattice of $V_{\mathbf{Q}}$. Let v be a point of $V_{\mathbf{R}}$ whose orbit under G is closed and whose isotropy group in $G_{\mathbf{R}}$ is stable under θ. Then $v \cdot \pi(\mathfrak{S}) \cap \Gamma$ is finite.*

By 2.3, $v \cdot \pi(G_{\mathbf{R}})$ is closed. Moreover $\pi(T)$ is diagonalizable *over* $\mathbf{Q}$ [26, Proposition 5], which means that a maximal eigenspace of $\pi(A)$ has a basis in $V_{\mathbf{Q}}$, hence that it is "rational with respect to Γ". Thus, taking the above remark into account, all conditions under which the proof of 5.3 is valid, are fulfilled.

5.5. EXAMPLE. Let $G = \mathbf{SL}(n, \mathbf{R})$, θ be the involution $g \to {}^t g^{-1}$, and $\mathfrak{S}$ a standard Siegel domain (4.4). Let V be the space of symmetric real $n \times n$ matrices, and Γ the lattice of integral symmetric matrices. For π, we take the usual representation $F \to {}^t X \cdot F \cdot X = F[X]$. Here A is the group of positive diagonal matrices with det 1, and $\pi(A)$ is diagonal with respect to the standard basis of Γ; the maximal eigenspaces of $\pi(A)$ (which are in fact 1-dimensional) are then rational with respect to Γ. Let c be a strictly positive number, p, q two positive integers whose sum is n, and v the diagonal matrix with p entries equal to c, and q entries equal to $-c$. Then G_v is the proper orthogonal group of the quadratic form v, and is clearly stable under θ. Moreover, the orbit $v \cdot G$ of v consists of all quadratic forms with determinant $(-1)^q \cdot c^n$ and signature (p, q), hence is closed. The lemma applies, and shows that $v[\mathfrak{S}] \cap \Gamma$ is finite. But the matrices $v[g] (g \in \mathfrak{S})$ are those of the forms of determinant $(-1)^q \cdot c^n$ and signature (p, q), which are reduced in the sense of Hermite (if we agree to call reduced those positive quadratic forms whose matrix is of the type ${}^t X \cdot X$ (with $X \in \mathfrak{S}$). This is a somewhat bigger set than the one considered by Hermite). The lemma in this case means therefore that the integral reduced forms with given non-zero determinant and signature are finite in number, a well known statement of Hermite [14, p. 127]. Since $\mathbf{SL}(n, \mathbf{R}) = \mathfrak{S} \cdot \mathbf{SL}(n, \mathbf{Z})$, it implies that the number of proper classes of integral quadratic forms with a given non-zero determinant is finite.

If we consider, instead of π, the natural representation of $\mathbf{SL}(n, \mathbf{R})$ in the space of homogeneous forms of degree $m \geqq 2$, then the lemma also applies, and shows that the number of classes of integral forms belonging to some closed orbit is finite. Such generalizations of Hermite's result have been given by C. Jordan (Journal Ec. Polytechnique, 48 (1880), pp. 151–168), and H. Poincaré (*ibidem*, 51 (1882), pp. 45–91).

6. A fundamental set for arithmetic subgroups

6.1. Let G be an algebraic group defined over $\mathbf{Q}$. Then $G_{\mathbf{Z}}$ is a discrete subgroup of $G_{\mathbf{R}}$. It is known [29], and will follow from 6.3, that if we change the imbedding, that is, if we replace G by its image G' under a rational injective homomorphism ρ, defined over $\mathbf{Q}$, then $G'_{\mathbf{Z}}$ is commensurable with $\rho(G_{\mathbf{Z}})$. (We recall that two subgroups of a group are commensurable if their intersection has finite index in both.) Therefore the commensurable class of $G_{\mathbf{Z}}$ in $G_{\mathbf{R}}$ has an intrinsic meaning.

6.2. PROPOSITION. *Let G be a connected algebraic group defined over*

$\mathbf{Q}$, $\pi: G \to \mathbf{GL}(V)$ *a rational representation defined over* $\mathbf{Q}$. *Then the subgroup of* $G_{\mathbf{Z}}$ *leaving a lattice* Γ *in* $V_{\mathbf{Q}}$ *invariant has finite index in* $G_{\mathbf{Z}}$.

Let H be the identity component of $G \cap \mathbf{SL}(n, \mathbf{C})$. Then $H_{\mathbf{Z}}$ has finite index in $G_{\mathbf{Z}}$, we may therefore assume $G \subset \mathbf{SL}(n, \mathbf{C})$. In this case, the coefficients $\pi(g)_{\mu\nu}$ of $\pi(g)$, with respect to a basis of Γ, are polynomials in those of g, with rational coefficients (2.2). Let $g'_{ij} = g_{ij} - \delta_{ij}$. Then

$$y_{\mu\nu}(g) = \pi(g)_{\mu\nu} - \delta_{\mu\nu} = P_{\mu\nu}(g'_{11}, \cdots, g'_{nn}) , \qquad (1 \leqq \mu, \nu \leqq \dim V) ,$$

where the $P_{\mu\nu}$ are polynomials with rational coefficients and no constant term. Let m be a common multiple of the denominators of the $P_{\mu\nu}$, and M be the congruence subgroup of the elements in $G_{\mathbf{Z}}$ which are $\equiv \mathrm{Id} \bmod \mathrm{m}$. Then M has finite index in $G_{\mathbf{Z}}$, and $\pi(g)_{\mu\nu} \in \mathbf{Z}$ for $g \in M$.

6.3. COROLLARY. *There exists a lattice in* $V_{\mathbf{Q}}$ *containing* Γ *which is invariant under* $G_{\mathbf{Z}}$. *If* π *is faithful,* $G_{\mathbf{Z}}$ *is commensurable with the subgroup of* G *leaving* Γ *invariant.*

We keep the notation of the previous proof. The subgroup M is invariant in $G_{\mathbf{Z}}$, therefore, for any $g \in G_{\mathbf{Z}}$, the lattice $\Gamma \cdot \pi(g)$ is invariant under M. Then the sum of the lattices $\Gamma \cdot \pi(g_i)$, where g_i runs through a system of representatives of $G_{\mathbf{Z}}/M$, is a lattice invariant under $G_{\mathbf{Z}}$. The second assertion follows from 6.2 applied to π and to π^{-1}.

6.4. COROLLARY. *Let* $G = H \cdot N$ *be the semi-direct product of a subgroup* H *and of an invariant subgroup* N, *both defined over* $\mathbf{Q}$. *Then* $H_{\mathbf{Z}} \cdot N_{\mathbf{Z}}$ *has finite index in* $G_{\mathbf{Z}}$.

We may assume G to be connected. The map $g = h \cdot n \to h (h \in H, n \in N)$ is a rational homomorphism defined over $\mathbf{Q}$, hence (6.2) $G_{\mathbf{Z}}$ has a subgroup M of finite index whose image is in $H_{\mathbf{Z}}$. Then $M \subset H_{\mathbf{Z}} \cdot N_{\mathbf{Z}}$.

6.5. THEOREM. *Let* $G \subset \mathbf{GL}(n, \mathbf{C})$ *be a reductive algebraic group defined over* $\mathbf{Q}$, $a \in \mathbf{SL}(n, \mathbf{R})$ *such that* $a \cdot G_{\mathbf{R}} \cdot a^{-1}$ *is self-adjoint* (*see* 1.9), *and* $\mathfrak{S}$ *a standard Siegel domain of* $\mathbf{GL}(n, \mathbf{R})$ (*see* 4.4). *Then there exist finitely many elements* $b_1, \cdots, b_m \in \mathbf{SL}(n, \mathbf{Z})$ *such that the interior* U *of* $\bar{U} = \bigcup_{i=1}^{i=m} (a^{-1} \cdot \mathfrak{S} \cdot b_i) \cap G_{\mathbf{R}}$ *has the following properties:*

(i) $G_{\mathbf{R}} = U \cdot G_{\mathbf{Z}}$;

(ii) $K \cdot U = U$ *for a suitable maximal compact subgroup* K *of* $G_{\mathbf{R}}$;

(iii) $U^{-1} \cdot U \cap x \cdot G_{\mathbf{Z}} \cdot y$ *is finite for any* $x, y \in G_{\mathbf{Q}}$.

The group $G_{\mathbf{Z}}$ *is finitely generated.*

By 3.8, we may find a rational representation $\pi: \mathbf{GL}(n, \mathbf{C}) \to \mathbf{GL}(V)$, defined over $\mathbf{Q}$, for which there exists $v \in V_{\mathbf{Q}}$ whose isotropy group is G and whose orbit is closed. Using 6.3, we take in $V_{\mathbf{Q}}$ a lattice Γ invariant under $\mathbf{GL}(n, \mathbf{Z})$; replacing v by a multiple if necessary, we many assume $v \in \Gamma$. The group $G_{\mathbf{R}}$ is real algebraic, reductive (3.2); by 1.9, there exists $a \in \mathbf{SL}(n, \mathbf{R})$ such that $a \cdot G_{\mathbf{R}} \cdot a^{-1} = G'_{\mathbf{R}}$ is self-adjoint. Let $v' = v \cdot \pi(a^{-1})$. Then $v' \cdot \pi(\mathbf{GL}(n, \mathbf{C})) = v \cdot \pi(\mathbf{GL}(n, \mathbf{C}))$ is closed, hence (2.3) so is $v' \cdot \pi(\mathbf{GL}(n, \mathbf{R}))$. The isotropy group of v' in $\mathbf{GL}(n, \mathbf{R})$ is $G'_{\mathbf{R}}$, and is by construction invariant under the Cartan involution $\theta: g \to {}^t g^{-1}$, which underlies the definition of the standard Siegel domain $\mathfrak{S}$. Moreover, in this case, A is the subgroup of diagonal matrices with positive real eigenvalues, and it belongs to the algebraic torus $D(n)$ of all diagonal matrices of $\mathbf{GL}(n, \mathbf{C})$, which is defined over $\mathbf{Q}$. Consequently, all conditions of 5.4 are fulfilled, and we may assert that $v' \cdot \pi(\mathfrak{S}) \cap \Gamma$ is finite. *A fortiori*, $v' \cdot \pi(\mathfrak{S}) \cap v \cdot \pi(\mathbf{SL}(n, \mathbf{Z}))$ is finite. Let then $b_1, \cdots, b_m \in \mathbf{SL}(n, \mathbf{Z})$ be such that

$$v \cdot \pi(\mathfrak{S}) \cap v \cdot \pi\big(\mathbf{SL}(n, \mathbf{Z})\big) \subset \{v \cdot \pi(b_1^{-1}), \cdots, v \cdot \pi(b_m^{-1})\} . \tag{1}$$

Let now

$$H = \{g \in \mathbf{GL}(n, \mathbf{R}) \mid v' \cdot \pi(g) = v\} . \tag{2}$$

Then

$$H = a \cdot G_{\mathbf{R}} = G'_{\mathbf{R}} \cdot a . \tag{3}$$

Let $h \in H$. By the classical reduction theory (4.5),

$$h = s \cdot b \qquad \big(s \in \mathfrak{S},\ b \in \mathbf{SL}(n, \mathbf{Z})\big) .$$

The equality $v' \cdot \pi(h) = v$ yields $v' \cdot \pi(s) = v \cdot \pi(b^{-1})$, hence, by (1), there exists an index i, $(1 \leqq i \leqq m)$ such that

$$v' \cdot \pi(s) = v \cdot \pi(b^{-1}) = v \cdot \pi(b_i^{-1}) .$$

We have then $b_i^{-1} \cdot b \in G \cap \mathbf{SL}(n, \mathbf{Z}) \subset G_{\mathbf{Z}}$, hence $b \in b_i G_{\mathbf{Z}}$, and

$$\begin{aligned} &H \subset \textstyle\bigcup_i \mathfrak{S} \cdot b_i \cdot G_{\mathbf{Z}} , \\ &G_{\mathbf{R}} = a^{-1} \cdot H \subset \textstyle\bigcup_i a^{-1} \cdot \mathfrak{S} \cdot b_i \cdot G_{\mathbf{Z}} , \\ &G_{\mathbf{R}} = \bar{U} \cdot G_{\mathbf{Z}}, \qquad \big(\bar{U} = \textstyle\bigcup_i (a^{-1} \cdot \mathfrak{S} \cdot b_i) \cap G_{\mathbf{R}}\big) . \end{aligned} \tag{4}$$

The equality (4) is *a fortiori* true if we replace $\mathfrak{S}$ by a standard Siegel domain containing $\mathfrak{S}$ in its interior, hence (i) is proved.

The group $K' = G'_{\mathbf{R}} \cap \mathbf{O}(n)$ is maximal compact in $G'_{\mathbf{R}}$ (1.10), and $K = a^{-1} \cdot K' \cdot a$ is maximal compact in $G_{\mathbf{R}}$. Clearly, $K' \cdot (\mathfrak{S} b_i \cap G'_{\mathbf{R}}) \subset \mathfrak{S} b_i \cap G'_{\mathbf{R}}$, and therefore $K \cdot \bar{U} = \bar{U}$, $K \cdot U = U$.

Let now $x, y \in G_{\mathbf{Q}}$, and $u \in U^{-1} \cdot U \cap x \cdot G_{\mathbf{Z}} \cdot y$. There exist, then, two indices $i, j (1 \leqq i, j \leqq m)$ such that

$$u \in (a^{-1} \cdot \mathfrak{S} \cdot b_i)^{-1} \cdot (a^{-1} \cdot \mathfrak{S} \cdot b_j) = b_i^{-1} \cdot \mathfrak{S}^{-1} \cdot \mathfrak{S} \cdot b_j ,$$

whence

$$b_i \cdot u \cdot b_j^{-1} \in \mathfrak{S}^{-1} \cdot \mathfrak{S} ,$$

and the finiteness of the number of the possible u's follows from Siegel's theorem (4.5). For $x = y = e$, this implies that $U^{-1} \cdot U \cap G_Z$ is finite, in other words that U meets only a finite number of its right translates under G_Z. Since $G_Z \cap (G_R)^0$ is of finite index in G_Z, the finite generation of G_Z follows from the following well known elementary lemma:

6.6. Lemma. *Let H be a group which operates on a connected topological space M, and U an open set such that $U \cdot H = M$. Then $J = \{h \in H \mid U \cdot h \cap U \neq \emptyset\}$ is a set of generators for H.*

In fact let H' be the subgroup generated by J. Then $U \cdot H'$ is open. If $U \cdot h \cap U \cdot h' \neq \emptyset (h \in H, h' \in H')$, then $U \cdot h \cdot h'^{-1} \cap U \neq \emptyset$, hence $h \in J \cdot h' \subset H'$, from which it follows first that $U \cdot H' = M$, and then that $H' = H$.

