Flows on homogeneous spaces: a review

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Introduction

Let G be a Lie group and C be a closed subgroup of G. The quotient G/C is also called a homogeneous space. To each $a \in G$ corresponds a translation T_a given by $xC \mapsto axC$. More generally any subgroup A of G yields an action on G/C where each $a \in A$ acts as the translation T_a as above. Similarly, given an automorphism ϕ of G such that $\phi(C) = C$ we can define a transformation $\overline{\phi} : G/C \to G/C$ by $\overline{\phi}(xC) = \phi(x)C$ for all $x \in G$; these transformations are called automorphisms of G/C. The composite transformations of the form $T_a \circ \overline{\phi}$, where T_a is a translation and $\overline{\phi}$ is an automorphism, are called affine automorphisms.

In the sequel we will be interested mainly in the homogeneous spaces G/C which admit a finite Borel measure invariant under the action of G. For a Lie group G we shall denote by $\mathcal{F}(G)$ the class of all closed subgroups C such that G/C admits a finite G-invariant measure. Discrete subgroups from the class $\mathcal{F}(G)$ are called *lattices*.

The transformations and actions as above form a rich class of 'dynamical systems'. The study of the asymptotic behaviour of their orbits is of great significance not only because of their intrinsic appeal but also on account of various applications to varied subjects such as Number theory, Geometry, Lie groups and their subgroups etc. As may be expected the systems have been studied from various angles; most concepts of dynamics have been considered in the context of these systems, with varying degrees of success in understanding them. My aim in this article is to give an exposition of the results concerning mainly the asymptoptic behaviour of the orbits such as their closures, their distribution in the ambient space etc. and discuss their applications, especially to problems of diophantine approximation.

§ 1 Homogeneous spaces - an overview

Let me begin with some examples which are particular cases of the general class of systems introduced above. I will also recall special features of various classes of homogeneous spaces, which it would be convenient to bear in mind in the sequel.

1.1. Rotations of the circle: Let $G = \mathbb{R}$ and $C = \mathbb{Z}$. Then $G/C = \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the circle group, and the translations in the above sense are the usual rotations of the circle. Also, the usual angle measure is invariant under all rotations. (Alternatively one can think of the same systems setting $G = \mathbb{T}$ and C to be the trivial subgroup). The translation corresponding to an irrational number t is called an 'irrational rotation'; it is a rotation by an angle which is an irrational multiple of π.

1.2. Affine transformations of tori: By a torus we mean the quotient $\mathbb{R}^n/\mathbb{Z}^n$ or equivalently the cartesian power $\mathbb{T}^n$, n being any natural number. Any $A \in GL(n, \mathbb{Z})$, namely an integral $n \times n$ matrix with determinant ± 1 defines an automorphism $\overline{A}$ of $\mathbb{R}^n/\mathbb{Z}^n$ by $\overline{A}(v + \mathbb{Z}^n) = (Av) + \mathbb{Z}^n$, A being viewed as a linear transformation via the standard basis of $\mathbb{R}^n$. Conversely any automorphism of $\mathbb{R}^n/\mathbb{Z}^n$ arises in this way. Composing translations and automorphisms we get what are called affine automorphisms; see [CFS] and [Wa] for details.

1.3. Nilflows and solvflows: Let G be a nilpotent Lie group and let $C \in \mathcal{F}(G)$. Then G/C is called a nilmanifold and the action of any subgroup of G on G/C is called a nilflow. Similarly when G is solvable G/C is called a solvmanifold and the action of any subgroup on G/C is called a solvflow.

A solvmanifold G/C admits a finite G-invariant measure if and only if it is compact (see [R], Theorem 3.1). If G is nilpotent then for any $C \in \mathcal{F}(G)$, the connected component C^0 of the identity in C is a normal subgroup of G (see [R], Corollary 2 of Theorem 2.3); in this case G/C can be viewed as a homogeneous space of the Lie group G/C^0, by the subgroup C/C^0; as the latter is a discrete subgroup, while studying the flows as above for nilpotent Lie groups there is no loss of generality in assuming C to be discrete, namely a lattice.

Let me also recall here that if G is a nilpotent Lie group and C is a lattice in G then $[G, G]C$ is a closed subgroup; see [Mal] and [R] for detailed results on the structure of lattices in nilpotent Lie groups. The factor homogeneous space $G/[G, G]C$ is called the maximal torus factor of G/C. Many properties of flows on G/C turn out to be characterisable in terms of the corresponding (factor) action on the torus factor.

In the year 1960-61 a conference on *Analysis in the large* was held at Yale University and a number of interesting results on nilflows and solvflows came out as a result of the interaction of the participants; see [AGH]. The study was pursued later extensively by L. Auslander and other authors, proving several results on the structure of solvmanifolds and the dynamical behaviour of orbits of solvflows (see [Au1], [Au2]). Through the following sections I shall recall only those of the results which are closely related to the theme of the exposition.

A particular class of solvflows which are not nilflows turns out to be of special interest especially on account of certain results of J. Brezin and C. C. Moore [BM]. A solvable Lie group is called *Euclidean* if it is a covering group of a Lie group of the form $K \cdot V$, semidirect product, where V is a vector group (topologically isomorphic to $\mathbb{R}^n$ for some n) and K is a compact abelian subgroup of $GL(V)$ (acting as automorphisms of V). A compact solvmanifold is called *Euclidean* if it can be written as a homogeneous space of a Euclidean solvable group. A compact solvmanifold is Euclidean if and only if it is finitely covered by a torus. A general compact solvmanifold admits a unique maximal Euclidean quotient; that is, given a solvable Lie group G and a closed subgroup C of G such that G/C is compact there exists a unique minimal closed subgroup L containing C, such that G/L is Euclidean; if M is the largest closed normal subgroup of G contained in L (the intersection of all conjugates of L) then G/M is a Euclidean solvable group; see [BM] for details on these and other properties of Euclidean solvmanifolds. In many ways this Euclidean factor plays the same role for solvmanifolds as the maximal torus factor for nilmanifolds.

1.4. Geodesic and horocycle flows: Let G be the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$, where I is the identity matrix, and Γ be a discrete subgroup of G. Suppose that Γ contains no elements of finite order. Then G/Γ can be realised canonically as the space of unit tangents of the surface $S = \mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half plane and G acts as its group of isometries with respect to the Poincaré metric (see [B]). Further, the geodesic flow corresponding to S is given (after suitable identification) by the action of the one-parameter subgroup of G which is the image of the subgroup $D = \{\mathrm{diag}\,(e^t, e^{-t}) \mid t \in \mathbb{R}\}$ in $PSL(2, \mathbb{R})$. Conversely any surface of constant negative curvature can be realised as $\mathbb{H}/\Gamma$ for a discrete subgroup Γ of $PSL(2, \mathbb{R})$ with no element of finite order (namely the fundamental group of S) and the associated geodesic flow corresponds to the action on G/Γ of the one-parameter subgroup as above. Thus the geodesic flows belong to the class of flows on homogeneous spaces. Also, Γ is a lattice in G if and only if the surface is of finite Riemannian area.

A particular flow plays a crucial role in the classical study of the geodesic flow of a surface of constant negative curvature, by G.A. Hedlund, E. Hopf and others. It is called the horocycle flow; to be precise there are two horocycle flows associated to each geodesic flow, the so called 'contracting' and 'expanding' horocycle flows, but viewed as flows themselves they are equivalent. In our notation the flow can be given as the action of the (the image in G of the) one-parameter subgroup $\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$, on G/Γ as above; when G/Γ is the unit tangent bundle of a surface of constant negative curvature this action coincides with the classical horocycle flow defined geometrically (see [He], [Man], [Ve], [Gh]).

Geodesic flows on n-dimensional Riemannian manifolds of constant negative curvature and finite (Riemannian) volume were studied through flows on homogeneous spaces by Gelfand and Fomin [GF]: Let $G = SO(n,1)$, the special orthogonal group of a quadratic form of signature $(n,1)$, Γ be a lattice in G and $\{g_t\}$ be a one-parameter subgroup of G whose action on the Lie algebra of G is diagonalisable over $\mathbb{R}$ (there exists such a one-parameter subgroup and it is unique upto conjugacy and scaling). Let M be a compact subgroup of G centralised by g_t for all $t \in \mathbb{R}$. The flow induced by $\{g_t\}$ on G/Γ factors to a flow on $M\backslash G/\Gamma$. Any geodesic flow of a manifold of constant negative curvature and finite Riemannian volume can be realised as such a flow, with M as the unique maximal compact subgroup of the centraliser of $\{g_t\}$ and Γ a suitable lattice; this may be compared with the one-dimensional case above. In a similar vein the geodesic flows on all locally symmetric spaces were considered by F. Mautner [Mau].

1.5. Modular homogeneous space: One particular homogeneous space and the flows on it are of great significance in many applications to Number theory. This consists simply of taking $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$, the subgroup consisting of integral matrices in G. The homogeneous space has a finite G-invariant measure; that is, Γ is a lattice in G (see [R], Corollary 10.5). On the other hand (unlike the lattices in solvable groups) the quotient space is noncompact. It turns out that as far as behaviour of orbits of flows and the proofs of the results are concerned, these examples involve most of the intricacies of the general case.

The following model for the above homogeneous space is one of the main reasons for the interest in it from a number-theoretic point of view. Consider the set $\mathcal{L}_n$ of lattices in $\mathbb{R}^n$ with unit discriminant (volume of any fundamental domain for the lattice). The action of $SL(n,\mathbb{R})$ on $\mathbb{R}^n$ induces an action on $\mathcal{L}_n$ which can be readily seen to be transitive. Further, $SL(n,\mathbb{Z})$ is precisely the isotropy subgroup for the lattice generated by the standard basis. Hence via the action $\mathcal{L}_n$ may be identified with $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. On $\mathcal{L}_n$ there is an intrinsically defined topology, in which two lattices are near if and only if they have bases which are near (see [Ca] for details); this topology can be seen to correspond to the quotient topology on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ under the identification as above. A well-known criterion due to K. Mahler asserts that a sequence of lattices $\{\Lambda_i\}$ in $\mathcal{L}_n$ is divergent (has no convergent subsequence) if and only if there exists a sequence $\{x_i\}$ in $\mathbb{R}^n - (0)$ with $x_i \in \Lambda_i$ for all i and $x_i \to 0$ as $i \to \infty$ (see [R], Corollary 10.9). The invariant measure on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is also related to the Lebesgue measure on $\mathbb{R}^n$ in an interesting way (see [Si]), which plays an important role in certain applications to diophantine approximation (e.g. Corollary 8.3, below).

1.6. Lattices in real algebraic groups: A subgroup H of $GL(n,\mathbb{R})$, $n \geq 2$, is said to be algebraic if there exists a set $\mathcal{P}$ of polynomials in the n^2 (coordinate) variables such that $H = \{g = (g_{ij}) \in GL(n,\mathbb{R}) \mid P(g_{ij}) = 0$ for all $P \in \mathcal{P}\}$; further H is said to be defined over $\mathbb{Q}$ if $\mathcal{P}$ can be chosen to consist of polynomials with rational coefficients. A theorem of Borel and Harish-Chandra gives a necessary and sufficient condition for $H \cap GL(n,\mathbb{Z})$, where H is an algebraic subgroup of $GL(n,\mathbb{R})$ defined over $\mathbb{Q}$, to be a lattice in H. The condition involves there being no "nontrivial characters defined over $\mathbb{Q}$". Without going into the technical definitions let me only mention that, in particular, if H is an algebraic subgroup defined over $\mathbb{Q}$ such that the unipotent matrices contained in H and a compact subgroup of H together generate H then $H \cap GL(n,\mathbb{Z})$ is a lattice in H. This gives an abundant class of homogeneous spaces. Also, as we shall see later, they are of importance in the study of orbit closures of unipotent one-parameter subgroups, in the case of the modular homogeneous spaces as above.

