Ratner's Theorem

This is a preliminary list of talks. Please choose what topic you would like to give a talk about and let me (wienhard@math.princeton.edu) know soon. There are two blocks of 2 or 3 talks, whoever is interested in one of these should work together. Also for the other talks, you might prepare them jointly with someone else who is interested.

For anyone who wants to get a little bit of an overview, you might want to have a look at Witte Morris [8] Chapter 1, Dani [2].

1. Overview (Maryam Mirzakhani)

2. Basics from Ergodic Theory (Yi Wang)

This talk should define and discuss: ergodic measure, ergodic decomposition of measures, Birkhoff's ergodic theorem (a consequence: distinct ergodic measures are mutually singular), uniquely ergodic measures. Please state and explain the theorems. Please give many examples, in particular also of not so well behaved systems. If you have time it would be nice to see/sketch a proof, but examples are more important.

2.1. **References.** Witte Morris [8] Chapter 3.1, 3.3, Eskin-Kleinbock [6] p.3-5., Einsiedler-Ward [4] Chapter 2, perhaps Starkov [10] Preliminaries A. A general reference for ergodic theory is Petersen's book [9].

3. The case of $G = \mathrm{SL}(2,\mathbf{R})$ (Sam Ruth, Kevin Wilson, Phil Isett, Peter Varju)

These 3 talks should discuss Ratner's theorem in the case of homogeneous spaces under $\mathrm{SL}(2,\mathbf{R})$:

Recall the geometry of the hyperbolic plane, define geodesic and horocyclic flow on the unit tangent bundle, identification with $\mathrm{PSL}(2,\mathbf{R})$. Define Haar measure and show invariance under the flows. Ergodicity of the geodesic and the horocyclic flow on a finite volume hyperbolic surface. State perhaps Mautner property for $\mathrm{SL}(2,\mathbf{R})$. Prove Hedlund's theorem: every orbit of the horocyclic flow on a finite volume hyperbolic surface is either periodic or dense, first compact case, then non-compact case. Classification of ergodic measures which are invariant under horocyclic flow.

3.1. **References.** Ghys [7] Sections 1-3, Starkov [10] Chapter 2 §14–§16, Bekka-Mayer [1] Chapter II§1, II§3, IV, Eskin-Kleinbock [6] Section 2. For Mautner property see also Section 3.4. in Einsiedler [5].

4. Flows on Homogeneous Spaces (Francesco Cellarosi)

This talk should discuss some examples of homogeneous spaces and flows on them, e.g. the space of lattices $\mathrm{SL}(n, \mathbf{Z})\backslash\mathrm{SL}(n, \mathbf{R})$, non-ergodicity of geodesic flow for $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$. It should serve as a preparation for the next talks.

4.1. **References.** Bekka-Mayer [1] Chapter II §4, Chapter V §2, Starkov [10] Chapter 1.

5. The case of $H = \mathrm{SL}(2, \mathbf{R})$ (Po-Lam Yung, Jonathan Luk)

These 2 talks discuss Ratner's theorem in the case of $\mathrm{SL}(2, \mathbf{R})$-invariant measures on general homogeneous spaces:

Discuss some of the background needed in the proof: Lie groups and Lie algebras, finite dimensional representations of $\mathrm{SL}(2, \mathbf{R})$, Mautner phenomenon for $\mathrm{SL}(2, \mathbf{R})$. Discuss, explain and illustrate the proof.

5.1. **References.** Einsiedler [5], the article also includes most of the background needed.

6. The general case (Ilya Vinagradov)

This talk should discuss some of the ideas and problems of the proof of Ratner's Theorem in the general case. There will be no time to give the technical details of the proof, so please concentrate on the outline of the proof, comparing everything to the case of $\mathrm{SL}(2, \mathbf{R})$-invariant measures, and focus on the important ideas, e.g. non-divergence of unipotent flows

6.1. **References.** Witte Morris [8] Chapter 5, Starkov [10] Chapter 2 §12, §17–19 (in particular the example of $\mathrm{SL}(2, \mathbf{C})$; Dani [2], there are also useful remarks in Ghys [7] e.g. Sections 2.4., 3.6. About non-divergence there are things in Eskin-Kleinbock [6]

7. Application I: Oppenheim conjecture/Values of quadratic forms (Zhiren Wang)

Discuss Oppenheim's conjecture. Assume either the general statement of Ratner's theorem or reduce to the case of $\mathrm{SO}(2, 1) \cong \mathrm{SL}(2, \mathbf{R})$-orbits on homogeneous spaces under $\mathrm{SL}(3, \mathbf{R})$.

7.1. **References.** Ghys [7], Section 4.2, Starkov [10] Chapter 4 §28, Bekka-Mayer [1] Chapter VI §1 (with background from Chapter V), Dani-Margulis [3].

8. Application II: Diophantine Approximation or something else (Alireza Salehi-Golsefidy)

8.1. **References.** e.g. Starkov [10], Chapter 4 §29–30.

References

1. M. Bachir Bekka and Matthias Mayer, *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, London Mathematical Society Lecture Note Series, vol. 269, Cambridge University Press, Cambridge, 2000. MR MR1781937 (2002c:37002)
2. S. G. Dani, *Flows on homogeneous spaces: a review*, Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 63–112. MR MR1411216 (98b:22023)
3. S. G. Dani and G. A. Margulis, *Values of quadratic forms at integral points: an elementary approach*, Enseign. Math. (2) **36** (1990), no. 1-2, 143–174. MR MR1071418 (91k:11053)
4. M. Einsiedler and T. Ward, *Ergodic theory: with a view towards number theory*, unfinished notes.
5. Manfred Einsiedler, *Ratner's theorem on* $\mathrm{SL}(2, \mathbf{R})$*-invariant measures*, Jahresber. Deutsch. Math.-Verein. **108** (2006), no. 3, 143–164. MR MR2265534 (2008b:37048)
6. A. Eskin and D. Kleinbock, *Unipotent flows and applications*, Lecture Notes for Clay Institute Summer school, 2007.
7. Étienne Ghys, *Dynamique des flots unipotents sur les espaces homogènes*, Astérisque (1992), no. 206, Exp. No. 747, 3, 93–136, Séminaire Bourbaki, Vol. 1991/92. MR MR1206065 (94e:58101)
8. Dave Witte Morris, *Ratner's theorems on unipotent flows*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2005. MR MR2158954 (2006h:37006)
9. Karl Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, Cambridge, 1983. MR MR833286 (87i:28002)
10. Alexander N. Starkov, *Dynamical systems on homogeneous spaces*, Translations of Mathematical Monographs, vol. 190, American Mathematical Society, Providence, RI, 2000, Translated from the 1999 Russian original by the author. MR MR1746847 (2001m:37013b)