6.7. Remarks.

(1) Let us call fundamental set an open subset of G_R satisfying properties (i), (ii), (iii). In view of the inclusion properties of standard Siegel domains listed at the end of 4.5, the class C of fundamental sets constructed in 6.5 has the following properties: it covers G_R, any finite union of such sets is contained in a set of C, any such set contains the closure of an element of C.

The main part of 6.5 will be extended to algebraic groups in 6.12. It will be shown later that, at any rate for a suitable a, the set U has finite Haar measure when G is semi-simple.

(2) Let K be a maximal compact subgroup such that $K \cdot U = U$, and π the natural projection of G_R onto $P = K\backslash G_R$. Clearly, $U' = \pi(U)$ has the following properties: (i′) $P = U' \cdot G_Z$; (iii′) for $x, y \in G_Q$, the set of $g \in G_Z$ for which $U' \cap U' \cdot x \cdot g \cdot y \neq \emptyset$ is finite. Conversely, the inverse image of a set U' in P having the properties (i′), (iii′) has the properties (i), (ii), (iii), so that the construction of fundamental sets in G_R or in $K\backslash G_R$ are essentially equivalent questions. In the special case where G is the orthogonal group of an integral non-degenerate indefinite quadratic form F, $K\backslash G_R$ is the space of majorizing positive forms of F in the sense of Hermite. If we take, furthermore, the natural representation of $\mathbf{GL}(n, \mathbf{C})$ in the space of symmetric matrices, then our construction reduces to that of Hermite (always with the minor difference that instead of using an arbitrary Siegel domain $\mathfrak{S}$ of $\mathbf{GL}(n, \mathbf{R})$, Hermite uses the inverse image of the space of reduced positive forms in his sense, that is, the domain given by 4.5a′). In this case, the two properties of U' above, and the resulting finite generation of G_Z, are also proved by Hermite [14, pp. 201–

233, § VIII].

6.8. LEMMA. *Let $M \subset G$ be reductive subgroups of* $\mathbf{GL}(n, \mathbf{C}) = G'$, *both defined over* $\mathbf{Q}$. *Let* $\pi: G' \to \mathbf{GL}(W)$ *be a rational representation, defined over* $\mathbf{Q}$, Γ *a lattice in* $W_{\mathbf{Q}}$ *invariant under* $G'_{\mathbf{Z}}$, $w' \in \Gamma$ *a point whose orbit under* G' *is closed and whose isotropy group in* G' *is* M. *Then* $w' \cdot G_{\mathbf{R}} \cap \Gamma$ *consists of a finite number of orbits of* $G_{\mathbf{Z}}$.

By 1.9, there exists $a \in \mathbf{SL}(n, \mathbf{R})$ such that $a \cdot G_{\mathbf{R}} \cdot a^{-1}$ and $a \cdot M_{\mathbf{R}} \cdot a^{-1}$ are self-adjoint. By 6.5, there exists a finite number of elements $b_i \in G'_{\mathbf{Z}}$ such that $G_{\mathbf{R}} = \bigcup_i (G_{\mathbf{R}} \cap a^{-1} \cdot \mathfrak{S} \cdot b_i) \cdot G_{\mathbf{Z}}$ where $\mathfrak{S}$ is a standard Siegel domain of $\mathbf{GL}(n, \mathbf{R})$. It suffices therefore to show that

$$w \cdot (G_{\mathbf{R}} \cap a^{-1} \cdot \mathfrak{S} \cdot b_i) \cap \Gamma$$

is finite, hence, *a fortiori*, that $w \cdot a^{-1} \cdot \mathfrak{S} \cap \Gamma$ is finite. Let $w' = w \cdot a^{-1}$. Then $G'_{w'} = a \cdot M \cdot a^{-1}$, hence $(G'_{w'})_{\mathbf{R}}$ is self-adjoint. Further, the group A of diagonal matrices which underlies the definition of $\mathfrak{S}$ belongs to an algebraic torus which is diagonal over $\mathbf{Q}$ hence 5.4 applies and yields the finiteness of $w \cdot a^{-1} \cdot \mathfrak{S} \cap \Gamma = w' \cdot \mathfrak{S} \cap \Gamma$.

6.9. THEOREM. *Let G be a reductive algebraic group defined over* $\mathbf{Q}$, $\pi: G \to \mathbf{GL}(V)$ *a rational representation defined over* $\mathbf{Q}$, Γ *a lattice in* $V_{\mathbf{Q}}$ *invariant under* $G_{\mathbf{Z}}$, *and X a closed orbit of G. Then $X \cap \Gamma$ consists of a finite number of orbits of* $G_{\mathbf{Z}}$.

Since G^0 has finite index in G, we may assume G to be connected. We assume $X \cap \Gamma \neq \varnothing$ (otherwise there is nothing to prove), take $v \in \Gamma \cap X$, and put $H = G_v$. The group H and the orbit $X = v \cdot G$ are defined over $\mathbf{Q}$. Since X is closed, H is reductive (3.8). We prove first:

(*) There exists a rational representation $\pi': G \to \mathbf{GL}(W)$ defined over $\mathbf{Q}$, a point $w \in W_{\mathbf{Q}}$ such that $G_w = H$, $w \cdot \pi'(G) = X'$ is closed, and that $X' \cap \Gamma'$ consists of a finite number of orbits of $G_{\mathbf{Z}}$ for any lattice $\Gamma' \subset W_{\mathbf{Q}}$ invariant under $G_{\mathbf{Z}}$.

Let $G' = \mathbf{GL}(n, \mathbf{C})$. The group H being reductive, defined over $\mathbf{Q}$, there exists by 3.8 a rational representation $\rho: G' \to \mathbf{GL}(W)$ defined over $\mathbf{Q}$, and a point $w \in W_{\mathbf{Q}}$ such that $w \cdot \rho(G')$ is closed and $H = G'_w$. We claim that the restriction π' of ρ to G fulfills our conditions. The orbit $X' = w \cdot \pi'(G)$ is closed because if we identify $w \cdot \rho(G')$ with $H \backslash G'$, the orbit X' is the inverse image of a point in the projection $H \backslash G' \to G \backslash G'$. Let now $\Gamma' \subset W_{\mathbf{Q}}$ be a lattice invariant under $G_{\mathbf{Z}}$. It is contained in a lattice invariant under $G'_{\mathbf{Z}}$ (6.3), therefore it is enough to consider $X' \cap \Gamma'$ for lattices invariant under $G'_{\mathbf{Z}}$. Of course, $X' \cap \Gamma' \subset X' \cap W_{\mathbf{R}}$, which consists of a finite number of closed orbits of $G_{\mathbf{R}}$ (2.3); therefore, it suffices to show that for $w' \in X' \cap \Gamma'$, the intersection $w' \cdot \pi'(G_{\mathbf{R}}) \cap \Gamma'$ consists of a finite number of orbits of $G_{\mathbf{Z}}$. There exists $g \in G$ such that $w' = w \cdot \pi'(g)$, hence $G'_{w'} = g^{-1} \cdot H \cdot g \subset G$. The group $G'_{w'}$ is reductive, since H is, and defined over $\mathbf{Q}$, since $w' \in W_{\mathbf{Q}}$. Moreover $w' \cdot \rho(G') = w \cdot \rho(G')$ is closed. The fact that $w' \cdot \pi'(G_{\mathbf{R}}) \cap \Gamma'$ is the union of a finite number of orbits of $G_{\mathbf{Z}}$ is now a consequence of 6.8 (with $M = G'_{w'}$).

The maps $g \to v \cdot \pi(g)$ and $g \to w \cdot \pi'(g)$ induce isomorphisms of $H\backslash G$ with X and X', defined over $\mathbf{Q}$, whence an equivariant isomorphism $\varphi: X \to X'$, defined over $\mathbf{Q}$. Let $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ be coordinates in V and W with respect to bases of Γ and Γ' respectively. The function $y_i(\varphi(x))(x \in X)$ is a regular function, defined over $\mathbf{Q}$. Since X is an affine algebraic set, $y_i(\varphi(x))$ may be written as a polynomial $P_i(x_1, \cdots, x_r)$ in the x_i's, with rational coefficients $(1 \leqq i \leqq s)$. If q is a common multiple of the denominators of those coefficients, then $q \cdot y_i(\varphi(x))$ is integral whenever $x_1, \cdots, x_r$ are, therefore

$$\varphi(X \cap \Gamma) \subset X' \cap \frac{1}{q}\Gamma' .$$

The lattice $(1/q)\Gamma'$ is of course also invariant under $G_{\mathbf{Z}}$, therefore $(1/q)\Gamma' \cap X'$ consists of a finite number of orbits of $G_{\mathbf{Z}}$ by (*). Since φ is G-equivariant, its restriction to $X \cap \Gamma$ is $G_{\mathbf{Z}}$-equivariant, hence $X \cap \Gamma$ is also a finite number of orbits of $G_{\mathbf{Z}}$.

6.10. In order to go from the reductive to the general case in the two following sections, we shall use the following facts: a connected algebraic group G defined over a field k (of characteristic 0) is the semi-direct product of a reductive group H and of an invariant unipotent group N, both defined over k.[4] If G is unipotent, defined over $\mathbf{Q}$, then $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact. The latter fact is completely elementary; in fact, G being nilpotent, defined over $\mathbf{Q}$, its Lie algebra has a basis (x_i) $(i = 1, \cdots, \dim G)$ consisting of matrices with integral coefficients, such that for each $j \geqq 1$, the elements $x_i (i \geqq j)$ span an ideal. Let, further, m be the degree of the ambient linear group. Then $x_i^m = 0$ $(i = 1, \cdots, \dim G)$, and $g_i = \exp(m! \cdot x_i) \in G_{\mathbf{Z}}$. Using induction on $\dim G$, it is immediately seen that the quotient of $G_{\mathbf{R}}$ by the subgroup generated by the g_i's is compact.

6.11. COROLLARY. *Let G, G' be algebraic groups defined over $\mathbf{Q}$ and $\mu: G \to G'$ a surjective rational homomorphism defined over $\mathbf{Q}$, with finite kernel (an isogeny). Then $\mu(G_{\mathbf{Z}})$ and $G'_{\mathbf{Z}}$ are commensurable.*

Here again, it is enough to prove this when G and G' are connected. It

[4] The corresponding statement for Lie algebras is proved in [7b, Chap. V, § 4, Prop. 5]. For the global version, see G. D. Mostow, Amer. J. Math. 78 (1956), 200-221, Theorem 6.1.

follows from 6.3 that we may assume $\mu(G_Z)\subset G'_Z$. By adding one coordinate (see 2.1), we may have G' closed in $\mathbf{M}(n', \mathbf{C})$.

Let now G be reductive. We let G act on $V = \mathbf{M}(n', \mathbf{C})$ by right translations $x \to x \cdot \mu(g)$, and get in this way a rational representation $G \to \mathbf{GL}(V)$ defined over $\mathbf{Q}$. The subgroup G'_Z is the intersection of the closed orbit G' of G with the lattice $\mathbf{M}(n', \mathbf{Z})$ which is invariant under G_Z, and belongs to V_Q. By 6.9, G'_Z consists of a finite number of cosets of $\mu(G_Z)$, hence $[G'_Z: \mu(G_Z)] < \infty$.

If G is unipotent, G_R/G_Z is compact; then so is $G'_R/\mu(G_Z)$; this space is a covering of G'_R/G'_Z with a discrete fibre which has $[G'_Z: \mu(G_Z)]$ elements. This number must be finite.

In the general case, $G = H \cdot N$, $G' = H' \cdot N'$ with H, $H' = \mu(H)$ reductive, defined over $\mathbf{Q}$, N, $N' = \mu(N)$ unipotent, invariant, defined over $\mathbf{Q}$ (6.10). By the above $\mu(H_Z \cdot N_Z)$ has finite index in $H'_Z \cdot N'_Z$, and 6.11 follows from 6.4.

6.12. THEOREM. *Let G be an algebraic group defined over* $\mathbf{Q}$. *There exists an open set U in G_R such that $U \cdot G_Z = G_R$, and that for any $x, y \in G_Q$, the intersection $U^{-1} \cdot U \cap x \cdot G_Z \cdot y$ is finite. The group G_Z is finitely generated.*

It is enough to prove this for G connected. If G is reductive, see 6.5. If G is unipotent, then (6.10) there exists a relatively compact open subset U such that $G_R = U \cdot G_Z$. Then U clearly satisfies our second condition.