1.7. Homogeneous spaces of semisimple Lie groups: We shall now look at another class of homogeneous spaces, of which those in Sections 1.4 and 1.5 are particular cases. Let G be a semisimple Lie group and $C \in \mathcal{F}(G)$. Then G has a unique maximal compact connected normal subgroup, say G_0. Clearly G_0C is a closed subgroup of G and G/G_0C is a homogeneous space which is a factor of G/C. In studying flows on G/C, in many ways it is adequate to consider the factor actions on G/G_0C. Thus it is enough to consider homogeneous spaces of groups which have no nontrivial compact factor groups (namely such that any surjective homomorphism on to a compact group is trivial), a condition which holds for the quotient group G/G_0 as above. (By a theorem of H. Weyl the condition is equivalent to there being no nontrivial compact connected normal subgroup in G; it is however customary to express the condition in terms of factors rather than normal subgroups).

Now let G be a semisimple Lie group with no nontrivial compact factors and let $C \in \mathcal{F}(G)$. By Borel's density theorem (see [R]; see also [D6] and [D8] for more general versions of Borel's density theorem) C^0, the connected component of the identity in C is a normal subgroup of G. Hence (as seen in the case of nilpotent Lie groups) there is no loss of genetality in assuming that C is a lattice. Further, Borel's density theorem also shows that any lattice contains a subgroup of finite index in the center of G (see [R], Corollary 5.17). Hence passing to a quotient the center may be assumed to be finite.

Lattices in semisimple Lie groups with no nontrivial compact factors can be 'decomposed' into 'irreducible' lattices: A lattice Γ is said to be *irreducible* if for any closed normal subgroup F of positive dimension $F\Gamma$ is dense in L. The assertion about decomposition is the following: given a semisimple Lie group G with no nontrivial compact factors and a lattice Γ in G there exist closed normal subgroups $G_1, ..., G_k$, for some $k \geq 1$, such that G is locally the direct product of $G_1, ..., G_k$ (that is, $G = G_1 G_2 \cdots G_k$ and the pairwise intersections of G_i's are discrete central subgroups) and $G_i \cap \Gamma$ is an irreducible lattice in G_i for each $i = 1, ..., k$ (see [R], Theorem 5.22); if Γ' is the product of $G_i \cap \Gamma$, $i = 1, ..., k$, then it follows that Γ' is a lattice in G and hence, in particular, of finite index in Γ. Thus, upto a combination of finite coverings and factoring modulo finite central subgroups the homogeneous space G/Γ is a product of the spaces $G_i/G_i \cap \Gamma$, in each of which the lattice involved is irreducible. It may also be noted that for each i, $G/(\Pi_{j\neq i}G_j)\Gamma$ is an irreducible factor of G/Γ.

It is an important fact about lattices in semisimple Lie groups G that if G has trivial center and no compact factors and $\mathbb{R}$-rank at least 2 then any irreducible lattice in G is 'arithmetic'. I will not go into the details of this (see [Z2] and [Mar5] for general theory in this respect); though it plays a role in some proofs, it will not be directly involved in the discussion of the results. Let me only mention that it means that the class of homogeneous spaces corresponding to the lattices as above is closely related to the examples in section 1.6 above.

1.8. General homogeneous spaces: Just as a general Lie group is studied via its special subgroups and factors which are simpler, homogeneous spaces and flows on them are also studied by reducing to simpler cases which I have discussed above. There are various results which make this possible. While it is tempting to recall here some of the results are involved in the sequel, they would probably seem rather technical and devoid of context at this point. I will therefore recall them only as and when necessary. The reader is referred especially to [R], [Au1], [Au2], [BM] and [Mar7] for the general theory.

§ 2 Ergodicity

Let (X, μ) be a measure space and H be a group which acts (measurably) on X preserving the measure μ. The action is said to be *ergodic* if for any measurable set E such that $\mu(hE\Delta E) = 0$ for all $h \in H$ (namely almost H-invariant) either $\mu(E) = 0$ or $\mu(X - E) = 0$. When X and H are locally compact second countable topological spaces and the action is continous, as will be the case in the sequel, for any Borel measure μ ergodicity is equivalent to either $\mu(E)$ or $\mu(X-E)$ being 0 for every H-invariant Borel set E (namely E such that $hE = E$ for all $h \in H$).

The ergodicity condition is of importance in the study of asymptotic behaviour of orbits in view of the following results.

2.1. Lemma (Hedlund): *Let X be a second countable Hausdorff topological space and let H be a group of homoemorphisms of X. Let μ be a Borel measure on X invariant under the action of H. Suppose that $\mu(\Omega) > 0$ for all nonempty open subsets of X. Then there exists a Borel subset Y such that $\mu(X-Y) = 0$ and for all $y \in Y$ the orbit Hy is dense in X.*

Proof: It is easy to see that if $\{\Omega_i\}_{i=1}^{\infty}$, is a basis for the topology then the assertion holds for the set $Y = \bigcap_i H\Omega_i$.

Remark: By a similar argument one can also see that if H as in the lemma is cyclic, say generated by a homeomorphism ϕ, then there exists a set Z of full measure such that for $z \in Z$, $\{\phi^i(z) | i = 1, 2, ...\}$ is dense in X.

Thus when there is ergodicity with respect to a measure of full support then the orbits of almost all points are dense. A considerably stronger implication of ergodicity to asymptotic behaviour of orbits comes from the following so called individual ergodic theorem of Birkhoff.

2.2. Theorem: *Let (X, μ) be a measure space such that $\mu(X) = 1$. Then for any measurable transformation ϕ preserving the measure μ and any $f \in L^1(X, \mu)$ the sequence of functions*

$$S_k(f)(x) = \frac{1}{k}\Sigma_{i=0}^{k-1} f(\phi^i(x))$$

converges almost everywhere and in L^1.

Similarly, if $\{\phi_t\}$ is a measurable flow on X (that is, $\{\phi_t\}_{t \in \mathbb{R}}$ is a one-parameter group of measurable transformations of X and $(t,x) \to \phi_t(x)$ is measurable) preserving μ then for any $f \in L^1(X, \mu)$ the functions S_T, $T > 0$, defined by

$$S_T(f)(x) = \frac{1}{T}\int_0^T f(\phi_t(x))\mathrm{d}t$$

converge almost everywhere and in L^1, as $T \to \infty$.

In either of the cases the limit f^* is invariant under the action (that is, $f^*(\phi(x)) = f^*(x)$ a.e. or $f^*(\phi_t(x)) = f(x)$ a.e. for all $t \in \mathbb{R}$, respectively) and $\int f^* \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.

Observe that when the action is ergodic with respect to μ then for any f as in the theorem f^* has to be the constant $\int f \, \mathrm{d}\mu$ a.e.. The convergence of averages as above for a point x signifies how the orbit of x (under the transformation ϕ or the flow $\{\phi_t\}$) is 'distributed' in the ambient space. This will be formulated rigorously later. Here let me discuss only a special case that is typical for orbits of an ergodic transformation (or flow). Let X be a locally compact second countable space. A sequence $\{x_i\}$ in X is said to be *uniformly distributed* with respect to a measure μ on X if for any bounded continuous function f on $\operatorname{supp}\mu$ (the support of μ)

$$\frac{1}{k}\Sigma_{i=0}^{k-1} f(x_i) \longrightarrow \int_X f \, d\mu,$$

as $k \to \infty$. Similarly a curve $\{x_t\}_{t\geq 0}$ in X is said to be uniformly distributed with respect to a measure μ on X if

$$\frac{1}{T}\int_0^T f(x_t)\, dt \longrightarrow \int_X f \, d\mu,$$

as $T \to \infty$, for all bounded continuous functions f on $\operatorname{supp}\mu$. It can be verified that if a sequence $\{x_i\}$ is uniformly distributed with respect to a measure μ with full support then for any Borel subset E such that the boundary of E in X is of μ measure 0,

$$\frac{\#\{i \mid 0 \leq i \leq n-1 \text{ and } x_i \in E\}}{n} \longrightarrow \mu(E),$$

as $n \to \infty$ (# stands for the cardinality of the set following the symbol). A similar assertion also holds for a uniformly distributed curve $\{x_t\}$, with the cardinality replaced by the Lebesgue measure of the set of t such that $x_t \in E$. The ergodic theorem readily implies the following:

2.3. Corollary: *Let X be a locally compact second countable Hausdorff space and let ϕ be a homeomorphism of X (respectively, let $\{\phi_t\}$ be a continuous one-parameter flow on X). Let μ be a probability measure invariant and ergodic with respect to ϕ (respectively $\{\phi_t\}$). Then there exists a Borel subset Y of X such that $\mu(X - Y) = 0$ and for any $y \in Y$, the sequence $\{\phi^i(y)\}$ (respectively the curve $\{\phi_t(y)\}_{t\geq 0}$) is uniformly distributed with respect to the measure μ.*

Let me now come to the question of proving ergodicity. To begin with let us consider a general set up. Let (X, μ) be a measure space with $\mu(X) < \infty$ and let T be an invertible measure-preserving transformation of X. Then one defines a unitary operator U_T on $L^2(X, \mu)$ by setting $(U_T f)(x) = f(T^{-1}x)$ for all $f \in L^2(X, \mu)$ and $x \in X$. It is easy to see that T is ergodic if and only if any $f \in L^2(X, \mu)$ such that $U_T f = f$ a.e. is constant a.e. or, equivalently, 1 is an eigenvalue of U_T with multiplicity one. This observation has been one of the major tools of proving ergodicity for many systems, ever since it was introduced by Koopman in 1931 (see [H] for some historical details). It is easy to apply it to the class of rotations of the circle, using Fourier series, and conclude that a rotation is ergodic (with respect to the angle measure) if and only if it is an irrational rotation. Similarly for affine transformations of tori using Fourier expansions one proves the following ([CFS], [H], [Wa]).

2.4. Theorem: *Let $\phi = T_a \circ \overline{A}$ be an affine transformation of $\mathbb{T}^n$, $n \geq 2$, where $a \in \mathbb{T}^n$ and $A \in GL(n, \mathbb{Z})$. Then ϕ is ergodic with respect to the Haar measure on $\mathbb{T}^n$ if and only if the following holds: no root of unity other than 1 is an eigenvalue of A and the subgroup generated by a and $\{x^{-1}\phi(x) \mid x \in \mathbb{T}^n\}$ is dense in $\mathbb{T}^n$.*

When a is the identity element the condition for ergodicity reduces to no root of unity being an eigenvalue of A.

Theorem 2.4 generalises to nilmanifolds in a simple form as follows (see [Gr] and [P]):

2.5. Theorem: *Let G be a nilpotent Lie group and Γ be a lattice in G. An affine automorphism ϕ of G/Γ is ergodic if and only if its factor $\overline{\phi}$ on $G/[G,G]\Gamma$ is ergodic.*

Let us now consider a general homogeneous space $X = G/C$ where G is a Lie group and $C \in \mathcal{F}(G)$. Through the rest of the section we shall consider only actions of subgroups of G, by translations; results for groups of affine automorphisms can be deduced with more technical work which we shall not go into. Let m denote the G-invariant probability measure on G/C. Then we can define a unitary representation of G over $L^2(X, m)$ by $(U_g f)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in L^2(X, m)$ and $x \in X$. Observe that for $a \in G$, U_a is the unitary operator associated to the translation T_a in the sense as above. Thus to prove T_a to be ergodic it is enough to show that U_a does not fix any nonzero vector orthogonal to all constant functions in $L^2(X, m)$. In doing this one can use the representation $g \mapsto U_g$ of G. A detailed study along these lines has led to satisfactory criteria for ergodicity of subgroup actions. Before going to the general results let me describe some simple observations which enable proving ergodicity for a fairly wide class of actions.

2.6. Lemma (Mautner phenomenon): *Let G be a topological group and $g \mapsto U_g$ be a (strongly continuous) unitary representation of G over a Hilbert space $\mathcal{H}$. Let $a, b \in G$ be such that $a^i b a^{-i}$ converges to the identity element as $i \to \infty$. Then any element of $\mathcal{H}$ which is fixed by U_a is also fixed by U_b; that is, for $p \in \mathcal{H}$, $U_a(p) = p$ implies that $U_b(p) = p$.*

The above lemma was noted first in [AG]; see also [D1]. More generally the following is also true [Mar7]:

2.7. Lemma: *Let G, $\mathcal{H}$ and U be as in Lemma 2.6. Let F be any subgroup of G. Let $x \in G$ be such that for any neighbourhood Ω of the identity in G, x is contained in the closure of $F\Omega F$. Then $U_x(p) = p$ for any $p \in \mathcal{H}$ such that $U_f(p) = p$ for all $f \in F$.*

The lemma follows immediately from the fact that $g \mapsto \| U_g(p) - p \|^2 = 2\| p \|^2 - 2\mathrm{Re} < U_g(p), p >$ is a continuous function on G which is constant on the double cosets of the subgroup F; the latter assertion follows from the last expression and the unitarity of the representation U.

Let G be a Lie group and let e be the identity element in G. For $a \in G$ we denote by $U^+(a)$ and $U^-(a)$ the subgroups defined by

$$U^+(a) = \{g \in G \mid a^i g a^{-i} \to e \text{ as } i \to \infty\}$$

and

$$U^-(a) = \{g \in G \mid a^{-i} g a^i \to e \text{ as } i \to \infty\};$$

these are 'horospherical subgroups' associated to a (see § 3 for more about them). Let A_a be the closed subgroup generated by $U^+(a)$ and $U^-(a)$; we shall call this the *Mautner subgroup* associated to a. It can be shown that A_a is a normal subgroup of G. If $A_a = G$ for an element a then Lemma 2.6 implies that the translation of G/C by a is ergodic for any $C \in \mathcal{F}(G)$; any $f \in L^2$ which is fixed by U_a is also fixed by $U_{a^{-1}}$ and hence by $U^+(a)$ and $U^-(a)$ by Lemma 2.6 and so by the closed subgroup generated by them. One immediate consequence of this is the following.

2.8. Corollary: *Let $G = SL(n, \mathbb{R})$ and $C \in \mathcal{F}(G)$. Let $a \in G$ be a matrix which has an eigenvalue λ (possibly complex) such that $|\lambda| \neq 1$. Then the translation T_a of G/Γ is ergodic. In particular, the geodesic flows of surfaces with constant negative curvature and finite area are ergodic.*

The analogous statement also follows for any irreducible lattice in any semisimple Lie group with no nontrivial compact factors; in this case we demand that the linear transformation $\mathrm{Ad}\, a$ of the Lie algebra have an eigenvalue of absolute value other than 1 (for $SL(n, \mathbb{R})$ this condition is equivalent to the one in the above Corollary).

More generally the Mautner phenomenon yields the following.

2.9. Corollary: *Let G be a Lie group and $C \in \mathcal{F}(G)$. Let $a \in G$ and A_a be the Mautner subgroup associated to a. Let $X = G/C$ and m be the G-invariant probability measure on X. Then we have the following:*

i) if $f \in L^2(X, m)$ and $U_a f = f$ then $U_g f = f$ for all $g \in A_a$.

ii) if $G' = G/A_a$, $C' = \overline{CA_a}/A_a$ and $a' = a\overline{CA_a}$ then the translation T_a of $X = G/C$ is ergodic if and only if the translation $T_{a'}$ of G'/C' by a' is ergodic.

This reduces the question of ergodicity of translations to the special case when the Mautner subgroup is trivial. It can be seen that this condition is equivalent to all the eigenvalues of $\mathrm{Ad}\, a$ being of absolute value 1. Lemma 2.7

can be used to deal with some part of this case also. I will however illustrate this only with the following simple example.

2.10. Proposition: *Let $G = SL(2, \mathbb{R})$, Γ be a lattice in G and $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then T_a is ergodic. In particular, the horocycle flow is ergodic.*

Proof: Let F be the cyclic subgroup generated by a. A simple computation shows that for any rational number α and $\epsilon > 0$ there exists a $\theta \in \mathbb{R}$ such that $0 < |\theta| < \epsilon$ and $\begin{pmatrix} \alpha & 0 \\ \theta & \alpha^{-1} \end{pmatrix} \in F \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} F$. This shows that for any neighbourhood Ω of the identity in G the closure of $F\Omega F$ contains all diagonal matrices. Hence by Lemma 2.7 any F-invariant L^2 function on G/Γ is also invariant under all diagonal matrices and hence constant by Corollary 2.8. This completes the proof.

Using a similar argument and the Jacobson-Morosov Lemma (see [J]) one can prove the following assertion for cyclic subgroups F; the general assertion can then be deduced using Lemma 7.1 of [Mo1].

2.11. Theorem: *Let G be a semisimple Lie group with finite center and no compact factors and Γ be an irreducible lattice in G. Let F be a subgroup of G. Then the action of F on G/Γ is ergodic if and only if $\overline{F}$ is noncompact.*

This result is essentially due to C. C. Moore [Mo1] though he did not put it in this form; on the other hand the results in [Mo1] are stronger inasmuch as they deal with mixing properties as well. The reader is also referred to [D1], where similar results are proved, in a more general context, along the line of argument indicated above. For a general semisimple group the conditions for ergodicity can now be seen to be the following.

2.12. Theorem: *Let G be a connected semisimple Lie group. Let K be the maximal compact connected normal subgroup and let G' be the smallest normal subgroup such that the quotient is compact. Let $C \in \mathcal{F}$ and $a \in G$. Then the translation T_a of G/C is ergodic if and only if $G'C$ is dense in G and the factor transformation of T_a on any irreducible factor of G/KC is ergodic.*

Let me now proceed to describe the results in the general case. They are obtained by a closer analysis of the spectrum of the associated unitary operator (see [Au], [Mo1], [D3], [BM], [Z2]).

A subgroup F of a Lie group L is said to be Ad-*compact* if the group of linear transformations $\{\mathrm{Ad}\, x : \mathcal{L} \to \mathcal{L} \mid x \in F\}$, where $\mathcal{L}$ is the Lie algebra of L, is contained in a compact subgroup of $GL(\mathcal{L})$. Now let G a be a Lie group. Then for any subgroup F of G there exists a unique minimal normal closed subgroup M_F such that FM_F/M_F is Ad-compact as a subgroup of G/M_F (see [Mo2]); I shall call M_F the Mautner-Moore subgroup associated to F. The general ergodicity criterion is given by the following theorem due to C. C. Moore:

2.13. Theorem (Moore, 1980): *Let G be a Lie group, F be a subgroup of G and let M_F be the Mautner-Moore subgroup associated to F. Then the following holds:*

i) If $g \mapsto U_g$ is a unitary representation of G over a Hilbert space $\mathcal{H}$ and $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{H}$ invariant under U_f for all $f \in F$ then $U_g(v) = v$ for all $v \in \mathcal{V}$ and $g \in M_F$.

ii) If $C \in \mathcal{F}(G)$ then the action of F on G/C is ergodic if and only if the action of $\overline{FM_F}$ on G/C is ergodic if and only if the action of $\overline{FM_F}/M_F$ on the factor G'/C' is ergodic, where $G' = G/M_F$ and $C' = \overline{CM_F}/M_F$.

iii) If $C \in \mathcal{F}(G)$ and $M_F C$ is dense in G then the action of F on G/C is ergodic.

Assertions (ii) and (iii) may be easily seen to follow from assertion (i). A proof of (i) for one-parameter subgroups may be found in [Mo2]; the case of (infinite) cyclic subgroups F can be deduced from that of one-parameter subgroups by embedding a suitable power of the generating element in a one-parameter subgroup (see [D3], § 6 for an idea of this) and the general case may be concluded from the latter using Lemma 7.1 of [Mo1].

One can see that if G is a nilpotent Lie group and $C \in \mathcal{F}(G)$ then for $a \in G$, T_a is not ergodic unless $[G, G] \subset M_a$. Therefore for translations Theorem 2.5 is a special case of the above general result. For a semisimple Lie group the criterion yields Theorems 2.11 and 2.12.

We shall now see a characterisation of ergodicity of translations in terms ergodicity of its factors on simpler homogeneous spaces. It may be recalled that in a connected Lie group G there is a smallest closed normal subgroup R, namely the solvable radical, such that G/R is a semisimple Lie group. On the other hand there also exists a smallest closed normal subgroup S such that G/S is solvable; if we set $G_0 = G$ and inductively define G_{i+1}, $i \geq 0$, to be the closure of $[G_i, G_i]$, then by dimension considerations G_i is the same subgroup for all large i and the common subgroup can be seen to have the desired property.

2.14. Theorem: *Let G be a connected Lie group and $C \in \mathcal{F}(G)$. Let R be the solvable radical of G and S be the smallest closed normal subgroup such that G/S is solvable. Then for $a \in G$, $T_a : G/C \to G/C$ is ergodic if and only if the translations $T_a : G/\overline{RC} \to G/\overline{RC}$ and $T_a : G/\overline{SC} \to G/\overline{SC}$ are ergodic; (the quotient spaces $G/\overline{RC}$ and $G/\overline{SC}$ can be realised as homogeneous spaces of G/R and G/S and when this is done the condition is equivalent to T_{aR} and T_{aS} being ergodic).*

A simple argument for deducing this from Theorem 2.13 is given by A. N. Starkov in [St2]. The criterion was proved earlier in [D3] under an additional condition of 'admissability' of C, that there exist a Lie group F, a lattice Γ in F and a continuous homomorphism $\psi : F \to G$ such that $\overline{\psi(\Gamma)} = C$; the condition holds trivially for a lattice. In [BM], where a detailed analysis of conditions for ergodicity and spectrum of flows was carried out for flows on homogeneous of finite volume, the above assertion was proved under a weaker 'admissability condition' that there exists a solvable Lie subgroup A of G containing the radical such that AC is closed (or equivalently $A/A \cap C$ is compact). Subsequently it was proved independently by D. Witte [Wi1] and A. N. Starkov [St1] that the latter admissability condition in fact holds for all $C \in \mathcal{F}(G)$; see also [Z3] for a more general result in this regard.

The characterisation in terms of factors would be complete with the following theorem of Brezin and Moore [BM]; (actually in [BM] only flows induced by one-parameter subgroups are considered, but the result for translations follows from it via suitable embeddings).

2.15 Theorem: *Let G be a connected solvable Lie group, $C \in \mathcal{F}(G)$ and $a \in G$. Then the translation T_a of G/C is ergodic if and only if the factor on the maximal euclidean quotient is ergodic.*

In view of Theorems 2.11, 2.12, 2.14 and 2.15 to complete the story of ergodicity of the translations it only remains to know criteria in the case of euclidean solvmanifolds. These are indeed completely understood and are described in [BM] (see Corollary 5.3 there). I will however not go into the details, since it would involve introducing more notation which does not seem worthwhile at this stage, but content myself by commenting that the criterion involves 'rational structures' and is comparable to the case of tori except that it is more elaborate.