In the general case, we use the decomposition $G = H \cdot N$ of 6.10. Let A and B be open subsets in H and N satisfying our two conditions in H and N, with B, moreover, relatively compact. We assert that $U = A \cdot B$ verifies our contention. In fact

$$G = H \cdot N = A \cdot H_Z \cdot N \subset A \cdot N \cdot H_Z \subset A \cdot B \cdot N_Z \cdot H_Z \subset A \cdot B \cdot G_Z ,$$

which proves the first condition. The group $H_Z \cdot N_Z$ has finite index in G_Z (6.4). Therefore, our second assertion is equivalent to the finiteness of $U^{-1} \cdot U \cap (x \cdot H_Z \cdot N_Z \cdot y)$ for arbitrary $x, y \in G_Q$. The group G being isomorphic to $H \cdot N$ over $\mathbf{Q}$, we also have $G_Q = H_Q \cdot N_Q$, and may write $x = a \cdot b$, $y = c \cdot d (a, c \in H_Q, b, d \in N_Q)$. Let $h \in H_Z$, $n \in N_Z$. We have

$$A \cdot B \cdot x \cdot h \cdot n \cdot y = A \cdot a \cdot h \cdot c \cdot B' \cdot n'$$
$$(B' = (a \cdot h \cdot c)^{-1} \cdot B \cdot a \cdot b \cdot h \cdot c \ ; \quad n' = c^{-1} \cdot n \cdot c \cdot d) ,$$

hence $A \cdot B \cap A \cdot B \cdot x \cdot h \cdot n \cdot y \neq \emptyset$ is equivalent to

$$A \cdot a \cdot h \cdot c \cap A \neq \emptyset , \qquad B' \cdot n' \cap B \neq \emptyset . \tag{1}$$

By our assumption on A, the possible h's are finite in number. Since B is relatively compact, $B'^{-1} \cdot B$ is also relatively compact, hence its intersection with the discrete set $c^{-1} \cdot N_{\mathbf{Z}} \cdot c \cdot d$ is finite. Thus there is a finite number of possibilities for h, and for each of them, a finite number of possible n's.

The above implies in particular that $U^{-1} \cdot U \cap G_{\mathbf{Z}}$ is finite. The finite generation of $G_{\mathbf{Z}}$ then follows from 6.6, as in 6.5.

6.13. Remark. As was mentioned in the introduction, the construction of U in 6.5 is a generalization of Hermite's procedure in the case of indefinite quadratic forms [14]. As is well known, the latter had been adapted, notably by Siegel, to many other cases, the most inclusive one being that of the automorphism group of a rational involutorial semi-simple algebra [24, 29]. This case represents essentially all classical groups with center reduced to the identity. However, U is constructed there in the symmetric space $K\backslash G_{\mathbf{R}}$, rather than in $G_{\mathbf{R}}$, but this is a minor difference (see 6.7). This implies, of course, the finite generation of $G_{\mathbf{Z}}$. The finiteness of the volume of $G_{\mathbf{R}}/G_{\mathbf{Z}}$, or, equivalently, of $(K\backslash G_{\mathbf{R}})/G_{\mathbf{Z}}$, in that case is also proved in [24], generalizing earlier results of Siegel.

In order to construct U, we take a rational representation of the ambient linear group $\mathbf{GL}(n, \mathbf{C})$, such that the representation space has a rational point with closed orbit and isotropy group G, whose existence follows from 2.5 and 3.5b. In this respect, we point out that 3.5a is used only in proving 6.9, and there only to ascertain that H is reductive. This last fact is obvious in 6.11, where $H = (e)$ and follows from [4] in 11.6, where H runs through the centralizers of semi-simple elements; as these are the only two applications of 6.9 made in this work, we see that, except for 6.9 in full generality, this paper can be made independent of 3.5a.

7. The finiteness of the volume for semi-simple groups

In this paragraph, we often write b^a for $a \cdot b \cdot a^{-1}$, where a, b are elements of a group.

For the sake of reference, we first sketch the proof of an elementary lemma on nilpotent Lie groups (see also [13, Lemma 1]):

7.0. Lemma. *Let N be a connected, simply connected, real or complex nilpotent Lie group, $\mathfrak{n}^{(i)}$ a strictly decreasing sequence of ideals of $\mathfrak{n}$ such that $[\mathfrak{n}, \mathfrak{n}^{(i)}] \subset \mathfrak{n}^{(i+1)}$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ two mutually complementary subspaces such that $\mathfrak{n}^{(i)} = \mathfrak{n}_1 \cap \mathfrak{n}^{(i)} + \mathfrak{n}_2 \cap \mathfrak{n}^{(i)}$ $(i = 0, \cdots; \mathfrak{n}^{(0)} = \mathfrak{n})$. Then $(x, y) \to \exp x \cdot \exp y (x \in \mathfrak{n}_2, y \in \mathfrak{n}_1)$ is an analytic homeomorphism of $\mathfrak{n}_2 \times \mathfrak{n}_1$ onto N.*

Let s be the biggest index such that $\mathfrak{n}^{(s)} \neq 0$. Then $\mathfrak{n}^{(s)}$ is central, generates a closed simply connected central subgroup $N^{(s)}$ of N, and

$N/N^{(s)}$ is simply connected. Let $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$ be supplementary subspaces to $\mathfrak{n}^{(s)} \cap \mathfrak{n}_1$ and $\mathfrak{n}^{(s)} \cap \mathfrak{n}_2$ in $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively. Proceeding by induction on s, we may assume the lemma to be true for $\mathfrak{n}_1/(\mathfrak{n}_1 \cap \mathfrak{n}^{(s)})$ and $\mathfrak{n}_2/(\mathfrak{n}_2 \cap \mathfrak{n}^{(s)})$ in $\mathfrak{n}/\mathfrak{n}^{(s)}$. This implies immediately that $(x, y) \to \exp x \cdot \exp y$ is an analytic homeomorphism of $\mathfrak{n}_2' \times \mathfrak{n}_1'$ onto $M = \exp \mathfrak{n}_2' \cdot \exp \mathfrak{n}_1'$, and that $(m, z) \to m \cdot z$ is a homeomorphism of $M \times N^{(s)}$ onto N. The lemma then follows readily from the following facts; the exponential is an analytic homeomorphism of $\mathfrak{n}$ onto N, $\exp(a + b) = \exp a \cdot \exp b$ if $[a, b] = 0$ $(a, b \in \mathfrak{n})$, and $\mathfrak{n}^{(s)}$ is central.

7.1. Let G be a real algebraic semi-simple Lie group, θ a Cartan involution of G, and $G = K \cdot A \cdot N$ an Iwasawa decomposition of G compatible with θ (1.11). We use the notation of 4.1, except for the fact that Σ will denote the set of all positive roots for the ordering defined by N. Moreover, given $x = k \cdot a \cdot n (k \in K, a \in A, n \in N)$, we put $H(x) = \log a$ and $\nu(x) = n$.

Let M be the centralizer and M^* the normalizer of A in K. Then $M^*/M = W$ is a finite group, the "restricted" Weyl group of G. It acts by inner automorphisms on A or $\mathfrak{a}$, and this representation is faithful.

7.2. LEMMA. *We keep the above notation. Then $M^*/(M^* \cap G^0) = G/G^0$ and each coset of M^* modulo $M^* \cap G^0$ contains an element which normalizes N. The group G is the disjoint union of the subsets Nm_wM_wAN, where m_w runs through a system of representatives of the elements w of W.*

By Bruhat's lemma, [10], G^0 is the disjoint union of the subsets $Nm_w(M \cap G^0)A \cdot N$, where w runs through $M(M^* \cap G^0)/M$. Our second assertion follows from this and the first assertion. By a theorem of E. Cartan (see e.g. [12, Lemma 33]), given $k \in K$, there exists $k' \in K^0$ such that $kAk^{-1} = k'Ak'^{-1}$. Therefore M^* meets each connected component of G. Moreover, $M(M^* \cap G^0)/M$ is transitive on the Weyl chambers of $\mathfrak{a}$, therefore each coset of M^* modulo $M^* \cap G^0$ contains an element which leaves the positive Weyl chamber invariant, hence normalizes N. This proves the first assertion.

REMARK. The above is also valid in a semi-simple Lie group with finitely many connected components, whose identity component has a finite center.

7.3. Let $w \in W$, $\Sigma(w)$ be the set of positive roots whose transforms under w are negative, and $\Phi(w)$ the set of roots which are linear combinations of elements of $\Sigma(w) \cup (-\Sigma(w))$. Then

$$\mathfrak{g}_w = \sum\nolimits_{\alpha \in \Phi(w)} (\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}])$$

is clearly a subalgebra. Let $\mathfrak{n}_w = \mathfrak{g}_w \cap \mathfrak{n}$, $\mathfrak{a}_w = \mathfrak{g}_w \cap \mathfrak{a}$, $\mathfrak{m}_w = \mathfrak{g}_w \cap \mathfrak{m}$, $\mathfrak{k}_w = \mathfrak{g}_w \cap \mathfrak{k}$. As usual, for $\alpha \in \mathfrak{a}^*$, h_α is the element of $\mathfrak{a}$ defined by $(h_\alpha, h) = \alpha(h)$, where $(\ ,\ )$ is the scalar product defined by the Killing form. Then $\mathfrak{g}_w$ is semi-simple,

$$\mathfrak{g}_w = \mathfrak{k}_w + \mathfrak{a}_w + \mathfrak{n}_w \tag{1}$$

is an Iwasawa decomposition of $\mathfrak{g}_w$, and

$$\mathfrak{a}_w = \textstyle\sum_{\alpha \in \Sigma(w)} \mathbf{R} \cdot h_\alpha = \sum_{\alpha \in \Phi(w)} \mathbf{R} \cdot h_\alpha \,. \tag{2}$$

In fact, it is elementary and known that if $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then

$$\begin{aligned} &B(x, \theta(x)) \neq 0 && (x \neq 0)\,, \\ &[x, \theta(x)] = h_\alpha B(x, \theta(x)), \quad [x, y] - h_\alpha B(x, y) \in \mathfrak{m}\,, \end{aligned} \tag{3}$$

where B is the Killing form, from which we deduce (2) and

$$\mathfrak{m}_w + \mathfrak{a}_w = \textstyle\sum_{\alpha \in \Phi(w)} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \quad \mathfrak{g}_w = \mathfrak{m}_w + \mathfrak{a}_w + \sum_{\alpha \in \Phi(w)} \mathfrak{g}_\alpha\,, \tag{4}$$

so that (1) follows from

$$\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{g}_\alpha + \theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha + (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}\,. \tag{5}$$

The algebra $\mathfrak{g}_w$ is stable under θ, hence reductive (1.4). By (4), $\mathfrak{m}_w + \mathfrak{a}_w$ belongs to the derived algebra $\mathscr{D}\mathfrak{g}_w$ of $\mathfrak{g}_w$. By (2), for each $\alpha \in \Phi(w)$, there exists $h \in \mathfrak{a}_w$ which does not annihilate α, hence $\mathfrak{g}_\alpha = [h, \mathfrak{g}_\alpha] \subset \mathscr{D}\mathfrak{g}$. Thus $\mathfrak{g}_w = \mathscr{D}\mathfrak{g}_w$, and $\mathfrak{g}_w$ is semi-simple.

Up to 7.7, G is a real algebraic semi-simple Lie group, θ a Cartan involution of G, $G = K \cdot A \cdot N$ an Iwasawa decomposition compatible with θ, Σ the set of the positive roots in the ordering defined by N. The Siegel domains of G are always defined with respect to the given Iwasawa decomposition.

7.4. LEMMA. *Let $\mathfrak{S}$ be a Siegel domain of G, $x \in G$ and t a positive real number. Then $\mathfrak{S}x \cap K \cdot A_t \cdot N$ is contained in a Siegel domain of G.*

For any Siegel domain $\mathfrak{S}'$ and elements $a \in A$, $n \in N$, the sets $\mathfrak{S}' \cdot a$ and $\mathfrak{S}' \cdot n$ belong to Siegel domains. In view of Bruhat's lemma (7.2) we may assume $x = m^{-1}$, $m \in M^*$. Let w be the element of W defined by m. We use the notation of 7.3. Let further $\mathfrak{n}'_w$ be the sum of the $\mathfrak{g}_\alpha$ where $\alpha \in \Sigma$, $\alpha \notin \Phi(w)$, $\mathfrak{a}'_w$ be the subspace of $\mathfrak{a}$ on which the elements of $\Phi(w)$ are all zero. Let A_w, A'_w, N_w, N'_w be the exponentials of $\mathfrak{a}_w, \mathfrak{a}'_w, \mathfrak{n}_w, \mathfrak{n}'_w$. Then $A = A_w \cdot A'_w$, and the group A'_w centralizes the analytic group $G(w)$ with Lie algebra $\mathfrak{g}_w$. Let further $\alpha_1, \cdots, \alpha_s$ be the positive roots arranged in increasing order. Then

$$\mathfrak{n}^{(i)} = \textstyle\sum_{j \geqq i} \mathfrak{g}_{\alpha_j}\,, \qquad \mathfrak{n}_1 = \mathfrak{n}_w,\ \mathfrak{n}_2 = \mathfrak{n}'_w\,,$$

satisfy the assumptions of 7.0, hence $(n, n') \to \exp n \cdot \exp n'$ is an analytic homeomorphism of $\mathfrak{n}_w \times \mathfrak{n}'_w$ onto $N = N_w \cdot N'_w$. Let $y \in \mathfrak{S}$ and $z = y \cdot m^{-1}$. We may write

$$(1) \quad \begin{aligned} y &= k \cdot a \cdot n & (k \in K, a \in A, n \in N)\,, \\ a &= a_1 \cdot a_2, \quad n = n_1 \cdot n_2 & (a_1 \in A_w, a_2 \in A'_w, n_1 \in N_w, n_2 \in N'_w)\,. \end{aligned}$$

In the sequel, we say that an element, which is a function of y, *is bounded* if it stays within some compact set when y varies subject to our conditions. We prove first that a_1 is bounded. We have

$$z = k \cdot a \cdot n \cdot m^{-1} = k \cdot n^a \cdot a \cdot m^{-1} = k \cdot n^a \cdot m^{-1} \cdot m \cdot a \cdot m^{-1}\,,$$
$$z = k \cdot m^{-1} \cdot n^{ma} \cdot w(a)\,,$$

hence

$$(2) \qquad H(z) = H(n^{ma}) + \log w(a)\,.$$

By 4.2, n^a is bounded, hence so is n^{ma}. There exists therefore $t_1 \geqq t$ such that

$$(3) \qquad \alpha(\log w(a)) \leqq t_1 \qquad (\alpha \in \Sigma)\,.$$

Let now $\alpha \in \Sigma(w)$. Then ${}^t w(\alpha) < 0$, and, by (3)

$$(4) \qquad -\alpha(\log a) = -{}^t w(\alpha)(\log w(a)) \leqq t_1\,.$$

By definition of a Siegel domain, there exists t_0 such that $\alpha(\log a) \leqq t_0$, for $\alpha > 0$. For t'' big enough, we have therefore

$$(5) \qquad |\,\alpha(\log a)\,| \leqq t'' \qquad (\alpha \in \Sigma(w))\,.$$

But $\alpha(\log a) = \alpha(\log a_1)$ for $\alpha \in \Sigma(w)$, and the roots $\alpha \in \Sigma(w)$ span the dual of $\mathfrak{a}_w$, therefore (5) implies that $\log a_1$ varies in a compact set. Thus a_1 is bounded, as was contended. But then, $w(a_1)$ is also bounded, and (3) shows the existence of t_3 such that

$$(6) \qquad \alpha(\log w(a_2)) \leqq t_3 \qquad (\alpha \in \Sigma)\,.$$

Since a_2 commutes with n_1, we may write

$$(7) \qquad z = k \cdot m^{-1} \cdot (m \cdot a_1 \cdot n_1 \cdot m^{-1}) \cdot w(a_2) \cdot n'_2 \qquad (n'_2 = m \cdot n_2 \cdot m^{-1})\,.$$

The element n is bounded (by definition of a Siegel domain), hence so are n_1, n_2 and n'_2. The product $a_1 \cdot n_1$ belongs to $G(w)$ and ${}^t w(\Sigma(w)) = - \Sigma(w^{-1})$; hence

$$m \cdot a_1 \cdot n_1 \cdot m^{-1} \in m \cdot G(w) \cdot m^{-1} = G(w^{-1})\,,$$

$$(8) \quad \begin{aligned} & m \cdot a_1 \cdot n_1 \cdot m^{-1} = k_3 \cdot a_3 \cdot n_3\,, \\ & \qquad (k_3 \in K \cap G(w^{-1}), a_3 \in A \cap G(w^{-1}), n_3 \in N \cap G(w^{-1}))\,, \end{aligned}$$

where a_3 and n_3 are bounded, since the left hand side is. The element a_2

commutes with $G(w)$, hence $w(a_2)$ commutes with $G(w^{-1})$, and we have by (7)

$$z = k \cdot m^{-1} \cdot k_3 \cdot a_3 \cdot w(a_2) \cdot n_3 \cdot n'_2 ;$$

therefore,

$$H(z) = \log a_3 + \log w(a_2) .$$

We have already seen that a_3, n_3 and n'_2 are bounded. Taking (6), into account, this shows the existence of $s > 0$ and of a compact set ω' in N such that

$$z \in K \cdot A_s \cdot \omega' ,$$

which proves the lemma.

7.5. LEMMA. *Let G_1 be an algebraic semi-simple subgroup of G, and assume it satisfies the following condition:* (a) $\theta(G_1) = G_1$, $G_1 = K_1 \cdot A_1 \cdot N_1 (K_1 = K \cap G_1, A_1 = A \cap G_1, N_1 = N \cap G_1)$ *is an Iwasawa decomposition of G_1, and the restrictions to $\mathfrak{a}_1$ of the positive roots on $\mathfrak{a}$ are $\geqq 0$ for the ordering associated to $\mathfrak{n}_1$. Let $\mathfrak{S}$ be a Siegel domain of G, and $x \in G$. Then there exists a Siegel domain $\mathfrak{S}_1$ of G_1 (for the Iwasawa decomposition $K_1 \cdot A_1 \cdot N_1$) and a finite number of elements $x_1, \cdots, x_q \in G_1$ such that $\mathfrak{S}x \cap G_1 \subset \bigcup_i \mathfrak{S}_1 \cdot x_i$.*

In this proof, the Siegel domains of G_1 are defined with respect to the decomposition $K_1 \cdot A_1 \cdot N_1$. Let

$$\mathfrak{n}_0 = \textstyle\sum_{\alpha \in \Sigma; \alpha(\mathfrak{a}_1)=0} \mathfrak{g}_\alpha , \qquad \mathfrak{n}' = \textstyle\sum_{\alpha \in \Sigma; \alpha(\mathfrak{a}_1) \neq 0} \mathfrak{g}_\alpha .$$

Then, clearly, $\mathfrak{n}_0$ is a subalgebra and $\mathfrak{n}'$ an ideal of $\mathfrak{n}$. By 7.0, $(n, n') \to \exp n \cdot \exp n'$ is an analytic homeomorphism of $\mathfrak{n}_0 \times \mathfrak{n}'$ onto N. For any $\alpha \in \Sigma$ whose restriction to $\mathfrak{a}_1$ is not zero, let $\mathfrak{g}'_\alpha$ be a supplementary subspace to $\mathfrak{g}_1 \cap \mathfrak{g}_\alpha$ in $\mathfrak{g}_\alpha$, and let $\mathfrak{n}_2$ be the sum of all the $\mathfrak{g}'_\alpha$. Then $\mathfrak{n}' = \mathfrak{n}_1 + \mathfrak{n}_2$, where $\mathfrak{n}_1$ is a subalgebra, and it follows from 7.0 that $(n_2, n_1) \to \exp n_2 \cdot \exp n_1$ is an analytic homeomorphism of $\mathfrak{n}_2 + \mathfrak{n}_1$ onto N'. Thus $(n_0, n_2, n_1) \to \exp n_0 \cdot \exp n_2 \cdot \exp n_1$ is an analytic homeomorphism of $\mathfrak{n}_0 + \mathfrak{n}_2 + \mathfrak{n}_1$ onto $N = N_0 \cdot N_2 \cdot N_1$.

By 7.2, we have $x = a \cdot u \cdot m^{-1} \cdot v (a \in A; u, v \in N; m \in M^*)$. Since $\mathfrak{S} \cdot a \cdot u$ is contained in a Siegel domain, we may assume $x = m^{-1} \cdot v$. Let us write $v = v' \cdot v'' (v' \in N_0 \cdot N_2; v'' \in N_1)$. Then $\mathfrak{S}x \cap G_1 = (\mathfrak{S} \cdot m^{-1} \cdot v' \cap G_1) \cdot v''$. We may therefore assume $v = v' \in N_0 \cdot N_2$. In this case, 7.5 will follow from the more precise lemma:

7.6. LEMMA. *We keep the above notation and assume*

$$x = m^{-1} \cdot v \qquad (m \in M^*, v \in N_0 \cdot N_2) .$$

Let M_1^* be the normalizer and M_1 the centralizer of A_1 in K_1, and

$x_i (1 \leqq i \leqq \operatorname{ord} M_1^*/M_1)$ a set of representatives of the cosets of M_1^* modulo M_1. Then there exists a Siegel domain $\mathfrak{S}_1$ of G_1 such that $\mathfrak{S} \cdot x \cap G_1 \subset \bigcup_i \mathfrak{S}_1 \cdot x_i$.

Let $y = k \cdot a \cdot n (k \in K, a \in A, n \in N)$ be an element of $\mathfrak{S}$ such that $z = y \cdot x \in G_1$. Then

$$z = k \cdot a \cdot n \cdot m^{-1} \cdot v = k_1 \cdot a_1 \cdot n_1 \qquad (k_1 \in K_1, a_1 \in A_1, n_1 \in N_1) .$$

We first show that $a_1 \cdot n_1 \cdot a_1^{-1}$ is bounded. We have

$$\begin{aligned} z = k \cdot n^a \cdot a \cdot m^{-1} \cdot v &= k \cdot m^{-1} \cdot m \cdot n^a \cdot m^{-1} \cdot m \cdot a \cdot m^{-1} v \\ &= k \cdot m^{-1} \cdot n^{ma} \cdot a^m \cdot v , \\ z \in K \cdot n^{ma} \cdot a^m \cdot v = K \cdot e^{H(u)} \cdot \nu(u) \cdot a^m \cdot v & \qquad\qquad (u = n^{ma}) , \end{aligned}$$

where $\nu(u)$ is the component in N of u (see 7.1), whence

$$a_1 = e^{H(u)} \cdot a^m , \qquad n_1 \cdot v^{-1} = (a^m)^{-1} \cdot \nu(u) \cdot a^m . \tag{1}$$

By 4.2, n^a is bounded, hence so are $u = n^{ma}$, $\nu(u)$, and therefore by (1), also $a_1 \cdot (a^m)^{-1}$, and

$$a_1 \cdot n_1 \cdot v^{-1} \cdot a_1^{-1} = (a_1 \cdot n_1 \cdot a_1^{-1}) \cdot (a_1 \cdot v^{-1} \cdot a_1^{-1}) . \tag{2}$$

It is clear from the definitions that a_1 normalizes N_1 and $N_0 \cdot N_2$, therefore the two factors on the right hand side belong to N_1 and $N_0 \cdot N_2$ respectively. Since the map $(n', n'') \to n' \cdot n''$ is a homeomorphism of $N_1 \times N_0 \cdot N_2$ onto N, we see that both factors on the right hand side of (2) are bounded.

Let, for $c > 0$,

$$V_{i,c} = \{z \in \mathfrak{S}x \cap G_1 \mid z \cdot x_i^{-1} \in K \cdot A_c \cdot N\} .$$

We want to prove the existence of $t > 0$ such that

$$\mathfrak{S}x \cap G_1 \subset \bigcup_i V_{i,t} . \tag{3}$$

The restricted Weyl group W_1 of G_1 is transitive on the Weyl chambers, therefore, given z, there exists an i such that $x_i \cdot a_1 \cdot x_i^{-1} \in A_1^-$, where $A_1^- = A_{1,0}$ denotes the exponential of the negative Weyl chamber. We have

$$z \cdot x_i^{-1} = k_1 \cdot a_1 \cdot n_1 \cdot x_i^{-1} = k_1 \cdot n_1^{a_1} \cdot a_1 \cdot x_i^{-1} \in K_1 \cdot n_1^{x_i \cdot a_1} \cdot a_1^{x_i} ,$$

$$z \cdot x_i^{-1} \in K_1 \cdot e^{H(u)} \cdot a_1^{x_i} \cdot (a_1^{x_i^{-1}} \cdot \nu(u) \cdot a_1^{x_i}) , \qquad\qquad (u = n_1^{x_i \cdot a_1}) ,$$

which implies

$$H(z \cdot x_i^{-1}) = H(u) + H(x_i \cdot a_1 \cdot x_i^{-1}) .$$

But $x_i \cdot a_i \cdot x_i^{-1} \in A_1^-$ by assumption, hence, taking 7.5(a) into account, $\alpha(x_i \cdot a_1 \cdot x_i^{-1}) \leqq 0$ for $\alpha \in \Sigma$. Moreover, $n_1^{a_1}$ is bounded, as was proved above, hence u is bounded. There exists, therefore, $t_i > 0$ such that $H(z \cdot x_i^{-1}) \subset \log A_{1,t_i}$ for all $z \in \mathfrak{S}x \cap G_1$ for which

$$x_i \cdot a_1 \cdot x_i^{-1} = x_i \cdot e^{H(z)} \cdot x_i^{-1} \in A_1^- .$$

This proves (3). In view of the condition (a) of 7.5, $A_{1,t} \subset A_t$, and it follows from 7.4 that we may find a Siegel domain $\mathfrak{S}'_i$ of G such that $z \cdot x_i^{-1} \in \mathfrak{S}'_i$ when $z \in V_{i,t}$; we have then $V_{i,t} \subset \mathfrak{S}'_i \cdot x_i$ or equivalently

(4) $$V_{i,t} \subset (\mathfrak{S}'_i \cap G_1) \cdot x_i .$$

But in view of 7.5(a), $\mathfrak{S}'_i \cap G_1$ is a Siegel domain of G_1. Since a finite union of Siegel domains (with respect to a fixed Iwasawa decomposition) is contained in a Siegel domain, the lemma follows from (3), (4).

7.7. LEMMA. *Let G' be a real algebraic semi-simple subgroup of G. Then there exists $a \in G$ such that $G_1 = a \cdot G' \cdot a^{-1}$ verifies condition* (a) *of* 7.5.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated to θ. By 1.1 and 1.8, there exists $b \in G$ such that $G_2 = b \cdot G' \cdot b^{-1}$ is stable under θ. Then $\mathfrak{g}_2 = (\mathfrak{k} \cap \mathfrak{g}_2) + (\mathfrak{p} \cap \mathfrak{g}_2)$ is a Cartan decomposition of the Lie algebra $\mathfrak{g}_2$ of G_2. Let $\mathfrak{a}_2$ be a maximal subalgebra of $\mathfrak{p} \cap \mathfrak{g}_2$ and $\mathfrak{a}'$ a maximal subalgebra of $\mathfrak{p}$ containing $\mathfrak{a}_2$. By E. Cartan's conjugacy theorem (see e.g. [12, Lemma 33]), there exists $k \in K$ such that $k \cdot \mathfrak{a}' \cdot k^{-1} = \mathfrak{a}$. The group $G_3 = k \cdot G_2 \cdot k^{-1}$ is then still invariant under θ, and $\mathfrak{g}_3 \cap \mathfrak{a} = \mathfrak{a}_3$ is a maximal subalgebra of $\mathfrak{p}_3 = \mathfrak{g}_3 \cap \mathfrak{p}$. Let us now choose orderings on $\mathfrak{a}^*$ and $\mathfrak{a}_3^*$ such that the restriction to $\mathfrak{a}_3$ of a positive element of $\mathfrak{a}^*$ is $\geqq 0$, for instance take the lexicographic orderings with respect to a basis whose first elements span $\mathfrak{a}_3$. Let then $m \in M^*$ be such that $\operatorname{Ad} m$ transforms the positive Weyl chamber of $\mathfrak{a}$ for this ordering into the positive Weyl chamber for the ordering defined by N. Then $a = m \cdot k \cdot b$ fulfills our conditions.