2.16. Remark: In the following sections we will be largely concerned with actions of subgroups generated by unipotent elements; an element u of a Lie group G is said to be unipotent if $\mathrm{Ad}\, u$ is a unipotent linear transformation, that is, $(\mathrm{Ad}\, u - I)^n = 0$ for some n, where I is the identity transformation (equivalently, 1 is the only eigenvalues of $\mathrm{Ad}\, u$). It may be worthwhile to note some additional properties in this case with regard to ergodicity. Let G be a connected Lie group and $C \in \mathcal{F}(G)$. Let U be a closed subgroup of G generated by unipotent elements (as elements of G). Then there exists a closed normal subgroup L of G containing U such that LC is closed and the action of U on LC/C is ergodic; note that the action on G/C consists of disjoint union of closed invariant sets on each of which the action corresponds to the U action on LC/C. This can be easily deduced from Theorem 2.12. In view of the observation in studying the actions we need to consider only ergodic ones. Now let G, C and U be as above and suppose that the U-action on G/C ergodic. Then the connected component C^0 of the identity in C is a normal subgroup of G (see [Wil] for the ideas involved; there U is taken to be cyclic but the argument generalises). Since it involves no loss of generality to go modulo a closed normal subgroup, this reduces the study to the case when C is a lattice.

In ergodic theory one also studies various other notions, such as mixing, entropy, Kolmogorov mixing, Bernoullicity etc. and these have also been studied for the systems under consideration. Some of the papers referred to above indeed deal with these properties as well; many papers have results on mixing, which is a property related to the spectrum. I will however not discuss the results here. The reader is referred to the survey articles [D13] and [Mar7] for some details in this regard. Let me however recall for reference in the sequel that a measure-preserving transformation T of a probability space (X, μ) is said to be *weakly mixing* if the constant functions are the only eigenfunctions of U_T and that this condition is equivalent to the cartesian square $T \times T$ of T (acting on $X \times X$) is ergodic.

Let me conclude this section on ergodicity with some remarks about what happens when ergodicity does not hold. While almost all orbits are dense when ergodicity holds, *no* orbit is dense if ergodicity does not hold. This can easily be read off from the criteria as above. In fact when a translation or flow is not ergodic, there exist invariant nonconstant C^∞ functions (see [BM]). It is shown in [St3] that given a flow on a homogeneous space of finite volume, induced by a one-parameter subgroup, the space can be partioned in to smooth invariant submanifolds such that all of them have smooth measures invariant and ergodic with respect to the flow; further each of the submanifolds is finitely covered by a homogeneous space of a Lie group by a lattice and the restriction of the flow to the submanifolds is the image of a flow on the homogeneous space, induced by a one-parameter subgroup of the Lie group. This enables one to reduce the study of flows on homogeneous spaces as above to the ergodic case.

§ 3 Dense orbits; some early results

In the previous section we saw the conditions for ergodicity of group actions by translations (and certain affine automorphisms) of homogeneous spaces with finite invariant measure and it was also noted that when ergodicity holds the orbits of almost all points are dense in the homogeneous space and, when the subgroup is either cyclic or a one-parameter subgroup, almost all of them (not necessarily the same set) are also uniformly disributed with respect to the probability measure invariant under the ambient Lie group. We now address the next set of questions arising naturally from this: are all orbits dense? if not, can we describe the closures of the orbits which are not dense?

what can we say about the distribution of an individual orbit if it is not uniformly distributed?

Through the remaining sections the aim will be to discuss a class of systems with a remarkably orderly behaviour in this respect.

I will begin by recalling that for an irrational rotation all orbits are dense and uniformly distributed with respect to the angle measure. This may be deduced by observing that (because of commutativity) any two orbits are translates of each other so once some orbit is dense (respectively, uniformly distributed) the same has to hold for all the orbits; the result can also be found in books on Number Theory, with different proofs. In the same way for a translation of the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ by $(\alpha_1, ..., \alpha_n)$, where $\alpha_1, ..., \alpha_n \in \mathbb{R}$, all orbits are dense and uniformly distributed in the torus (with respect to the Haar measure) whenever the translation is ergodic, namely when $\alpha_1, ..., \alpha_n$ and 1 are linearly independent over $\mathbb{Q}$. The density assertion is known as Kronecker's theorem and the uniform distribution assertion is a theorem of H. Weyl (see [CFS], [Wa] for details).

An action of a group G on a locally compact space X is said to be *minimal* if there is no proper closed nonempty G-invariant subset; clearly an action is minimal if and only if orbits of all points are dense in X. A homeomorphism is said to be minimal if the corresponding action of the cyclic group is minimal. Our observation above means that every ergodic translation of a torus is minimal.

If we move from translations to affine automorphisms of tori ergodicity no longer implies all orbits being dense (leave alone uniformly distributed). In fact for any automorphism $\overline{A}$, $A \in GL(n, \mathbb{Z})$, the orbit of any point of the form $q + \mathbb{Z}^n \in \mathbb{R}^n/\mathbb{Z}^n$, where q is a vector with rational coordinates, is periodic (finite) and hence can not be dense even if $\overline{A}$ is ergodic. In general, apart from the periodic orbits there are also a whole lot of other orbits which are not dense (see [DGS]). Nevertheless for some affine automorphisms the situations is different:

3.1. Theorem (Furstenberg, 1961): *Let $A \in GL(n, \mathbb{Z})$ be a unipotent matrix and let $a \in \mathbb{R}^n$ be such that the affine automorphism $T = T_a \circ \overline{A}$ of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is ergodic. Then all orbits of $T_a \circ \overline{A}$ are uniformly distributed (and in particular dense) in $\mathbb{T}^n$.*

The result is proved by showing that the Haar measure is the only probability measure on $\mathbb{T}^n$ invariant under an affine automorphism T as in the theorem. A homeomorphism T of a locally compact topolological space X is called *uniquely ergodic* if it admits only one invariant probability measure. Thus an affine automorphism as in the above theorem is uniquely ergodic. A straightforward argument shows that if a homeomorphism of a compact space is uniquely ergodic and μ is the the unique invariant probability measure then the orbits of all points of $\operatorname{supp}\mu$ are uniformly distributed with respect to μ (see [F1] or [CFS], for instance).

Theorem 3.1 yields an ergodic-theoretic proof of the following classical result of H.Weyl on the distribution of the fractional parts of values of a polynomial; see [CFS], Ch.7, Section 2, for a proof.

3.2. Theorem: *Let $P(t) = a_0t^n + a_1t^{n-1} + ... + a_n$ be a polynomial of degree $n \geq 1$, with real coefficients. Suppose that at least one of $a_k, 0 \leq k \leq n-1$, is irrational. Then the sequence $\xi_n = \langle P(n) \rangle$ of fractional parts of $P(n)$ is uniformly distributed in the interval* $[0, 1]$.

In [Gr] L. Green proved that any ergodic nilflow is minimal. An interesting application of this to diophantine approximation was given in [AH], proving the following analogue of one of Weyls results (cf. [We], Theorem 14) on uniform distribution; we recall here that a set of integers is said to be relatively dense if the difference between successive integers in the set is bounded above.

3.3 Theorem: *Let $P_i(t) = \Sigma_{j=1}^n a_{ij}t^j$, where a_{ij} are integers such that $\Sigma_j |a_{ij}| > 0$ for each $i = 1, ...n$. Let $\alpha_1, ..., \alpha_n$ be real numbers which together with 1 form a linearly independent set over $\mathbb{Q}$. Then for any $\epsilon > 0$ and $\theta_1, ..., \theta_n \in \mathbb{R}$ there exists a relatively dense set M of integers such that for each $i = 1, ..., n$ and $m \in M$, $\alpha_i P_i(m) - \theta_i$ differs from an integer by at most ϵ.*

Furstenberg's theorem was generalised by W. Parry to affine automorphisms of nilmanifolds, proving unique ergodicity of $T = T_a \circ \overline{A}$ when T is ergodic and A is unipotent (see [P]).

In [AB] Auslander and Brezin obtained a generalisation of Weyl's criterion for uniform distribution of sequences on tori in the setting of solvmanifolds; their approach is more general than the present one, in that they consider also avarages along more general sequences of sets rather than the intervals in $\mathbb{Z}$ or $\mathbb{R}$ in our discussion. In the course of their study they prove in particular that for a nilpotent Lie group G and any lattice Γ in G the action of any subgroup H of G on G/Γ is uniquely ergodic whenever it is ergodic; for cyclic subgroups and, with a little argument, for connected Lie subgroups this follows also from the result of Parry. As for the case of a general solvable case, while the analogue of Weyl's criterion proved in [AB] would apply in particular to actions of cyclic and one-parameter subgroups, the precise implications to uniform distribution (or density) of orbits do not seem to have been analysed in literature, in terms of the 'position' of the subgroup in the ambient group.

Let us now come to the case of semisimple groups, starting with the simplest case of $SL(2, \mathbb{R})$. In this case results on orbit closures which can be compared to what I have recalled for affine transformations goes back to a classical paper of G. A. Hedlund [He] on the horocycle flow; see also [Man] and [Gh] for exposition and proof.

3.4. Theorem (Hedlund, 1936): *Let $G = SL(2, \mathbb{R})$ and Γ be a lattice in G. For $t \in \mathbb{R}$ let $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and let $H = \{h_t \mid t \in \mathbb{R}\}$. Then every H-orbit on G/Γ is either periodic or dense in G/Γ. If G/Γ is compact then all orbits are dense while if G/Γ is noncompact then the set of points with periodic orbits is nonempty and consists of finitely many immersed cylinders, each of which is dense in G/Γ.*

Thus in the compact quotient case the action is minimal. When the quotient G/Γ is noncompact there are finitely many 'cusps'; these correspond to the cusps of the surface of constant negative curvature, they being finitely many when the area is finite (see [B]). To each cusp there corresponds a one-parameter family of periodic orbits, one periodic orbit with each value of the period, which together form an immersed cylinder in G/Γ.

Hedlund's ideas were followed up in a paper of L. Greenberg [Gre], where the author also noted the following simple *duality principle,* which interrelates results on orbits of flows on finite-volume homogeneous spaces with those for lattice actions on certain 'large' homogeneous spaces and in turn for actions on linear spaces, as we shall see below.

3.5. Proposition: *Let G be a Lie group, C and H be a closed subgroups of G. Then for $g \in G$ the H-orbit of gC is dense in G/C if and only if the C-orbit of $g^{-1}H$ is dense in G/H.*

The proof is immediate, both the statements being equivalent to the set HgC being dense in G. Via this observation the study of subgroup actions on homogeneous spaces can be related to the study orbits of lattices (or more generally finite covolume subgroups) on linear spaces as follows: Let G be a Lie group and $C \in \mathcal{F}(G)$ and consider a linear action of G on a finite-dimensional $\mathbb{R}$-vector space V. One would then like to understand, for $v \in V$, the closure of the orbit Cv. Fix a point $v_0 \in V$ and let $v \in Gv_0$. The Proposition implies in particular that for $g \in G$, Cgv_0 is dense in $Gv = Gv_0$ if the orbit $Hg^{-1}C/C$ of $g^{-1}C$ is dense in G/C, H being the isotropy subgroup $\{x \in G \mid xv_0 = v_0\}$ of v_0. Greenberg also proved the following theorem, which via the duality principle generalises the compact quotient case of Theorem 3.4.; in the sequel I shall follow the terminology that a lattice Γ in a Lie group G is said to be *uniform* if the quotient G/Γ is compact and *nonuniform* otherwise.