7.8. THEOREM. *Let G be a semi-simple algebraic Lie group, defined over* **Q**. *Then $G_{\mathbf{R}}/G_{\mathbf{Z}}$ has finite Haar measure.*

We may assume G to be contained in $\mathbf{SL}(n, \mathbf{C})$ (see 2.1), and also to be connected, since $(G^0)_{\mathbf{R}}$ has finite index in $G_{\mathbf{R}}$. We now apply 7.7, with G' and G replaced respectively by $G_{\mathbf{R}}$ and $\mathbf{SL}(n, \mathbf{R})$, θ and $K \cdot A \cdot N$ by the standard Cartan involution and Iwasawa decomposition of $\mathbf{SL}(n, \mathbf{R})$, and take $a \in \mathbf{SL}(n, \mathbf{R})$ such that $G' = a \cdot G \cdot a^{-1}$ is self-adjoint and satisfies condition (a) of 7.5. Let $\mathfrak{S}$ be a standard Siegel domain of $\mathbf{SL}(n, \mathbf{R})$. By 6.5, there exist finitely many elements $b_i \in \mathbf{SL}(n, \mathbf{Z})$ such that $G_{\mathbf{R}} = \bar{U} \cdot G_{\mathbf{Z}}$ with

$$\bar{U} = \bigcup_i (a^{-1} \cdot \mathfrak{S} \cdot b_i) \cap G_{\mathbf{R}} .$$

We want to prove that $\bar{U}$ has finite Haar measure on $G_{\mathbf{R}}$. It is enough to show that $(a^{-1} \cdot \mathfrak{S} \cdot b_i) \cap G_{\mathbf{R}}$ has finite measure, hence also that $\mathfrak{S} \cdot b_i \cdot a^{-1} \cap G'_{\mathbf{R}}$ has finite Haar measure on $G'_{\mathbf{R}}$. By 7.5 (with G and G_1 replaced by

$\mathbf{SL}(n, \mathbf{R})$ and $G'_{\mathbf{R}}$), $\mathfrak{S} \cdot b_i \cdot a \cap G'_{\mathbf{R}}$ is contained in a finite union of translates of a Siegel domain of $G'_{\mathbf{R}}$. Since a Siegel domain of a semi-simple Lie group has finite volume (4.3), our contention is proved.

8. Remarks on characters of algebraic groups

8.1. Let G be an algebraic group. The group of *rational characters* of G, that is of rational homomorphisms of G into $\mathbf{C}^*$, is denoted by $X(G)$. It is a finitely generated commutative group, free when G is connected [23; 26]. If K is a field of definition for G, then $X_K(G)$ will be the group of rational characters defined over K.

A rational homomorphism $f: G \to G'$ obviously induces a homomorphism $f^0: X(G') \to X(G)$, which maps $X_K(G')$ into $X_K(G)$ if K is a field of definition for G, G' and f.

Let G be reductive, connected, defined over K, $S = Z(G)^0$ the identity component of its center, and G' the derived group of G. Then the restriction map $X(G) \to X(S)$ is injective, and identifies $X(G)$ and $X_K(G)$ with subgroups of finite index of $X(S)$ and $X_K(S)$ respectively. In fact, we have $X(G') = 1$, since G' is semi-simple, hence the restriction is injective. Conversely, given χ in $X(S)$ or in $X_K(S)$, χ^m ($m =$ order of $G' \cap S$) is trivial on $S \cap G'$, and therefore extends to a character of G.

8.2. Let now $G = T$ be an algebraic torus, K a field of definition for T. Let $\Gamma(T)$ be the group of rational homomorphisms of $\mathbf{C}^*$ into T, and $\Gamma_K(T)$ the group of rational homomorphisms of $\mathbf{C}^*$ into T which are defined over K. Both $X(T)$ and $\Gamma(T)$ are free, of rank $n = \dim T$. Given $a \in \Gamma(T)$, $b \in X(T)$, the homomorphism $b \circ a: \mathbf{C}^* \to \mathbf{C}^*$ has the form $x \to x^m (m \in \mathbf{Z})$. The map $(a, b) \to \langle a, b \rangle = m$ is an integral valued non-degenerate bilinear form on $\Gamma(T) \times X(T)$, which puts these two groups in duality [8; Exp. 9, No. 5]. A rational homomorphism $f: T \to T'$ induces a homomorphism $f_0: \Gamma(T) \to \Gamma(T')$, and we have $\langle f_0 a, b \rangle = \langle a, f^0 b \rangle$ $(a \in \Gamma(T), b \in X(T'))$.

8.3. An algebraic torus T is said to *split over* K if it is defined over K and isomorphic over K to a product of groups $\mathbf{C}^*$. If $T \subset \mathbf{GL}(n, \mathbf{C})$, this is equivalent to the existence of $x \in \mathbf{GL}(n, K)$ such that $x \cdot T \cdot x^{-1}$ is diagonal [26, Prop. 5]. If T splits over K, then $X_K(T) = X(T)$, $\Gamma_K(T) = \Gamma(T)$, the subtori and the homomorphic images (over K) of T split over K. Therefore, if $a \in \Gamma_K(T)$, $a \neq 0$, then $a(\mathbf{C}^*)$ is a one-dimensional subtorus of T which splits over K, and if $\Gamma_K(T) = \Gamma(T)$, then T splits over K. That the two latter conditions are also equivalent to $X_K(T) = X(T)$ follows from the following lemma:

8.4. LEMMA. *Let T be an algebraic torus, K a field of definition for T. Then*

(a) *Given any subtorus S of T, defined over K, there exists a subtorus S', defined over K, such that $S \cdot S' = T$, and $S \cap S'$ is finite.*

(b) *$X_K(T)$ and $\Gamma_K(T)$ have equal ranks. In particular, $X_K(T) \neq \{0\}$ if and only if T contains a subtorus $S \neq (e)$ which splits over K.*

The torus T certainly splits over $\bar{K}$ [1, Chap. II], therefore there exists a finite Galois extension K' of K over which T splits. Let A be the Galois group of K' over K. It operates in a natural fashion on $\Gamma(T)$, $X(T)$, and we have

$$\langle \sigma(a), \sigma(b) \rangle = \langle a, b \rangle \qquad (a \in \Gamma(T), b \in X(T), \sigma \in A) .$$

The corresponding linear representations ρ, ρ' of A in $\Gamma(T) \otimes \mathbf{Q}$ and $X(T) \otimes \mathbf{Q}$ are therefore contragredient to each other. The fixed points of A in $\Gamma(T)$ and $X(T)$ are $\Gamma_K(T)$ and $X_K(T)$ respectively. Consequently, the ranks of $\Gamma_K(T)$ and $X_K(T)$ are equal to the dimension of the fixed point sets of ρ, ρ'. Since ρ and ρ' are contragredient to each other, these dimensions are equal, whence the first part of (b). The second one follows then from 8.3. Let now S be a subtorus of T, defined over K. Then $\Gamma(S)$ may be identified with a submodule of $\Gamma(T)$, which is invariant under A. Since A is finite, there is a subspace V of $\Gamma(T) \otimes \mathbf{Q}$, supplementary to $\Gamma(S) \otimes \mathbf{Q}$, and invariant under A. The images $\gamma(\mathbf{C}^*)$, $(\gamma \in \Gamma(T) \cap V)$, are subtori defined over K' which are permuted by A. They generate a subtorus S' of T, invariant under A, hence defined over K, and such that $\Gamma(S') = V \cap \Gamma(T)$. We have then $\Gamma(S) \cap \Gamma(S') = (0)$, and $\Gamma(S) + \Gamma(S')$ has finite index in $\Gamma(T)$, whence (a).

9. The finiteness of the volume

9.1. Let G be a Lie group, H a discrete subgroup. Then a Haar measure on G induces on G/H a measure μ, such that $g(\mu) = \chi(g) \cdot \mu$, where $\chi(g) = |\det \operatorname{Ad} g|$. If $\mu(G/H) < \infty$, then $\chi(g) = 1 (g \in G)$, μ is invariant and G is unimodular (any left invariant Haar measure is right invariant).

9.2. LEMMA. *Let G^*, G be connected algebraic groups defined over $\mathbf{Q}$, $\pi: G^* \to G$ an isogeny over $\mathbf{Q}$. Then $G^*_{\mathbf{R}}$ is unimodular and $G^*_{\mathbf{R}}/G^*_{\mathbf{Z}}$ has finite invariant measure, if and only if $G_{\mathbf{R}}$ is unimodular, and $G_{\mathbf{R}}/G_{\mathbf{Z}}$ has finite invariant measure.*

It is clear that $G_{\mathbf{R}}$ is unimodular if and only if $G^*_{\mathbf{R}}$ is so. Let N be the kernel of π. By 6.11, $G^*_{\mathbf{Z}}$ has a subgroup M of finite index, whose image $\pi(M)$ is a subgroup of finite index of $G_{\mathbf{Z}}$. We have $G^*_{\mathbf{R}}/M \cdot N \cong G_{\mathbf{R}}/\pi(M)$, and

$G^*_{\mathrm{R}}/G^*_{\mathrm{Z}}$ (resp. $G_{\mathrm{R}}/G_{\mathrm{Z}}$) has finite measure if and only if $G^*_{\mathrm{R}}/M\cdot N$ (resp. $G_{\mathrm{R}}/\pi(M)$) has, whence the lemma.

9.3. Let T be an algebraic torus, defined over $\mathbf{Q}$. We shall use here and in § 10 the fact that if $X_{\mathrm{Q}}(T)=1$, then $T_{\mathrm{R}}/T_{\mathrm{Z}}$ is compact, proved by Ono [22]. (It is not stated explicitly there, but is an immediate consequence of the compactness of the quotient $J(T)/T_{\mathrm{Q}}$ of the idele group of T by the principal ideles.)

On the other hand, if T is diagonal, then T_{Z} is obviously finite. By 8.3, it follows that T_{Z} is finite whenever T splits over $\mathbf{Q}$. In that case, $\mu(T_{\mathrm{R}}/T_{\mathrm{Z}})$ is of course infinite ($T\neq(e)$).

9.4. THEOREM. *Let G be an algebraic group, defined over* $\mathbf{Q}$. *Then G_{R} is unimodular, and $G_{\mathrm{R}}/G_{\mathrm{Z}}$ of finite invariant measure if and only if $X_{\mathrm{Q}}(G^0)=1$.*

It is clearly enough to prove this when G is connected. Assume first that $G=T$ is a torus. If $X_{\mathrm{Q}}(G)=1$, then $T_{\mathrm{R}}/T_{\mathrm{Z}}$ is compact by Ono's result (9.3). If $X_{\mathrm{Q}}(G)\neq 1$, then, by 8.4, we have $T=S\cdot S'$, where S, S' are subtori of strictly positive dimension, and S splits over $\mathbf{Q}$. We have a natural isogeny $S\times S'\to T$, and it follows from 9.2, 9.3 that $\mu(T_{\mathrm{R}}/T_{\mathrm{Z}})$ is infinite.

Let now G be reductive, $G=S\cdot G'$ its standard decomposition, where $S=Z(G)^0$, and G' is the derived group of G. Therefore G is the quotient of $S\times G'$ by a finite group. By 7.8, $G'_{\mathrm{R}}/G'_{\mathrm{Z}}$ has finite invariant measure. By 8.1, $X_{\mathrm{Q}}(G)$ and $X_{\mathrm{Q}}(S)$ have equal ranks. Our assertion in this case follows therefore from the above and 9.2.

In the general case, $G=H\cdot N$ is the semi-direct product of a reductive group H and of an invariant unipotent group N, both defined over $\mathbf{Q}$, and $N_{\mathrm{R}}/N_{\mathrm{Z}}$ is compact (6.10). Of course, $X(N)=1$, hence $X(G)=X(H)$, $X_{\mathrm{Q}}(G)=X_{\mathrm{Q}}(H)$.

Let $X_{\mathrm{Q}}(G)=1$. Then $X_{\mathrm{Q}}(H)=1$, and $H_{\mathrm{R}}/H_{\mathrm{Z}}$ has finite invariant measure by the above. Moreover, the determinant of $\mathrm{Ad}\,h\,|\,\mathfrak{n}\,(h\in H)$ is one, since $h\to\det(\mathrm{Ad}\,h\,|\,\mathfrak{n})$ is an element of $X_{\mathrm{Q}}(H)$. Therefore, G_{R} is unimodular, and a Haar measure on G_{R} is the product of Haar measures on H_{R} and N_{R}. We have $H_{\mathrm{R}}=A\cdot H_{\mathrm{Z}}$, $N_{\mathrm{R}}=B\cdot N_{\mathrm{Z}}$ with A, B open, of finite Haar measure, whence $G_{\mathrm{R}}=A\cdot N\cdot H_{\mathrm{Z}}=A\cdot B\cdot G_{\mathrm{Z}}$ with $A\cdot B$ of finite measure.

Assume now G_{R} to be unimodular, and $G_{\mathrm{R}}/G_{\mathrm{Z}}$ to have finite invariant measure. Since, G_{R}, H_{R}, N_{R} are unimodular, we must have $\det(\mathrm{Ad}\,h\,|\,\mathfrak{n})=1\,(h\in H)$, and the Haar measure on G_{R} is the product of Haar measures on H_{R} and N_{R}. The subgroup $H_{\mathrm{Z}}\cdot N_{\mathrm{Z}}$ has finite index in G_{Z} (6.4), and $G_{\mathrm{R}}/H_{\mathrm{Z}}\cdot N_{\mathrm{Z}}$ has finite measure. The projection $G_{\mathrm{R}}/N_{\mathrm{Z}}\to G_{\mathrm{R}}/N_{\mathrm{R}}\cong H_{\mathrm{R}}$ is a fiber map with compact fibre $N_{\mathrm{R}}/N_{\mathrm{Z}}$, which commutes with the action of

H_Z defined by right translations. Therefore H_R/H_Z must also have finite invariant measure; by the above, this implies $X_Q(H) = 1$, hence $X_Q(G) = 1$.

19. Closed conjugacy classes

10.1. PROPOSITION. *Let G be a real or complex algebraic group, $x \in G$, $y \in \mathfrak{g}$, and $C(x) = \{g \cdot x \cdot g^{-1}, g \in G\}$, $C(y) = \mathrm{Ad}\, G(y)$ the conjugacy classes of x and y. Then*

(a) *If x (resp. $\mathrm{ad}\, y$) is semi-simple, $C(x)$ (resp. $C(y)$) is closed.*

(b) *If G is reductive, and x (resp. $\mathrm{ad}\, y$) is not semi-simple, $G(x)$ (resp. $C(y)$) is not closed.*

PROOF OF (a) Going over to the complexification, if necessary, it is enough, by 2.3, to consider the case where G is complex algebraic. For an endomorphism A of a vector space, we denote by $C(A, \lambda)$ its characteristic polynomial $\det(A - \lambda \cdot \mathrm{Id.})$ and by $M_A(\lambda)$ its minimal polynomial, where λ is an indeterminate. Let

$$P_y = \{z \in \mathfrak{g}, C(\mathrm{ad}\, z, \lambda) = C(\mathrm{ad}\, y, \lambda), M_{\mathrm{ad} y}(\mathrm{ad}\, z) = 0\} .$$

This is an algebraic subset of $\mathfrak{g}$, clearly invariant under G. The minimal polynomial of $\mathrm{ad}\, z (z \in P_y)$ divides the minimal polynomial of $\mathrm{ad}\, y$, hence has only simple factors, and $\mathrm{ad}\, z$ is also semi-simple. In particular, the dimension of the centralizer $Z(z)$ of z in G is equal to the multiplicity of the eigen-value zero of $\mathrm{ad}\, z$. But $\mathrm{ad}\, z$ has the same eigenvalues as $\mathrm{ad}\, y$, hence $\dim Z(z) = \dim Z(y)$. The orbit $\mathrm{Ad}\, G(z)$ of z, whose dimension is equal to $\dim G - \dim Z(z)$, has therefore the same dimension as $\mathrm{Ad}\, G(y)$, and P_y is a disjoint union of orbits of the same dimension. Since the boundary of an orbit is a union of orbits of strictly smaller dimension [1, § 15], it follows that $\mathrm{Ad}\, G(z)$ is closed for every $z \in P_y$.

The proof for x is entirely analogous. We introduce the set

$$P_x = \{z \in G, C(\mathrm{Ad}\, z, \lambda) = C(\mathrm{Ad}\, x, \lambda), M_{\mathrm{Ad} x}(\mathrm{Ad}\, z) = 0\} .$$

This is an algebraic set, invariant under G. The dimension of $Z(z)$ will be equal to the multiplicity of the eigenvalue one of $\mathrm{Ad}\, z$, hence equal to $\dim Z(x)$, and P_x consists again of orbits of the same dimension.

PROOF OF (b) Let first $G = \mathbf{SL}(2, \mathbf{R})$ or $\mathbf{SL}(2, \mathbf{C})$ and $x \neq 1$ be unipotent (resp. $y \neq 0$ be nilpotent). After a suitable inner automorphism, we may assume

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \left(\text{resp. } y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) .$$

Let $g_t = \mathrm{diag}\,(t, t^{-1})$. Then

$$g_t \cdot x \cdot g_t^{-1} = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix} \qquad \left(\text{resp. } \operatorname{Ad} g_t(y) = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}\right),$$

hence the identity (resp. the origin) belongs to $\bar{C}(x)$ (resp. $\bar{C}(y)$).

Let now G be reductive, and $\operatorname{ad} y$ be not semi-simple. We may write $y = z + y'$, with z central, $y' \in \mathcal{D}\mathfrak{g}$, hence $\operatorname{ad} y' = \operatorname{ad} y$ not semi-simple. It is clearly enough to show that the conjugacy class of y' in the semi-simple part of $\mathfrak{g}$ is not closed; we may therefore assume G to be semi-simple. We have then $y = s + n$, with $[s, n] = 0$, $\operatorname{ad} s$ semi-simple, $\operatorname{ad} n$ nilpotent and not zero [5, § 6, No. 3]. The centralizer $\mathfrak{z}(s)$ of s in $\mathfrak{g}$ is reductive, its center consists of semi-simple elements [4, Prop. 4.1], $\mathcal{D}\mathfrak{z}(s)$ is semi-simple, and contains n. By the Jacobson-Morosow theorem [15], there exists a three-dimensional subalgebra $\mathfrak{m}$, isomorphic to $\mathfrak{sl}(2, \mathbf{R})$, or $\mathfrak{sl}(2, \mathbf{C})$, containing n. The analytic subgroup M generated by $\mathfrak{m}$ in $\operatorname{Ad} \mathfrak{g}$ is a homomorphic, locally isomorphic, image of $\mathbf{SL}(2, \mathbf{R})$ or $\mathbf{SL}(2, \mathbf{C})$. By the above, the conjugacy class of n in $\mathfrak{m}$ has zero in its closure; since M centralizes s, it follows that the semi-simple element s belongs to the closure of $\bar{C}(y)$.

For x, the proof is similar. We write $x = z \cdot x'$, with z central, semi-simple, and x' in the semi-simple part of G, and not semi-simple. It is enough to show that $C(x')$ is not closed, hence we may assume G to be semi-simple. We have $x = x_s \cdot x_u$ with $x_s \cdot x_u = x_u \cdot x_s$, x_s semi-simple, $x_u \neq e$ unipotent [1, § 8]. Applying the Jacobson-Morosow theorem to $\log x_u$, in the derived algebra of $\mathfrak{z}(x_s)$, we see that x_u belongs to a group M, which is a homomorphic, locally isomorphic image of $\mathbf{SL}(2, \mathbf{R})$ or $\mathbf{SL}(2, \mathbf{C})$, and centralizes x_s. By the above, the conjugacy class of x_u in M contains e, hence $\bar{C}(x)$ contains x_s.

10.2. REMARK. The proof of (b) shows more precisely that the semi-simple part of x (resp. y) belongs to $\bar{C}(x)$ (resp. $\bar{C}(y)$), and, in the real case, that it belongs to the closure of the conjugacy class of x (resp. y) with respect to the identity component G^0 of G, hence with respect to any open subgroup. Now, if G is a real (resp. complex) semi-simple Lie group, then $\operatorname{Ad} \mathfrak{g}$ is of finite index in a real algebraic group (resp. is an algebraic group). If G is moreover linear, it is of finite index in a real (resp. complex) algebraic group. Consequently, 10.1 yields the following:

10.3. COROLLARY. *Let G be a real or complex semi-simple Lie group, and $y \in \mathfrak{g}$. Then $\operatorname{Ad} G(y)$ is closed if and only if $\operatorname{ad} y$ is semi-simple. If G is linear, the conjugacy class in G of an element $x \in G$ is closed if and only if x is semi-simple.*

11. Groups of units with compact fundamental sets

11.1. LEMMA. *Let G be a locally compact separable group, M a locally compact separable space on which G operates continuously on the right, $m \in M$, and H a closed subgroup. If $m \cdot H$ is closed, and $H\backslash G$ is compact, then $m \cdot G$ is closed.*

It follows from our assumption that $G = H \cdot K$ with K compact. If $m \cdot h_j \cdot k_j \to p(h_j \in H, k_j \in K)$, then, assuming, as we may, $k_j \to k \in K$, we have $m \cdot h_j \to p \cdot k^{-1}$. Since $m \cdot H$ is closed, there exists $h \in H$ such that $p \cdot k^{-1} = m \cdot h$, whence $p = m \cdot h \cdot k \in m \cdot G$.

11.2. Let G act on itself by inner automorphisms. If H is discrete, the conjugacy class in H of $h \in H$ is of course a closed subset of G; therefore, if $H\backslash G$ is compact, then the conjugacy class in G of any element of H is closed. Lemma 11.1 was suggested by this remark, which is due to Selberg. Together with 10.3, it shows that if G is a linear semi-simple Lie group, H a discrete subgroup such that $H\backslash G$ is compact, then any element of H is semi-simple.

11.3. PROPOSITION. *Let G be a connected algebraic group defined over $\mathbf{Q}$, and $\pi: G \to \mathbf{GL}(V)$ a rational representation of G defined over $\mathbf{Q}$. If $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact, then the orbit under $G_{\mathbf{R}}$ of an element $v \in V_{\mathbf{Q}}$ is closed.*

By 6.3, there exists a lattice $\Gamma \subset V_{\mathbf{Q}}$ which contains v and is invariant under $G_{\mathbf{Z}}$. Therefore, $v \cdot \pi(G_{\mathbf{Z}})$ is a discrete set, and 11.3 follows from 11.1.

11.4. LEMMA. *Let G be a connected reductive algebraic group, k a field of definition for G. Then the following conditions are equivalent:*

(a) $X_k(G) = 1$, *and $\mathfrak{g}_k$ consists of semi-simple elements;*

(b) $X_k(G) = 1$, *and G_k consists of semi-simple elements:*

(c) $X_k(S) = 1$ *for every algebraic subtorus S of G which is defined over k.*

If k is a number field, and J its ring of integers, these conditions are equivalent to:

(d) $X_k(G) = 1$, *and G_J consists of semi-simple elements.*

(a) $\Rightarrow$ (b). The group G, being algebraic, contains the unipotent and semi-simple parts of its elements [1, Chap. II], therefore, if G_k contains a non-semi-simple element, it also contains a unipotent element $g \neq e$. But then $\log g$ is a non-zero nilpotent element of $\mathfrak{g}_k$.

(b) $\Rightarrow$ (c). Assume (c) to be false. By 8.4, there exists then a one-dimensional subtorus S of G which splits over k. It is not central, since otherwise 8.4 and 8.1 would imply $X_k(G) \neq 1$. The image of S in $\operatorname{Aut} \mathfrak{g}$

then also splits, which means that it can be diagonalized over k. The weights of the Lie algebra $\mathfrak{s}_k$ of S_k in $\mathfrak{g}_k$ are then elements of $(\mathfrak{s}_k)^*$, and $\mathfrak{g}_k$ is a direct sum of subspaces $\mathfrak{g}_{k,\alpha}(\alpha \in (\mathfrak{s}_k{}^*))$, where as usual, $\mathfrak{g}_{k,\alpha} = \{x \in \mathfrak{g}_k \mid [s, x] = \alpha(s)x,\ s \in \mathfrak{s}_k\}$. Since S is not central, $\mathfrak{g}_{k,\alpha} \neq 0$ for at least one $\alpha \neq 0$. But then it is well known that an element $x \in \mathfrak{g}_{k,\alpha}$ is nilpotent. If $x \neq 0$, then e^x is a unipotent element in G_k, different from the identity, which contradicts (b).

(c) $\Rightarrow$ (a). By 8.1, (c) implies that $X_k(G) = 1$. Assume now that $\mathfrak{g}_k$ does not consist of semi-simple elements. Being algebraic, it contains then a nilpotent element $x \neq 0$ [7a, p. 165], which necessarily belongs to $\mathscr{D}\mathfrak{g}$. Applying the Jacobson-Morosow theorem to $\mathscr{D}\mathfrak{g}_k$, we get a three dimensional subalgebra $\mathfrak{m} \subset \mathscr{D}\mathfrak{g}_k$, containing x, isomorphic over k to the Lie algebra of $\mathbf{SL}(2, k)$. There exists therefore in G an algebraic subgroup M, with Lie algebra $\mathfrak{m} \otimes \mathbf{C}$, defined over k, and which is the image of $\mathbf{SL}(2, \mathbf{C})$ under a rational homomorphism with finite kernel, defined over k. The image of the group of diagonal matrices in $\mathbf{SL}(2, \mathbf{C})$ is then a one dimensional subtorus of G, which splits over k. Let now k be a number field, J the ring of integers of k. Clearly, $(a) \Rightarrow$ (d). Assume now (a) to be false. As remarked at the beginning of the proof, there exists then $x \in \mathfrak{g}_k$ which is nilpotent and not zero. Let s be a positive rational integer such that $x^s = 0$, t an element of J such that $t \cdot x$ has integral coefficients, and $m = s! \cdot t$. Then e^{mx} is a unipotent element, different from the identity contained in G_J, in contradiction with (d).

11.5. COROLLARY. *Let G be a connected reductive algebraic group. If G satisfies the conditions of* 11.4, *then every connected subgroup of G which is defined over k is reductive, and satisfies those conditions.*

Let M be a connected algebraic subgroup of G, defined over k. As recalled in 6.10, $M = H \cdot N$ is the semi-direct product of a reductive group H and of a unipotent invariant subgroup N, both defined over k. In our case $N_k = (e)$ by the condition (b), hence $N = (e)$, and $M = H$ is reductive. Then it clearly satisfies (c).

11.6. THEOREM. *Let G be a reductive algebraic group, defined over* $\mathbf{Q}$. *Then $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact if and only if $X_{\mathbf{Q}}(G^0) = 1$ and $G_{\mathbf{Q}}$ consists of semi-simple elements.*

The group G^0 has finite index in G, therefore $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact if and only if $(G^0)_{\mathbf{R}}/(G^0)_{\mathbf{Z}}$ is. In the sequel, we assume G to be connected. Our assertion is then that the conditions (a) to (d) of 11.4 are equivalent to the compactness of $G_{\mathbf{R}}/G_{\mathbf{Z}}$.

Let first $G_{\mathbf{R}}/G_{\mathbf{Z}}$ be compact. If $X_{\mathbf{Q}}(G) \neq 1$, there exists a non-trivial one dimensional rational representation $\pi: G \to \mathbf{C}^*$, defined over $\mathbf{Q}$. But then $\pi(G) = \mathbf{C}^*$, and $\pi(G_{\mathbf{R}})$ contains the multiplicative group $\mathbf{R}^+$ of strictly positive real numbers. The orbit under $G_{\mathbf{R}}$ of any element in $\mathbf{Q}^*$ (in fact in $\mathbf{C}^*$) has the origin in its closure, and is not closed, in contradiction with 11.3. Thus $X_{\mathbf{Q}}(G) = 1$. The condition on $G_{\mathbf{Q}}$ follows from 10.1 and 11.3.