3.6. Theorem (Greenberg, 1963): *Let $G = SL(n, \mathbb{R})$ and let Γ be a uniform lattice in G. Then for any $v \in \mathbb{R}^n - \{0\}$ the Γ-orbit Γv is dense in $\mathbb{R}^n$.*

Similarly if $G = Sp(2n, \mathbb{R})$, the symplectic group realised canonically as a subgroup of $SL(2n, \mathbb{R})$, and Γ is a lattice in G then for the natural action of G on $\mathbb{R}^{2n}$ the Γ-orbit of any nonzero vector is dense in $\mathbb{R}^{2n}$.

Theorem 3.6 implies in particular that if $f(x_1, \ldots, x_n)$ is a real quadratic form in n variables then for any uniform lattice Γ in $SL(n, \mathbb{R})$, the set $\{f(\gamma_{11}, \gamma_{21}, \ldots, \gamma_{n1}) \mid \gamma = (\gamma_{ij}) \in \Gamma\}$ is dense in $\mathbb{R}$. The particular case of this with $n = 2$ and f a positive definite form was proved by K. Mahler [Ma], who conjectured it to be true for indefinite forms as well.

Many other number theoretic applications involve $SL(n, \mathbb{Z})$, which is a nonuniform lattice. Consider first its natural action on $\mathbb{R}^n$. In this case one does not expect all orbits to be dense, since the orbits of points with rational coordinates are in fact discrete. It was proved in [D], following the ideas of the paper of Greenberg that those and their scalar multiples are in fact the only exceptions. It was also proved that, for even n, the orbit of a point v under the subgroup $Sp(n, \mathbb{Z})$, consisting of integral $n \times n$ symplectic matrices, is dense if v is not a scalar multiple of a rational point; see Theorem 3.7 below for a more general result.

The reader would notice that in terms of the actions on homogeneous spaces Greenberg's result involves considering orbits of a rather large subgroup. ¿From the point of view of dynamics this suggests asking what happens for smaller subgroups. One class of subgroups to which the study was extended in the intermediate period, that needs to be mentioned, is the class of horospherical subgroups.

A subgroup U of a Lie group G is said to be *horospherical* if there exists a $a \in G$ such that

$$U = \{g \in G \mid a^i g a^{-i} \to e \text{ as } i \to \infty\},$$

e being the identity element in G; specifically U is called the horospherical subgroup corresponding to a. It is not difficult to see that a horospherical subgroup is always a connected (not necessarily closed) Lie subgroup. If G is a semisimple Lie group then a subgroup is horospherical if and only if it is the unipotent radical of a parabolic subgroup.

It may be noticed that the subgroup $\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$ of $SL(2, \mathbb{R})$, which defines the horocycle flow, is a horospherical subgroup in $SL(2, \mathbb{R})$ in the above sense. In the case of higher-dimensional manifolds of constant negative curvature and finite volume, the classical horospherical foliations associated to the geodesic flow are given by orbits of horospherical subgroups; in the notation as in § 1.4, if U is the horospherical subgroup corresponding to g_t, $t > 0$, then the images of U-orbits on G/Γ in $M\backslash SO(n, 1)/\Gamma$ give the horospherical foliation.

Generalising the minimality result of Hedlund for the horocycle flow on compact quotients of $SL(2, \mathbb{R})$ by lattices, it was proved by Veech in [V1] that if G is a connected semisimple Lie group with no nontrivial compact factors and Γ is a uniform lattice in G then the action of any maximal horospherical subgroup of G on G/Γ is minimal. In [V2] he proved another minimality result (see [V2], Theorem 1.3) which shows in particular that for G and Γ as in the preceding assertion the action of a horospherical subgroup of G is minimal whenever it is ergodic. It is noted that the argument uses ideas from Anosov's proof of density of leaves of horospherical foliations (see [An], Theorem 15), which itself is also a generalisation of Hedlund's theorem as above. Using a similar argument it was shown in [D2] that if G is any connected Lie group, Γ is a uniform lattice in G, U is the horospherical subgroup corresponding to a weakly mixing (see § 2) affine automorphism $T = T_a \circ \overline{A}$, where $a \in G$ and A is an automorphism of G such that $A(\Gamma) = \Gamma$, then the U-action on G/Γ is minimal; by the horospherical subgroup corresponding to $T_a \circ \overline{A}$ we mean the subgroup $\{g \in G | (\sigma_a \circ A)^i(g) \to e \text{ as } i \to \infty\}$, where σ_a is the inner automorphism corresponding to a and e is the identity element. While these results pertain only to minimality, various results were also obtained around the same time on unique ergodicity of horospherical flows, which in particular imply minimality; it would however be convenient to postpone going over them, to § 5 where I will be discussing invariant measures in greater detail.

For a general, not necessarily uniform, lattice Γ the orbit closures of actions of horospherical subgroups on G/Γ were considered in [DR], [D15], [D16] and [St5]; I will take up these results in the next section after introducing a certain perspective. Let me conclude this section with some results on diophantine approximation, in the spirit of Theorem 3.6 but involving nonuniform lattices, obtained in my joint paper [DR] with S. Raghavan.

3.7. Theorem: *Let $\Gamma = SL(n, \mathbb{Z})$ and $1 \leq p < n$. Let V be the cartesian product of p copies of $\mathbb{R}^n$, equipped with the componentwise action of Γ. Then for $v = (v_1, ..., v_p)$ the Γ-orbit is dense in V if and only if the subspace of $\mathbb{R}^n$ spanned by the vectors $v_1, ..., v_p$ does not contain any nonzero rational vector (vector with rational coordinates with respect to the natural basis of $\mathbb{R}^n$).*

It was also shown that if n is even and $Sp(n, \mathbb{Z})$ is the group of integral symplectic matrices (with respect to a nondegenerate symplectic form) the $Sp(n, \mathbb{Z})$-orbit of any symplectic p-tuple, where $1 \leq p \leq n/2$, is dense in the space of symplectic p-tuples; a p-tuple $(v_1, ..., v_p)$ is said to be symplectic if $\omega(v_i, v_j) = 0$ for all $1 \leq i, j \leq p$, ω being the symplectic form defining the symplectic group.

The theorem can be applied to study the values of systems of linear forms, over integral points and also over primitive integral points. We recall that an integral n-tuple is said to be *primitive* if the entries are not all divisible by a positive integer exceeding 1.

3.8. Corollary: *Let $\xi_1, ..., \xi_p$ be real linear forms in n variables, where $1 \le p < n$. Suppose that no nontrivial linear combination of $\xi_1, ..., \xi_p$ is a rational form (that is, if $\lambda_1, ..., \lambda_p \in \mathbb{R}$ are such that $\Sigma\, \lambda_i \xi_i$ is a form with rational coefficients, then $\lambda_i = 0$ for all i). Then for any $a_1, ..., a_p \in \mathbb{R}$ and $\epsilon > 0$ there exists a primitive integral n-tuple $(x_1, ..., x_n)$ such that*

$$|\,\xi_i(x_1, ..., x_n) \;-\; a_i| \;<\; \epsilon \text{ for all } i = 1, ..., p.$$

§ 4 Conjectures of Oppenheim and Raghunathan

One major aspect to be noticed in Hedlund's theorem is that for the systems considered, even when there are both dense as well as non-dense orbits (as in the noncompact quotient case) the closures of all orbits are 'nice' objects geometrically. This is also the case Greenberg's generalisation of Hedlund's theorem and the other similar results which I mentioned above. From the point of view of dynamics this is quite an unusual behaviour. One would naturally wonder how general this phenomenon is. The question got an added impetus from an observation of M. S. Raghunathan that a well-known conjecture going back to a paper of A. Oppenheim from 1929, on values of indefinite quadratic forms at integral points would be settled if an analogue of Hedlund's theorem is proved for the case $G = SL(3, \mathbb{R})$ and $\Gamma = SL(3, \mathbb{Z})$ for the action of the special orthogonal group $SO(2,1)$ corresponding to a quadratic form of signature $(2,1)$. Let me recall the conjecture and indicate the connection.

Let $Q(x_1, ..., x_n) = \Sigma_{ij}\, a_{ij}x_ix_j$ be a quadratic form with real coefficients. We are interested in the set of values of Q on integral n-tuples and especially in the question whether it is dense in $\mathbb{R}$. Assume for simplicity that Q is nondegenerate. It is clear that if Q is a definite form (either positive definite or negative definite) then the set of values under consideration is discrete. Similarly if the coefficients a_{ij} are all rational multiples of a fixed real number (equivalently if all the ratios are rational) then also the set as above is discrete. The conjecture in question was that if Q is indefinite and not a multiple of a rational quadratic form and $n \ge 3$ then the set of values at integral points is indeed dense; the idea of the conjecture can be traced back to a paper of A. Oppenheim in 1929, though the original statement is somewhat weaker. The problem was attacked by several number theorists, including H. Davenport, H. Heilbronn, B. J. Birch, H. Ridout, A. Oppenheim, G. L. Watson and others, mainly using the circle method and through many papers together by the sixties the conjecture was known to hold for all quadratic forms when $n \ge 21$ and in special cases in lower number of variables. There has also been some work on the problem in the recent years by number theoretic methods by H. Iwaniec, R. C. Baker and

H. P. Schlickewey. I will not go into the details of this and various nuances of the conjecture but would refer the interested reader to [Le], [Mar7], [Gh] and [B2]. I may mention however that the condition $n \geq 3$ can not be omitted; it is easy to see that the quadratic form $x_1(\lambda x_1 - x_2)$ does not take nonzero values arbitrarily close to 0 over integral x_1, x_2, if λ is a badly approximable number, namely an irrational number with bounded partial quotients.

If Q is a quadratic form satisfying the conditions of the conjecture then $Q(\mathbb{Z}^n) = Q(H\Gamma\mathbb{Z}^n)$, where $\Gamma = SL(n,\mathbb{Z})$ and H is the special orthogonal group corresponding to Q, namely $H = \{g \in SL(n,\mathbb{R}) \mid Q(gv) = Q(v) \text{ for all } v \in \mathbb{R}^n\}$; we view n-tuples as column vectors and consider the action of matrices by left multiplication. Now if it is shown that the orbit $H\Gamma/\Gamma$ of the H action on $SL(n,\mathbb{R})/\Gamma$ is either dense or closed in $SL(n,\mathbb{R})/\Gamma$, à la Hedlund, then it would follow that $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ unless $H\Gamma$ is closed. One can see that the latter holds only if Q is a multiple of a form with rational coefficients, a condition which is disallowed in the hypothesis of the conjecture. Thus the conjecture can be approached via the study of orbits of the special orthogonal groups on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$; while considering the totality of quadratic forms we fix a special orthogonal group and study the closures of all orbits.

As a strategy of attacking the problem via study of flows on homogeneous spaces Raghunathan formulated a more general conjecture. This pertains to orbit-closures of actions of unipotent elements and unipotent subgroups on G/Γ, where G is a Lie group and Γ is a lattice; recall that an element $u \in G$ is said to be *unipotent* if $\mathrm{Ad}\, u$ is a unipotent linear transformation of the Lie algebra of G; a subgroup U of G is unipotent if all elements of U are unipotent.

4.1. Conjecture. *Let G be a Lie group and Γ be a lattice in G. Let U be a unipotent subgroup of G. Then the closure of any U-orbit is a homogeneous set; that is, for any $x \in G/\Gamma$ there exists a closed subgroup F of G such that $\overline{Ux} = Fx$.*

(In [D7] where the conjecture first appeared in print the statement was formulated only for one-parameter subgroups U; the weaker form only reflects my choice for the write-up and does not fully convey what was meant. The lack of concern for generality in the formulation there was partly on account of knowing that the conjecture being true for one-parameter subgroups implies it being true for all connected unipotent subgroups; see [D14], Theorem 3.8, for an idea of the proof of this).