From now on, G is assumed to verify the conditions of 11.4, and we prove the compactness of $G_{\mathbf{R}}/G_{\mathbf{Z}}$ by induction on dim G. There is nothing to prove in dimension zero, therefore we may assume 11.6 to be true for all groups (connected or not) of dimension strictly smaller than dim G. In particular, in view of 11.5, $H_{\mathbf{R}}/H_{\mathbf{Z}}$ is compact for every proper algebraic subgroup H of G which is defined over $\mathbf{Q}$.

We have $G = S \cdot G'$ with $S = Z(G)^0$, G' semi-simple invariant, S and G' defined over $\mathbf{Q}$. If $0 < \dim S < \dim G$, then, by the above $S_{\mathbf{R}} = A \cdot S_{\mathbf{Z}}$, $G'_{\mathbf{R}} = B \cdot G'_{\mathbf{Z}}$ with A and B compact, hence $S_{\mathbf{R}} \cdot G'_{\mathbf{R}} = A \cdot B \cdot S_{\mathbf{Z}} \cdot G'_{\mathbf{Z}}$. Since $S_{\mathbf{R}} \cdot G'_{\mathbf{R}}$ has finite index in $G_{\mathbf{R}}$, and $S_{\mathbf{Z}} \cdot G'_{\mathbf{Z}} \subset G_{\mathbf{Z}}$, it follows that $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact. If $G = S$, see 9.3.

Let now $G = G'$ be semi-simple. Every element of $\mathfrak{g}_{\mathbf{Q}}$ is semi-simple (11.4), and therefore its conjugacy class under G is closed (10.1). The adjoint representation is consequently a locally faithful rational representation of G, defined over $\mathbf{Q}$, in which all rational points have closed orbits. Since, as pointed out above, $H_{\mathbf{R}}/H_{\mathbf{Z}}$ is compact for every proper algebraic subgroup of G which is defined over $\mathbf{Q}$, it will be enough, in order to conclude the proof of 11.6, to prove the following lemma.

11.7. LEMMA. *Let G be a connected semi-simple algebraic group defined over* $\mathbf{Q}$. *Assume that $H_{\mathbf{R}}/H_{\mathbf{Z}}$ is compact for every proper algebraic subgroup of G defined over* $\mathbf{Q}$, *and that there exists a locally faithful rational representation $\pi: G \to \mathbf{GL}(V)$ defined over* $\mathbf{Q}$ *in which all points of $V_{\mathbf{Q}}$ have closed orbits. Then $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact.*

The representation π being fully reducible, we may take out the trivial representations, and assume

$$G_x \neq G \qquad (x \in V, x \neq 0)\,. \tag{1}$$

We fix a lattice Γ in $V_{\mathbf{Q}}$ which is invariant under $G_{\mathbf{Z}}$ (see 6.3), and take coordinates in V with respect to a basis of Γ. Let P be the set of polynomials on V which are invariant under G. Since π is defined over $\mathbf{Q}$, we have $P = P' \otimes \mathbf{C}$, where P' is the set of invariant polynomials with rational coefficients. By the theorem of invariants, applied to $\pi: \mathfrak{g}_{\mathbf{Q}} \to \mathfrak{gl}(V_{\mathbf{Q}})$,, [5, § 6, No. 9], P' is a finitely generated algebra over $\mathbf{Q}$. It is therefore generated by 1, and by finitely many homogeneous polynomials $P_1, \cdots, P_s$, which we may assume to have integral coefficients and degrees $\geqq 1$.

Let $\sigma: V \to \mathbf{C}^s$ be the map $v \to (P_1(v), \cdots, P_s(v))$. It is continuous, maps Γ, $V_{\mathbf{Q}}$, $V_{\mathbf{R}}$ in $\mathbf{Z}^s$, $\mathbf{Q}^s$, $\mathbf{R}^s$ respectively. By assumption, any $v \in V_{\mathbf{Q}}$ has a closed orbit. Since the P_i's, together with 1, generate P over $\mathbf{C}$, and the invariant polynomials separate the closed orbits (3.3), we see that

$$v, v' \in V_{\mathbf{Q}}, \sigma(v) = \sigma(v') \Rightarrow v' \in v \cdot G . \tag{2}$$

The origin is a closed orbit, hence is the only closed orbit on which all P_i's are zero, and

$$\sigma(v) \neq 0 \qquad (v \in V_{\mathbf{Q}} - 0) . \tag{3}$$

The group G being semi-simple, connected, $\pi(G)$ consists of transformations of determinant one, and $\pi(G_{\mathbf{R}})$ leaves invariant the euclidean measure on V, identified with $\mathbf{R}^n$ by means of the basis chosen above. By Minkowski's classical idea, there exists a compact set C in $V_{\mathbf{R}}$ containing zero such that $C \cdot \pi(g) \cap \Gamma \neq \{0\}$ for any $g \in G_{\mathbf{R}}$. (For the sake of completeness, we recall the proof: let C_0 be a compact set in $V_{\mathbf{R}}$ with measure strictly greater than 1. Then the projection $V_{\mathbf{R}} \to V_{\mathbf{R}}/\Gamma$ is not injective on $C_0 \cdot \pi(g)$ whence the existence of $z \in \Gamma - 0$ such that $C_0 \cdot \pi(g) \cap (C_0 \cdot \pi(g) + z) \neq \emptyset$. Therefore $C = \{x - y; x, y \in C_0\}$ fulfills our condition.) The image $\sigma(C)$ of C is compact, $\sigma(C) \cap \mathbf{Z}^s$ is finite, and we may find finitely many elements $w_j \in \Gamma - 0 (1 \leqq j \leqq t)$ such that

$$\sigma(C) \cap \sigma(\Gamma - 0) = \{\sigma(w_1), \cdots, \sigma(w_t)\} .$$

By (2), the intersection $\sigma^{-1}(\sigma(v)) \cap \Gamma (v \in V_{\mathbf{Q}})$, belongs to the orbit of v, which is closed by assumption, hence (6.9) consists of a *finite* number of orbits of $G_{\mathbf{Z}}$. There exists therefore a finite number of elements $v_1, \cdots, v_m \in \Gamma - 0$ such that

$$\textstyle\bigcup_i v_i \cdot \pi(G_{\mathbf{Z}}) = \sigma^{-1}\big(\sigma(C) \cap \sigma(\Gamma - 0)\big) \cap \Gamma . \tag{4}$$

Let now $g \in G_{\mathbf{R}}$. There exist $c \in C$ and $v \in \Gamma - 0$ such that $c \cdot \pi(g^{-1}) = v$. The polynomials P_i being invariant under G, we have then

$$\sigma(v) = \sigma\big(c \cdot \pi(g^{-1})\big) = \sigma(c) \in \sigma(C) \cap \sigma(\Gamma - 0) ,$$

hence $v \in \sigma^{-1}(\sigma(C) \cap \sigma(\Gamma - 0))$; by (4), there exists an index $i (1 \leqq i \leqq m)$ such that

$$c \cdot \pi(g^{-1}) = v \in v_i \cdot \pi(G_{\mathbf{Z}}) .$$

Given $g \in G_{\mathbf{R}}$, we have thus found an index i, and $b \in G_{\mathbf{Z}}$, such that $v_i \cdot \pi(b \cdot g) \in C$. This shows that

$$G_{\mathbf{R}} = \textstyle\bigcup_i G_{\mathbf{Z}} \cdot X_i \qquad (X_i = \{g \in G_{\mathbf{R}} \mid v_i \cdot \pi(g) \in C\}) .$$

Let G_i be the isotropy group of $v_i (1 \leqq i \leqq m)$. The orbit $v_i \cdot \pi(G)$ is closed, hence so is $v_i \cdot \pi(G_{\mathbf{R}})$ (see 2.3); there exists a compact set $B_i \subset G_{\mathbf{R}}$ such that

$X_i = G_{i\mathbf{R}} \cdot B_i$ (5.2). We have then

$$G_{\mathbf{R}} = \bigcup_i G_{\mathbf{Z}} \cdot G_{i\mathbf{R}} \cdot B_i \qquad (B_i \text{ compact}) .$$

But G_i is defined over $\mathbf{Q}$, since $v_i \in V_{\mathbf{Q}}$, and is a proper subgroup since $v_i \neq 0$ (see (1)). Therefore $G_{i\mathbf{R}} = G_{i\mathbf{Z}} \cdot C_i$, with C_i compact, and finally $G_{\mathbf{R}} = G_{\mathbf{Z}} \cdot A$ with $A = \bigcup_i C_i \cdot B_i$ compact.

REMARK. The preceding proof is an adaptation of A. Weil's argument, pertaining to groups of automorphisms of algebras with involution "which do not represent zero" [30, Theorem 4.1.1], and was in part suggested by it.

11.8. THEOREM. *Let G be an algebraic group defined over* $\mathbf{Q}$. *Then $G_{\mathbf{R}}/G_{\mathbf{Q}}$ is compact if and only if $X_{\mathbf{Q}}(G^0) = 1$, and every unipotent element of $G_{\mathbf{Z}}$, or, equivalently, of $G_{\mathbf{Q}}$, belongs to the radical of G.*[5]

As in 11.6, we may restrict ourselves to the case of a connected group G. Then $G = H \cdot N$ is the semi-direct product of a reductive group H and of an invariant unipotent group N, both defined over $\mathbf{Q}$, and $N_{\mathbf{R}}/N_{\mathbf{Z}}$ is compact (see 6.10 for references). In view of 11.6, it will be enough to show:

(i) $G_{\mathbf{R}}/G_{\mathbf{Z}}$ is compact if and only if $H_{\mathbf{R}}/H_{\mathbf{Z}}$ is compact.

(ii) H verifies the conditions of 11.4 if and only if $X_{\mathbf{Q}}(G) = 1$, and every unipotent element of $G_{\mathbf{Q}}$ (resp. of $G_{\mathbf{Z}}$) belongs to N.

PROOF OF (i). Let $G_{\mathbf{R}}/G_{\mathbf{Z}}$ be compact. Since $H_{\mathbf{Z}} \cdot N_{\mathbf{Z}}$ is of finite index in $G_{\mathbf{Z}}$ (6.4), we have $G_{\mathbf{R}} = K \cdot H_{\mathbf{Z}} \cdot N_{\mathbf{Z}}$, with K compact, whence $H_{\mathbf{R}} = \pi(K) \cdot H_{\mathbf{Z}}$, where π is the natural projection of G onto H. Let now $H_{\mathbf{R}} = A \cdot H_{\mathbf{Z}}$ with A compact. Then

$$G_{\mathbf{R}} = A \cdot H_{\mathbf{Z}} \cdot N_{\mathbf{R}} = A \cdot N_{\mathbf{R}} \cdot H_{\mathbf{Z}} = A \cdot B \cdot N_{\mathbf{Z}} \cdot H_{\mathbf{Z}} ,$$

with B compact, since $N_{\mathbf{R}}/N_{\mathbf{Z}}$ is compact, whence $G_{\mathbf{R}} = K \cdot G_{\mathbf{Z}}$, with $K = A \cdot B$ compact.

PROOF OF (ii). A unipotent group has only the trivial rational character, hence $X(G)$ and $X_{\mathbf{Q}}(G)$ are naturally isomorphic to $X(H)$ and $X_{\mathbf{Q}}(H)$. Assume H to verify the conditions of 11.4. Then $X_{\mathbf{Q}}(G) = 1$. Moreover, since a rational representation maps unipotent elements into unipotent elements [1, Chap. II], the unipotent elements of $G_{\mathbf{Q}}$ belong to the kernel of π, that is to N.

Assume now that $X_{\mathbf{Q}}(G) = 1$, and that every unipotent element of $G_{\mathbf{Z}}$ belongs to N. We have then $X_{\mathbf{Q}}(H) = 1$, by the initial remark of the proof of (ii). If $x \in G_{\mathbf{Q}}$ is unipotent, then a suitable power x^m of x is a

[5] Another proof of Theorem 11.8 has been given by G. D. Mostow and T. Tamagawa, to appear in Ann. of Math.

unipotent element of $G_{\mathbf{Z}}$ (see the proof of (d) $\Rightarrow$ (a) in 11.4). It belongs to N by assumption, hence so does the one parameter group generated by $\log x^m = m \cdot \log x$, and therefore $x \in N$. Thus every unipotent element of $G_{\mathbf{Q}}$ is in N. Since $H_{\mathbf{Q}}$ contains the semi-simple and unipotent parts of its elements [1, Chap. II], we see that $H_{\mathbf{Q}}$ has only semi-simple elements.

12. Groups over number fields

12.1. *Notation.* K is a number field, J the ring of algebraic integers of K, Φ the set of distinct isomorphisms of K into $\mathbf{C}$, and $\bar{\sigma}$ the composition of $\sigma \in \Phi$ with the complex conjugation. Φ' will be a subset of Φ which contains exactly one representative of each pair $(\sigma, \bar{\sigma})$. As usual, x^σ is the image of $x \in K$ under σ, and K^σ the image of K. The completion of K^σ with respect to the absolute value in $\mathbf{C}$ is denoted by L_σ. Thus $L_\sigma = \mathbf{C}$ if $\sigma \neq \bar{\sigma}$, and $L_\sigma = \mathbf{R}$ otherwise. This notation is used throughout this paragraph.

12.2. Let $G \subset \mathbf{GL}(n, \mathbf{C})$ be a connected algebraic group defined over K, and I the ideal of polynomials on $\mathbf{M}(n, \mathbf{C})$, with coefficients in K, which vanish on G. The algebraic group defined by the ideal I^σ is denoted by G^σ. It has K^σ as a field of definition. For a subset ψ of Φ, we put

$$(1) \qquad G_\psi = \prod\nolimits_{\sigma \in \psi} G^\sigma \ , \qquad G_{\psi,r} = \prod\nolimits_{\sigma \in \psi} G^\sigma_{L_\sigma} \ .$$

In G_ψ, or in $G_{\psi,r}$ we identify G_J (resp. G_K) with the set of elements $(x^\sigma)_{\sigma \in \psi} (x \in G_J$, resp. $x \in G_K)$.