The minimality results mentioned above confirm the conjecture in their respective cases. The result from [DR] involved in Theorem 3.7 implies validity of the conjecture for certain specific horospherical subgroups of $SL(n,\mathbb{R})$ and $Sp(n,\mathbb{R})$ (unipotent radicals of parabolic subgroups of maximum dimension, in the first case) acting on the quotients of the groups by $SL(n, \mathbb{Z})$ and $Sp(n, \mathbb{Z})$ respectively. In [D15] the conjecture was verified for horospherical flows on not necessarily compact homogeneous spaces of all reductive Lie groups. The case of not necessarily reductive Lie groups was studied in [D12] (Appendix), [D16] and [St5] and certain partial results were obtained towards the conjecture. I will mention a couple of the results; though we will be seeing later some results which are much stronger in spirit, it does not seem easy to read off these results from them. In [D16] it was shown that if G is a connected Lie group, Γ is a lattice in G, R is the radical of G and U is a horospherical subgroup associated to an element of G and if U acts ergodically on G/Γ then an orbit $Ug\Gamma/\Gamma$, $g \in G$, is dense if and only if $Ug\overline{R\Gamma}/\overline{R\Gamma}$ is dense in $G/\overline{R\Gamma}$. Starkov [St5] proved that (under the same notation) if G/Γ is compact then for $x = g\Gamma \in G/\Gamma$, where $g \in G$, $\overline{Ux} = Fx$ for the subgroup F defined as follows: let U^- be the horospherical subgroup opposite to U (corresponding to the inverse element), M be the subgroup generated by U and U^- and set $F = \overline{M(g\Gamma g^{-1})}$ (it may be noted that M is a normal subgroup and hence F is indeed a subgroup).

Oppenheim's conjecture was settled by G. A. Margulis in 1986-87 (see [Mar3], [Mar4]). While it is based on the study of flows on homogeneous spaces, his proof proceeds somewhat differently than the strategy indicated above. He proved that for the action of $SO(2,1)$ on $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$ any relatively compact orbit is actually compact. By the Mahler criterion recalled in § 1.5 this implies that any quadratic form as in Oppenheim's conjecture takes arbitrarily small values. For a form which does not take the value 0 at any integral point, this would mean that 0 is a limit point of the set of values and then a result of Oppenheim implies that the set of values is actually dense. For forms admitting integral zeros Margulis produced a somewhat technical variation of the argument, showing that the conjecture holds in this case also. In [DM1] we proved the following stronger result.

4.2. Theorem: *For the action of $SO(2,1)$ on $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$, any orbit is either dense or closed.*

(A similar result was also proved there for all lattices satisfying a certain condition and we had mentioned that in fact the condition holds for all lattices. A proof of the latter was given in [DM4] but while the basic idea there is well founded, there are serious presentational errors which make the proof unsatisfactory; a proof of the relevant part may be found in [EMS]; the author also hopes to present it in detail at a suitable place).

The reader may notice that though $SO(2,1)$ is not a unipotent subgroup the conclusion in Theorem 4.2 is similar to that in Conjecture 4.1. The theorem would actually follow if one knew the validity of the conjecture. Nevertheless it is convenient to view both the statements as particular cases of the following conjecture formulated by Margulis in [Mar2] (see also [Mar7]).

4.3. Conjecture : *Let G be a Lie group and $C \in \mathcal{F}(G)$. Let H be a subgroup which is generated by the unipotent elements contained in it. Then the closure of any H-orbit on G/C is a homogeneous set.*

Though the conjecture is stated for all $C \in \mathcal{F}(G)$, using Remark 2.16 and some technical arguments it can be reduced to the case of lattices; in the sequel I shall discuss only the case of lattices. Theorem 4.2 verifies a particular case of the Conjecture 4.3. Evidently the latter is also satisfied in the cases where Conjecture 4.1 holds. We shall see more about the conjecture later.

Observe that Theorem 4.2 yields a proof of Oppenheim's conjecture by the argument indicated earlier. It also yields the following stronger result, in which we consider the values only on *primitive* integral n-tuples; this may be compared with Corollary 3.8 in the case of systems of linear forms.

4.4. Theorem: *Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$, $n \geq 3$, which is not a multiple of a form with rational coefficients. Let $\mathcal{P}$ denote the set of primitive integral n-tuples. Then $Q(\mathcal{P})$ is dense in $\mathbb{R}$.*

Subsequently in [DM3] we also produced a rather elementary proof for Theorem 4.4, involving only standard material on finite-dimensional vector spaces and topological groups; I should add that Zorn's lemma was used to conclude existence of minimal elements in certain families of compact subsets. In [D17] I constructed a variation of the argument not involving choice of minimal subsets. However I learnt later from A. Katok that (by an observation which he attributed to S. Simpson) the existence of minimal sets as required in the earlier proof could be obtained without the use of Zorn's lemma.

In [DM2] we verified Conjecture 4.1 for the case of $G = SL(3, \mathbb{R})$, Γ any lattice in G and $U = \{u_t\}$ a 'generic' unipotent one-parameter subgroup, namely such that $u_t - I$ has rank 2 for $t \neq 0$, I being the identity matrix. This result has the following consequence.

4.5. Corollary: *Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^3$. Let L be a linear form on $\mathbb{R}^3$ such that the plane $\{v \in \mathbb{R}^3 | L(v) = 0\}$ and the double cone $\{w \in \mathbb{R}^3 | Q(w) = 0\}$ intersect in a line and are tangential along the line. Suppose that no linear combination of Q and L^2 is a rational quadratic form. Then for any $a, b \in \mathbb{R}$ and any $\epsilon > 0$ there exists a primitive integral triple x such that*

$$|Q(x) - a| < \epsilon \text{ and } |L(x) - b| < \epsilon.$$

Margulis has pointed out in [Mar6] that Theorem 4.2 can be strengthened to the following.

4.6. Theorem: *If Γ is a lattice in $SL(3,\mathbb{R})$ and Ω is a nonempty open subset $SL(3,\mathbb{R})/\Gamma$ then there are only a finite number of $SO(2,1)$-orbits on $SL(3,\mathbb{R})/\Gamma$ disjoint from Ω and each of them is closed.*

This was applied to the problem of minima of rational quadratic forms. For a nondegenerate quadratic form Q on $\mathbb{R}^n$ let $m(Q)$ denote the infimum of $\{|Q(x)| \mid x \in \mathbb{Z}^n - (0)\}$ and let $\mu(B) = |d(Q)|^{-1/n}m(Q)$, where $d(Q)$ denotes the discriminant of Q. The set M_n of all numbers of the form $\mu(Q)$ where Q is a nondegenerate quadratic form on $\mathbb{R}^n$ is called the Markov spectrum. For $n = 2$ this is realted to the classical Markov numbers. It follows from Markov's work that $M_2 \cap (\frac{4}{9}, \infty)$ is a countable discrete subset of $(\frac{4}{9}, \infty)$; the intersection with the interval $[0, \frac{4}{9}]$ is an uncountable set with a complicated topological structure. From Theorem 4.6 Margulis deduced the following.

4.7. Corollary: *For $n \geq 3$, for any $\epsilon > 0$ the set $M_n \cap (\epsilon, \infty)$ is finite; further, there are only finitely many equivalence classes of quadratic forms Q with $\mu(Q) > \epsilon$.*

(It may be noted that the corollary is essentially about $n = 3$ or 4, since for $n \geq 5$ by Meyer's theorem $M_n = \{0\}$). The result was proved by J. W.S.Cassels and H. F. P. Swinnerton-Dyer earlier under the assumption of the Oppenheim conjecture being true. The reader is referred to [Mar6] and [Gh] for more details.

§ 5 Invariant measures of unipotent flows

In the last section we saw various results on closures of orbits but nothing was said how they are distributed in space, except for certain assertions about all orbits being uniformly distributed in the whole space. The general question of distribution of orbits involves the study of invariant measures of the flows. For any action the set of all invariant probability measures is a convex set and the ergodic ones (namely those with respect to which the action is ergodic) form its extreme points. Further, any invariant probability measure has an 'ergodic decomposition' as a direct integral of ergodic invariant probability measures (see [Ro], for instance). Therefore to understand the set of all finite invariant measures it is enough to classify the ergodic invariant measures. In this section I will describe the results in this regard. They will be applied in § 7 to the problem of distribution of orbits. Let me begin with the following theorem of Furstenberg which may be said to have been instrumental in initiating a full-fledged study of invariant measures and its application to proving Raghunathan's conjecture.

5.1. Theorem (Furstenberg, 1972): *Let $G = SL(2,\mathbb{R})$, Γ a uniform lattice in G and $U = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$. Then the G-invariant probability measure is the only U-invariant probability measure on G/Γ.*

As in the earlier cases this implies that the orbits of U are uniformly distributed with respect to the G-invariant probability measure on G/Γ. The result was generalised by Veech [V2] to actions of maximal horospherical subgroups of semisimple Lie groups on quotients of the latter by uniform lattices. R. Bowen [Bo] and R. Ellis and W. Perrizo [EP] showed that if G is a connected Lie group, Γ is a unifrom lattice in G and $a \in G$ is such that T_a is a weakly mixing (see § 2) translation of G/Γ then the action of the horospherical subgroup U corresponding to a on G/Γ is uniquely ergodic; namely the G-invariant probability measure is the only probability measure invariant under the U-action. Interestingly these proofs and that of Veech in the case of semisimple Lie groups are all substantially different from each other; while Bowen's proof is geometric the other two are analytical. Recall that the horospherical subgroups are nilpotent and hence in particular solvable groups. Since for the action of a solvable group any compact invariant subset supports an invariant measure, these results imply minimality the results recalled in § 3 for the flows under consideration.

Let us now return temporarily to the case of the horocycle flows on G/Γ where $G = SL(2, \mathbb{R})$ and Γ is a lattice in G. If G/Γ is noncompact, namely when Γ is a nonuniform lattice, by Hedlund's theorem we know that there exist continuous families of periodic orbits and thus, in contrast to the above situation, there is a large class of invariant probability measures; each periodic orbit supports an invariant probability measure. It turns out however that together with the G-invariant probability measure these measures account for all ergodic invariant measures. In [D4] I obtained a classification of invariant measures for a class of horospherical flows which in the case of the horocycle flow on $SL(2, \mathbb{R})/\Gamma$, implies the following.

5.2. Theorem : *Let $G = SL(2, \mathbb{R})$, Γ a lattice in G and $U = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$. Let μ be an ergodic U-invariant probability measure on G/Γ. Then either μ is G-invariant or there exists a $x \in G/\Gamma$ such that the U-orbit of x is periodic and the support of μ is contained in Ux.*

Let us now consider the question in a more general set up. Let G be a Lie group and Γ be a lattice in G. A canonical class of probability measures on G/Γ arises as follows. Let H be any closed subgroup of G such that $H \cap (g\Gamma g^{-1})$ is a lattice in H for some $g \in G$. Since $H/(H \cap g\Gamma g^{-1})$ can be identified in a natural way with $Hg\Gamma/\Gamma$, the H-orbit of $g\Gamma$, the latter admits a H-invariant probability measure; we shall think of it as a probability measure on G/Γ assigning zero measure to the complement of the orbit. Measures on G/Γ arising in this way are called *algebraic measures;* that is, μ is algebraic if there exists a closed subgroup H of G such that μ is H-invariant and its support is contained in a single orbit of H; such an orbit is always closed (see

[R], Theorem 1.13).