There exists an algebraic group $G' = R_{K/\mathbf{Q}} G$, the *group obtained from G by restriction of the groundfield from K to* $\mathbf{Q}$, which is defined over $\mathbf{Q}$, and is isomorphic over $\bar{K}$ to G_Φ. It is essentially unique up to isomorphism over $\mathbf{Q}$ [30, Chap. I], and is isomorphic to G_Φ by an isomorphism μ' of the form $(\mu^\sigma)_{\sigma \in \Phi}$, where $\mu \colon G' \to G$ is a rational homomorphism defined over K, which verifies

$$(2) \qquad \mu'(G'_{\mathbf{Z}}) = G_J \ , \qquad \mu'(G'_{\mathbf{Q}}) = G_K \qquad (G' = R_{K/\mathbf{Q}} G) \ .$$

The homomorphism μ also induces in a natural way an isomorphism of $X_K(G)$ onto $X_{\mathbf{Q}}(G')$. Let $\sigma \neq \bar{\sigma}$. The standard embedding of $\mathbf{GL}(n, \mathbf{C})$ into $\mathbf{GL}(2n, \mathbf{R})$ induces an isomorphism of $G^\sigma_{L_\sigma}$ onto the set of real points of an algebraic group $R_{\mathbf{C}/\mathbf{R}} G^\sigma \subset \mathbf{GL}(2n, \mathbf{C})$; from this, one deduces the existence of an isomorphism of real algebraic groups $\beta \colon G'_{\mathbf{R}} \to G_{\Phi',r}$, which also maps $G'_{\mathbf{Z}}$ and $G'_{\mathbf{Q}}$ onto G_J and G_K respectively.

In general, G_J is not discrete in $G^\sigma_{L_\sigma}$, however, it is clear from the above, that G_J is discrete in G_Φ or in $G_{\Phi'}$. *More generally, if ψ contains all $\sigma \in \Phi'$ for which $G^\sigma_{L_\sigma}$ is not compact, then G_B is discrete in $G_{\psi,r}$.*

To see this, it is enough to show that given $\delta > 0$, there are only finitely many $b \in J$ which occur as coefficients of matrices in G_J, and which verify $|b^\sigma| < \delta(b \in \psi)$. If $\bar{\sigma} \in \psi$, then $|b^\sigma| < \delta$. If $\sigma, \bar{\sigma} \notin \psi$, then $G^\sigma_{L_\sigma}$ is compact, hence $|b^\sigma| < \varepsilon$, where ε depends only on $G^\sigma_{L_\sigma}$. Our assertion follows then from the familiar fact that J has only finitely many elements b, all of whose conjugates $b^\sigma (b \in \Phi)$ are in absolute value under a given bound.

When this condition is fulfilled, G_J is called an arithmetically defined subgroup, or a group of units of $G_{\psi,r}$. In view of the possibility of restricting the groundfield, it is clear that there is no essential loss in generality in limiting oneself to the case $K = \mathbf{Q}$, $J = \mathbf{Z}$, and that the main results of the preceding paragraphs extend automatically to the groups of units considered here. We state this formally for some of them for the convenience of reference, and leave the reformulation of the others to the reader.

12.3. THEOREM. *We keep the notation of* 11.1. *Let G be a connected algebraic group defined over K, and ψ a subset of Φ' containing all σ for which $G^\sigma_{L_\sigma}$ is not compact. Then*

(a) *G_J is finitely generated.*

(b) *$G_{\psi,r}$ is the union of open subsets U having the following properties:*

(i) *$G_{\psi,r} = U \cdot G_J$;*

(ii) *$K \cdot U = U$ for a suitable maximal compact subgroup of $G_{\psi,r}$;*

(iii) *$U^{-1} \cdot U \cap x \cdot G_J \cdot y$ is finite for $x, y \in G_K$.*

(c) *$G_{\psi,r}/G_J$ has finite Haar measure if and only if $X_K(G) = 1$; it is compact if and only if $X_K(G) = 1$, and every unipotent element of G_K or, equivalently, of G_J, belongs to the radical of G.*

The first assertion follows from 12.2 and 6.5. When $\psi = \Phi'$, (b) and (c) follow from 12.2, 6.5, 6.7, 9.4 and 11.8. Let now $\psi \neq \Phi'$ and θ be the complement of ψ in Φ'. Then $G_{\theta,r}$ is compact, and $G_{\Phi',r} = G_{\psi,r} \times G_{\theta,r}$. Let U be an open set verifying the properties (i) to (iii) for $\psi = \Phi'$. Since $G_{\theta,r}$ is compact, and invariant, it belongs to all maximal compact subgroups of $G_{\Phi',r}$, and the maximal compact subgroups of $G_{\Phi',r}$ are the products of $G_{\theta,r}$ with the maximal compact subgroups of $G_{\psi,r}$. By (ii) we have $G_{\theta,r} \cdot U = U$, hence $U = G_{\theta,r} \times U'$ with U' open in $G_{\psi,r}$; it is then obvious that U' has the properties (i) to (iii).

Let us write A and B for the images of G_J in $G_{\psi,r}$ and $G_{\Phi',r}$ under the canonical imbeddings. We have then clearly $A \cdot G_{\theta,r} = B \cdot G_{\theta,r}$, therefore $G_{\psi,r}/A = G_{\Phi',r}/A \cdot G_{\theta,r} = G_{\Phi',r}/B \cdot G_{\theta,r}$ is the base space of a fibration of $G_{\Phi',r}/B$ with fibre $G_{\theta,r}$. Since $G_{\theta,r}$ is compact, $G_{\psi,r}/A$ is compact (or of finite measure) if and only if $G_{\Phi',r}/B$ is, and (c) follows from the above.

12.4. COROLLARY. *Let G be a connected algebraic group defined over*

K. We keep the assumptions of 11.3, *and assume moreover that there is at least one $\sigma \in \Phi$ for which $G^{\sigma}_{L_\sigma}$ is compact. Then $G_{\psi,r}/G_J$ is compact if and only if $X_K(G) = 1$.*

If $G^{\sigma}_{L_\sigma}$ is compact, then it is reductive, and all its elements are semi-simple. Of course, $x \in G_K$ is semi-simple if and only if x^σ is semi-simple $(\sigma \in \Phi)$, in our case, G_K consists therefore of semi-simple elements, and 12.4 follows from 12.3(c).

13. Appendix: Remarks on algebraic groups

This appendix contains some remarks about algebraic groups which are actually not needed in the paper, but bring natural complements to some auxiliary results proved in §§ 1, 8. In A, unlike in § 2, the universal field underlying the definition of an algebraic group may have arbitrary characteristic.

A. Algebraic tori

13.1. Let T be an algebraic torus, K a field of definition for T. Then there always exists a separable finite Galois extension K' of K over which T splits (see Ono, Ann. of Math. 74 (1961), 61–139, Prop. 1.2.1). This being taken into account, it is clear that 8.2, 8.3, and 8.4 go over without change to the general case. It follows from 8.4 that if S is a subtorus of T, defined over K, and if $\chi \in X_K(S)$, then there exists an integer m such that χ^m extends to a rational character of T. In fact we take S' as in 8.4a, and m such that χ^m is trivial on the finite group $S \cap S'$. Since the character groups are finitely generated, this can also be expressed by saying that the injection $i: S \to T$ induces a homomorphism i^0 of $X(T)$ (resp. $X_K(T)$) onto a subgroup of finite index of $X(S)$ (resp. $X_K(S)$).

13.2. PROPOSITION. *Let T be an algebraic torus defined over a field K. Then T contains two subtori T_c, T_a defined over K such that $X_K(T_c) = 1$, T_a splits over K, $T_c \cap T_a$ is finite and $T = T_c \cdot T_a$. If S is an algebraic torus and $f: S \to T$ a rational homomorphism, both defined over K, then $f(S_c) \subset T_c$ and $f(S_a) \subset T_a$.*

Let K' be a Galois extension of K over which T splits, and A the Galois group of K' over K. Let T_c be the identity component of the intersection of the kernels of the characters defined over K, and T_a be the subtorus generated by the images of the elements $\gamma \in \Gamma_K(T)$. They are defined over K', and invariant under A. Since K' is separable over K, T_c and T_a are defined over K. By 8.2, T_a splits over K. By 13.1, $X_K(T_c) = 1$. Thus T_c has no non-trivial subtorus which splits over K (8.4), and $T_c \cap T_a$ is finite. Lemma 8.4 also implies that $\dim T_c + \dim T_a = \dim T$, whence

$T = T_a \cdot T_c$. Let now $f: S \to T$ be a rational homomorphism defined over K. We have $f_0(\Gamma_K(S)) \subset \Gamma_K(T)$, hence $f(S_a) \subset T_a$. If $\chi \in X_K(T)$, then $\chi \circ f \in X_K(S)$, hence $S_c \in \operatorname{Ker}(\chi \circ f)$, and $f(S_c) \in \operatorname{Ker} \chi$, which implies $f(S_a) \subset T_a$.

B. Real algebraic reductive groups

13.3. PROPOSITION. *Let $T \subset \mathbf{GL}(n, \mathbf{C})$ be an algebraic torus defined over $\mathbf{R}$, and $\mathfrak{t}_{\mathbf{R}}$ the Lie algebra of $T_{\mathbf{R}}$. Then $T_{c,\mathbf{R}}$ is a torus, $T_{a,\mathbf{R}}$ is real diagonalizable, the Lie algebra of $T_{c,\mathbf{R}}$ (resp. $T_{a,\mathbf{R}}$) is the set of elements of $\mathfrak{t}_{\mathbf{R}}$ with purely imaginary (resp. real) eigenvalues.*

Let $\mathfrak{t}'$ (resp. $\mathfrak{t}''$) be the set of elements of $\mathfrak{t}_{\mathbf{R}}$ with real (resp. purely imaginary) eigenvalues. Then $\mathfrak{t}_{\mathbf{R}} = \mathfrak{t}' + \mathfrak{t}''$, and $\mathfrak{t}', \mathfrak{t}''$ are algebraic (see 1.4). The irreducible real algebraic group T' with Lie algebra $\mathfrak{t}'$ is then a real algebraic torus which splits over $\mathbf{R}$, and is contained in $T_{a,\mathbf{R}}$. The analytic subgroup of $\mathbf{GL}(n, \mathbf{R})$ generated by $\mathfrak{t}''$ is closed, and belongs to a compact group hence is a torus in the usual sense (compact connected commutative Lie group). Since a compact linear group is algebraic [7b, p. 230], T'' is also the irreducible real algebraic subgroup of $T_{\mathbf{R}}$ with Lie algebra $\mathfrak{t}''$. Clearly, every element of $X_{\mathbf{R}}(T)$ is trivial on T'', therefore $T' = T_{a,\mathbf{R}}$, $T'' = T_{c,\mathbf{R}}$.

13.4. The preceding proposition shows that the decomposition $\mathfrak{m} = \mathfrak{m}_{\mathfrak{k}} + \mathfrak{m}_{\mathfrak{p}}$ of a fully reducible commutative algebraic Lie algebra corresponds to the global decomposition of 13.3. In particular (13.3) it is compatible with rational representations, and has therefore an intrinsic meaning, independent of the imbedding. From this and 1.10, 1.11, it follows that the notion of Cartan involution of a real reductive algebraic group is independent from the imbedding in $\mathbf{GL}(n, \mathbf{R})$, up to birational isomorphism. Since a real representation of a semi-simple Lie algebra is always rational, the following proposition generalizes a fact mentioned in 1.1:

13.5. PROPOSITION. *Let G be a real algebraic, reductive group, θ a Cartan involution of G, and $\rho: G \to \mathbf{GL}(m, \mathbf{R})$ a rational representation. Then there exists a Cartan involution θ' of $\mathbf{GL}(m, \mathbf{R})$ such that $\rho(\theta(g)) = \theta'(\rho(g))$.*

Let $\mathfrak{c}$ be the center of $\mathfrak{g}$. The image $\mathfrak{g}'$ of $\mathfrak{g}$ is the direct product of $\rho(\mathfrak{c})$ and of $\rho(\mathscr{D}\mathfrak{g}) = \mathscr{D}\mathfrak{g}'$. By 13.4, $\rho(\mathfrak{c})$ is completely reducible, and therefore $\mathfrak{g}'$ is reductive in $\mathfrak{gl}(m, \mathbf{R})$. By a general theorem [7a, p. 140], $\mathfrak{g}'$ is algebraic. Let G' be the real algebraic subgroup of $\mathbf{GL}(m, \mathbf{R})$ with Lie algebra $\mathfrak{g}'$, and M be the centralizer of $\rho(\mathfrak{c})$. The group M is algebraic, and its Lie algebra $\mathfrak{m}$ is reductive [4]. Thus (1.2) G' and M are real algebraic, reductive. As was recalled in 1.1, the image of a Cartan decomposition of $\mathscr{D}\mathfrak{g}$

is a Cartan decomposition of $\mathcal{D}\mathfrak{g}'$. It follows therefore from 1.6 and the conjugacy of Cartan decompositions of $\mathcal{D}\mathfrak{g}'$ that we may find a Cartan involution θ'' of $\mathbf{GL}(m, \mathbf{R})$ which leaves M, G' invariant and induces the given Cartan involution of $\mathcal{D}\mathfrak{g}'$. By 13.4 and 1.4, the $\mathfrak{k}$- and $\mathfrak{p}$-parts of $\rho(\mathfrak{c})$ with respect to θ'' are necessarily the images of the $\mathfrak{k}$- and $\mathfrak{p}$-parts of θ. The Cartan involution θ'' verifies therefore $\rho\circ\theta = \theta''\circ\rho$ on $\mathfrak{g}$, hence also on G^0, and leaves G' invariant. It is defined by a positive non-degenerate quadratic form F which is invariant under the identity component of $\rho(K)$, where K is the fixed point set of θ in G. Let F' be the average of F, over $\rho(K)$. The corresponding Cartan involution θ' will then fulfill our conditions.

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