It should be noted that the class of closed subgroups H yielding algebraic measures depends on the lattice Γ. While for some lattices the only algebraic measures other than the G-invariant measure may come from compact subgroups of G (the latter are uninteresting from our point of view and may be considered trivial) for most lattices, including all non-uniform lattices as a matter of fact, there exist algebraic measures arising from proper noncompact subgroups of G. For example, when $G = SL(2, \mathbb{R})$ and Γ is a nonuniform lattice, the measures on periodic orbits of $H = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$ discussed earlier are algebraic measures of this kind. For $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$ there are noncompact closed subgroups of various intermediate dimensions intersecting Γ in a lattice; e.g. the subgroup consisting of all upper triangular unipotent matrices, the subgroup $SL(m, \mathbb{R})$, where $1 \leq m < n$, embedded (say) in the upper left corner in $SL(n, \mathbb{R})$ in the usual way, etc.; more generally if H is a real algebraic subgroup defined over $\mathbb{Q}$ and satisfying the condition as in the theorem of Borel and Harish-Chandra (see § 1.6) then we get an algebraic probability measure on the H-orbit of the point Γ (the coset of the identity element). We note that in particular if Q is a nondegenerate indefinite quadratic form on $\mathbb{R}^n$ with rational coefficients (with respect to the standard basis) then $SO(Q)$, the corresponding special orthogonal group, intersects Γ is a lattice and yields an algebraic measure supported on $SO(Q)\Gamma/\Gamma$.

Now let G be any Lie group, Γ be a lattice in G and let U be a subgroup of G. Collecting together algebraic measures arising from subgroups H containing U gives us a class of U-invariant measures. The key conjecture for the classification of invariant measures is the following.

5.3. Conjecture: *Let G be a Lie group, Γ be a lattice in G and U be a subgroup generated by the unipotent elements contained in it. Then any U-invariant ergodic probability measure is algebraic.*

For the case of $G = SL(2, \mathbb{R})$ and U the unipotent one-parameter subgroup corresponding to the horocycle flow, this statement is equivalent to Theorem 5.2. The conjecture was verified in [D7] for maximal horospherical subgroups of reductive Lie groups, where the conjecture was formulated for one-parameter subgroups (the case of one-parameter subgroups implies the corresponding statement for all connected unipotent subgroups).

In what constitutes a landmark development in the area, through a series of papers [R1], [R2], [R3] Marina Ratner proved Conjecture 5.3 for a large class of subgroups U, including all unipotent subgroups. She proved the following.

5.4. Theorem (Ratner, 1991): *Let G be a Lie group and Γ be a discrete subgroup of G. If U is a unipotent subgroup of G then for the U-action on G/Γ any finite ergodic U-invariant measures is algebraic. The same conclusion also holds for the action of any Lie subgroup U of G satisfying the following conditions:*

i) U^0 is generated by the unipotent one-parameter subgroups contained in it and

ii) U/U^0 is finitely generated and each coset of U^0 in U contains a unipotent element.

It may be emphasized that Γ is not assumed to be a lattice but just any discrete subgroup. The general proof is quite intricate and involves several steps. The reader may find it helpful to first go through [R5] and [Gh] where the basic ideas are explained dealing only with the case of the horocycle flows.

Recently Margulis and Tomanov have given another proof of the above theorem in the case of algebraic groups; their proof bears a strong influence of Ratner's original arguments but is substantially different in its approach and methods; see [MT].

Given a Lie group G and two unimodular closed subgroups H and Γ of G there is also a canonical duality between H-invariant measures on G/Γ and Γ-invariant measures on G/H. We note that up to Borel isomorphism G can be viewed as a cartesian product of G/H and H. Using this we can associate with any measure on G/H a measure on G which is invariant under the action of H by multiplication on the right. If the measure on G/H is Γ invariant then the corresponding measure on G is also Γ-invariant, under the left action. This and a corresponding observation for measures on G/Γ yields the one-one correspondence between the two classes of measures as above; see [F2] and [D4]. This correspondence is used in the proofs of some of the results on classification of invariant measures of actions of horopherical subgroups (see [F2], [V2], [D4] and [D7]). On the other hand via the duality Theorem 5.4 yields a description of Γ-invariant measures on G/U (notation as in Theorem 5.4). In turn this may be applied to describe, in the same fashion as for orbit-closures, the measures on vector spaces invariant under actions of certain discrete groups. The strategy in particular yields the following Corollary; I will not go into the details of other more technical results that can be obtained.

5.5. Corollary: *Let $\Gamma = SL(n, \mathbb{Z})$, $1 \leq p < n$ and consider the Γ-action on the p-fold cartesian product V of $\mathbb{R}^n$, as in Theorem 3.7. Let λ be a (locally finite) ergodic Γ invariant measure on V. Then the exists a Γ-invariant subspace W and $a \in V$ such that $\Gamma(W+a)$ is closed and λ is the measure supported on $\Gamma(W+a)$ whose restriction to $W+a$ is the translate of a Lebesgue* *measure on* W.

Various applications of the classification of invariant measures will be discussed in later sections. I will conclude the present section by describing a way of 'arranging' the invariant measures of unipotent flows, which is useful in these applications.

Let G and Γ be as before. Let $\mathcal{H}$ be the class of all proper closed subgroups H of G such that $H \cap \Gamma$ is a lattice in H and H contains a unipotent subgroup U of G acting ergodically on $H/H \cap \Gamma$. Then $\mathcal{H}$ is countable (see [R4], Theorem 1; see also Proposition 2.1 of [DM6] which is a variation of the assertion and implies it in view of Corollary 2.13 of [Sh1]). For any subgroups H and U of G let

$$X(H, U) = \{g \in G \mid Ug \subseteq gH\}.$$

Now let U be a subgroup of G generated by the unipotent elements contained in it and acting ergodically on G/Γ. Let μ be any ergodic U-invariant probability measure other than the G-invariant probability measure. By Theorem 5.4 μ is algebraic; thus there exists a closed subgroup L containing U such that μ is the L-invariant measure on a L-orbit, say $Lg\Gamma/\Gamma$. Since μ is not G-invariant L is a proper closed subgroup. Put $H = g^{-1}Lg$. Then $H \cap \Gamma$ is a lattice in H. Also $g^{-1}Ug$ is generated by unipotent elements, contained in H and acts ergodically on $H/H \cap \Gamma$. Hence $H \in \mathcal{H}$. Also, $Ug \subseteq Lg = gH$ and hence $g \in X(H, U)$. Therefore the support of μ, which is the same as $Lg\Gamma/\Gamma = gH\Gamma/\Gamma$, is contained in $X(H, U)\Gamma/\Gamma$. Thus for every ergodic invariant probability measure μ other than the G-invariant measure $\mu(\cup_{H \in \mathcal{H}} X(H, U)\Gamma/\Gamma) = 1$. Since by ergodic decomposition any invariant probability measure π can be expressed as a direct integral of ergodic invariant measures (see [Ro], for instance) this implies that $\pi(\cup_{H \in \mathcal{H}} X(H, U)\Gamma/\Gamma) > 0$, unless π is G-invariant. Since $\mathcal{H}$ is countable this yields the following characterisation of the G-invariant measure on G/Γ.

5.6. Corollary : *Let G, Γ and U be as above. Let π be a U-invariant probability measure on G/Γ such that $\pi(X(H, U)\Gamma/\Gamma) = 0$ for all $H \in \mathcal{H}$. Then π is G-invariant.*

§ 6 Homecoming of trajectories of unipotent flows

As a part of the strategy involved in one of his proofs of the arithmeticity theorem, Margulis proved that if $\{u_t\}$ is a unipotent one-parameter subgroup of $SL(n, \mathbb{R})$ then for any $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ such that the set $\{t \geq 0 | u_t x \in K\}$ is unbounded; that is, the trajectory of $\{u_t\}$ starting from x keeps returning to K (see [Mar1]). After studying finite invariant measures of maximal horospherical flows in [D4] I realised that Margulis' proof could be strengthened to conclude that for actions of unipotent subgroups all ergodic invariant measures are finite (see [D5] and [D10]); for the case of a one-parameter subgroup $\{u_t\}$ and x as above this involves proving that for a suitable compact subset K the set $\{t \geq 0 | u_t x \in K\}$ has positive upper density. The line of argument later yielded the following result (see [D14], Theorem 3.5; the semisimplicity condition in the statement there is redundant in view of Lemma 9.1 of [D4]; it came to be included in the hypothesis only in the flow of the general development there):

6.1. Theorem: *Let G be a connected Lie group and Γ be a lattice in G. Then for any $\epsilon > 0$ there exists a compact subset K of G/Γ such that for any unipotent one-paramater subgroup $\{u_t\}$ and $g \in G$ at least one of the following conditions holds:*

i) $l(\{t \in [0,T] | u_t g\Gamma \in K\}) \geq (1-\epsilon)T$ for all large T, or

ii) there exists a proper closed connected subgroup H of G such that $H \cap \Gamma$ is a lattice in H and $\{g^{-1}u_t g\}$ is contained in H.

It may be noticed that if the second alternative holds then the whole $\{u_t\}$-orbit of $g\Gamma$ is contained in the lower-dimensional homogeneous space $gH\Gamma/\Gamma \approx L/L \cap g\Gamma g^{-1}$, where $L = gHg^{-1}$; in view of this in various contexts one can use a suitable induction hypothesis and reduce the problem to considering only the situation as in condition (i). It is clear in particular that for any $\{u_t\}$ and any $g \in G$ as in the hypothesis there exists a compact subset K, depending on the $\{u_t\}$ and g, such that the conclusion as in (i) holds. The compact subset can be chosen to be common for all unipotent one-parameter subgroups and g from a given compact subset; that is, the following holds:

6.2. Theorem: *Let G be a connected Lie group and Γ be a lattice in G. Let a compact subset F of G/Γ and $\epsilon > 0$ be given. Then there exists a compact subset K of G/Γ such that for any unipotent one-parameter subgroup $\{u_t\}$ of G, $x \in F$ and $T \geq 0$ such that $u_s x \in F$ for some $s \in [0,T]$ we have*

$$l(\{t \in [0,T] \,|\, u_t x \in K\}) \geq (1-\epsilon)T.$$

This follows from Proposition 2.7 of [D14] together with the arithmeticity theorem; (see [DM6], Theorem 6.1; it may also be noted that it is enough to prove the assertion in the theorem for $x \in F$, which can then be applied to $u_s x$ and the one-parameter subgroups $\{u_t\}$ and $\{u_{-t}\}$ to get the desired result).

I may also mention that the closed subgroup as in the second alternative in Theorem 6.1 can be chosen from a special class of subgroups; for instance if G is semisimple (the general case can be reduced to this via Lemma 9.1 of [D4]) then H can be chosen to be contained in a 'Γ-parabolic' subgroup;

by a Γ-parabolic subgroup we mean a parabolic subgroup whose unipotent radical intersects Γ in a lattice. This follows from Theorem 1 of [DM4]; (as mentioned earlier, though the result is correct the proof in [DM4] involves some errors). A more general result in the direction of Theorem 6.3 has been proved in [EMS] where they consider multiparameter polynomial trajectories; I will however not go into the details.

The above theorems are used in studying the distribution of orbits of unipotent flows, as we shall see in the next section. I will devote the rest of this section to a discussion of various other applications of the theme. Let me begin with the following result derived in [D14] (see Theorem 3.9 there).

6.3. Corollary: *Let G be a connected Lie group and Γ be a lattice in G. There there exists a compact subset C of G such that the following holds: if U is a closed connected unipotent subgroup which is not contained in any proper closed subgroup H such that $H \cap \Gamma$ is a lattice in H then $G = C\Gamma U = U\Gamma C^{-1}$.*

The motivation for such a theorem is the following: if Γ is a uniform lattice in G then there exists a compact subset C of G such that $G = C\Gamma = \Gamma C^{-1}$. While this can not be true for a nonuniform lattice, the theorem means that it is true modulo any unipotent subgroup not contained in a subgroup H as in the hypothesis. This has the following interesting consequence.

6.4. Corollary: *Let G be a connected Lie group and Γ be a lattice in G. Consider a linear action of G on a finite-dimensional vector space V. Let $v \in V$ be such that the following conditions are satisfied:*

i) there exists a sequence $\{g_i\}$ in G such that $g_i v \to 0$, as $i \to \infty$, and

ii) the isotropy subgroup of v (that is, $\{g \in G | gv = v\}$) contains a connected unipotent subgroup U which is not contained in any proper closed subgroup H such that $H \cap \Gamma$ is a lattice in H.

Then there exists a sequence $\{\gamma_i\}$ in Γ such that $\gamma_i v \to 0$.

This may be viewed as diophantine approximation with matrix argument, with elements of lattices replacing integers. It can be applied for instance to various representations of $SL(n, \mathbb{R})$ and we get conditions for orbits of $SL(n, \mathbb{Z})$ to have points arbitrarily close to zero. It also implies (see [D14], Corollary 4.3) that if G is a connected semisimple Lie group with trivial center and Γ is a lattice in G then for any unipotent element u which is not contained in any proper closed subgroup such that $H \cap \Gamma$ is a lattice in H, the closure of the Γ conjugacy class $\{\gamma u \gamma^{-1} | \gamma \in \Gamma\}$ contains the identity element.

By one of the remarks above, Theorem 6.2 implies in particular that for the action of a unipotent one-parameter subgroup on G/Γ, where G is a Lie group and Γ is a lattice, every ergodic invariant (locally finite) measure is finite. It was proved in [D5] and [D10] (in the arithmetic and the general case respectively) that in fact any invariant measure is a countable sum of finite invariant measures. Using this and some more arguments Margulis obtained another proof of the theorem of Borel and Harish-Chandra on the finiteness of the invariant measures of certain homogeneous spaces. More generally he proved the following (see [Mar2]).

6.5. Theorem: *Let G be a connected Lie group and Γ be a lattice in G. Let H be connected Lie subgroup such that if R is the radical of H and N is the subgroup consisting of all unipotent elements of G contained in R then R/N is compact. If μ is a locally finite H-invariant mesure on G/Γ then there exists a sequence $\{X_i\}$ of H-invariant Borel subsets of G/Γ such that $\mu(X_i)$ is finite for all i and $\cup X_i = G/\Gamma$; if μ is ergodic then it is a finite measure. In particular, the H-invariant measure on any closed orbit of H on G/Γ is a finite measure.*

The results of this section are also used (see [DM1]), together with a lemma of Y. Dowker (see Lemma 1.6 in [DM1]), to conclude that for unipotent flows as above all minimal (closed nonempty) sets are compact (this is of course implied by Ratner's theorem verifying Raghunathan's conjecture, (Theorem 7.4 below), but cumulatively that involves a much longer argument). The reader is also referred to [Mar8] for a proof of the corresponding assertion for actions of more general unipotent groups.

§ 7 Distribution and closures of orbits

In this and the following sections I will discuss the distribution of orbits, using Theorem 5.4 on classification of invariant measures and Theorem 6.2. The reader is also referred to [D18] for a similar discussion, presented somewhat differently. I will begin with a cautionary note that when there are more than one invariant probability measures, an orbit need not be uniformly distributed with respect to any of them. Roughly speaking, over time the 'distribution' may oscillate between various invariant measures, possibly uncountably many of them. Rigorously this may be described as follows. Let T be a homeomorphism of a locally compact space X. Let $x \in X$ and for any natural number N let π_N be the probability measure which is the average of the N point masses supported at the points $x, Tx, ..., T^{N-1}x$; that is, for any Borel set E, $\pi_N(E) = N^{-1}\Sigma_{i=0}^{N-1} \chi_E(T^i(x))$, χ_E being the characteristic function of E. The orbit of x being uniformly distributed with respect to a measure μ is equivalent to π_N converging to μ (in the weak topology on the space of measures, with respect to bounded continuous functions) as $N \to \infty$. In considering convergence it is convenient to have the space to be compact and therefore if X is noncompact we view the measures as being on its one-point compactification, say $\tilde{X}$; if X is compact we set $\tilde{X} = X$. Then convergence of $\{\pi_N\}$ to a probability measure on X is equivalent to there being a unique limit in the space of probability measures on $\tilde{X}$ and its assigning zero mass to the point at infinity. It is easy to see that any limit point is a T-invariant probability measure. If X is compact and there is only one invariant probability measure, namely if the transformation is uniquely ergodic, then this immediately implies that the orbit is uniformly distributed. On the other hand if there are several invariant probability measures then different subsequences of $\{\pi_N\}$ may converge to different probability measures. This is what is meant by oscillation of the distribution of the orbit. For a general dynamical system this happens quite frequently (see e.g.[DGS], [DMu]).

Before proceeding further it may be noted that a similar discussion applies to one-parameter flows. If $\{\phi_t\}$ is a one-parameter subgroup of homeomorphisms of a locally compact space X and $x \in X$ then for any $T > 0$ we define a probability measure π_T by setting $\pi_T(E)$, for any Borel set E, to be $T^{-1}l(\{t \in [0,T] \mid \phi_t(x) \in E\})$, where l is the Lebesgue measure on $\mathbb{R}$. The orbit of x under $\{\phi_t\}$ being uniformly distributed with respect to a probability measure μ is equivalent to π_T converging to μ as $T \to \infty$.

In a general sense the task of proving uniform distribution of an orbit can be divided in to two parts; first to understand all invariant measures of the transformation or flow in question and then to identify which of them can occur as limit points of the families of measure associated to any particular orbit, as above.

Using Theorem 5.4 on the classification of invariant measures of the horocycle flow it was shown, in [D9] for $\Gamma = SL(2, \mathbb{Z})$ and in [DS] for a general lattice, that all nonperiodic orbits are uniformly distributed with respect to the invariant volume. That is,

7.1. Theorem: *Let $G = SL(2, \mathbb{R})$, Γ be any lattice in G and $U = \{u_t\}$, where $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for all $t \in \mathbb{R}$. Let $x \in G/\Gamma$ be such that $u_t x \neq x$ for any $t > 0$. Then the $\{u_t\}$-orbit of x is uniformly distributed with respect to the G-invariant probability measure. Also, for any $s \neq 0$, $\{u_s^i x\}_{i=0}^{\infty}$ is uniformly distributed with respect to the G-invariant probability measure.*

Before going over to the general case, let me note the following interesting consequence of the $SL(2, \mathbb{Z})$-case of the above theorem, proved in [D9].

7.2. Theorem: *Let α be an irrational number. For any $T > 0$ let $I(T)$ denote the set of all natural numbers k such that $k \leq T < k\alpha >$ and the integral part of $k\alpha$ is coprime to k. Then*

$$\frac{1}{T}\Sigma_{I(T)} <k\alpha>^{-1} \longrightarrow \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

as $T \to \infty$, where ζ is the Riemann zeta function.

The classification of invariant measures of flows made it possible to address the questions of closures and distribution of the orbits formulated earlier. Using Ratner's theorem together with a technique from [DM2] Nimish Shah [Sh1] proved Conjecture 4.1 in the cases when either G is a reductive Lie group of $\mathbb{R}$-rank 1 or $G = SL(3, \mathbb{R})$ (he proved Theorem 7.3 in these cases). More complete results in this respect were obtained by Ratner around the same time, by a totally different method, proving the following.

7.3. Theorem (Ratner, 1991): *Let G be a connected Lie group and Γ be a lattice in G. Let $U = \{u_t\}$ be a unipotent one-parameter subgroup of G. Then for any $x \in G/\Gamma$ there exists an algebraic probability measure μ such that the $\{u_t\}$-orbit of x is uniformly distributed with respect to μ.*

She proved also the corresponding assertion for cyclic subgroups generated by unipotent elements. From the theorem she deduced Raghuanthan's conjecture (see Conjecture 4.1) for unipotent subgroups and also the following stronger assertion.

7.4. Theorem (Ratner, 1991): *Let G and U be as in Theorem 5.4. Let Γ be a lattice in G. Then for any $x \in G/\Gamma$, $\overline{Ux}$ is a homogeneous subset with finite invariant measure; that is, there exists a closed subgroup F such that Fx admits a finite F-invariant measure and $\overline{Ux} = Fx$.*

Let G, Γ and U be as in Theorem 7.3. We call a point $x \in G/\Gamma$ *generic* for the U-action if there does not exist any proper closed subgroup F containing U such that Fx admits a finite F-invariant measure; a posteriori, in view of Theorem 7.3 this coincides with the usual notion of generic points when U is either cyclic or a one-parameter subgroup; in the sequel however we shall mean only the condition described here. Consider the orbit of a generic point x. In this case the assertion of Theorem 7.3 is that it is uniformly distributed with respect to the G-invariant probability measure, say m. Let me say a few words about how the assertion of the theorem is proved in this case; (for a non-generic point the theorem would follow if we consider an appropriate homogeneous subspace). For $T > 0$ let π_T be the probability measure on G/Γ given by $\pi_T(E) = T^{-1}\, l(\{t \in [0,t] | u_t x \in E\})$. We have to show that π_T converge to m as $T \to \infty$. Let X denote G/Γ if the latter is compact and the one-point compactification if it is noncompact. We view π_T as measures on X, in the obvious way. It is then enough to show that if $\{T_i\}_{i=1}^{\infty}$ is a sequence such that $T_i \to \infty$ and the sequence $\{\pi_{T_i}\}$ converges, to say π, in the space of probability measures on X then π is the measure assigning 0 mass to the complement of G/Γ in X and G-invariant on the subset G/Γ. Let such a sequence $\{T_i\}$ be given and let π be the limit of $\{\pi_{T_i}\}$. If G/Γ is noncompact and ∞ is the point at infinity then the desired condition $\pi(\infty) = 0$ follows from Theorm 6.2. Now consider π as a probability measure on G/Γ. Recall that any limit such as π is U-invariant. The G-invariance of π is deduced by proving that $\pi(X(H,U)\Gamma/\Gamma) = 0$ for all $H \in \mathcal{H}$ (carrying this out constitutes the major task in the proof) and applying Corollary 5.5.

For proving that $\pi(X(H,U)\Gamma/\Gamma) = 0$ for $H \in \mathcal{H}$, I will indicate a different approach than in Ratner's proof in [R4]. This approach is involved in many papers by now, including [DM2], [DM5], [Sh1], [Sh2], [MS], [EMS]; the arguments in [DS] also follow the same scheme, though in a simpler context. The main idea in this is that the sets $X(H,U)\Gamma/\Gamma$ are locally like affine submanifolds and a trajectory of $U = \{u_t\}$ corresponds to a polynomial trajectory and hence it does not spend too much time too close to the submanifold. To make sense out of this one associates to each $H \in \mathcal{H}$ a representation $\rho_H : G \to GL(V_H)$ over a finite-dimensional real vector space V_H, and a $p_H \in V_H$ such that the following holds: if $\eta_H : G \to V_H$ denotes the orbit map $g \mapsto \rho_H(g)p_H$ and A_H is the subspace of V_H spanned by $\eta_H(X(H,U))$ then $X(H,U) = \eta^{-1}(A_H) = \{g \in G \mid \eta_H(g) \in A_H\}$; specifically ρ_H may be chosen to be the h-th exterior power of the adjoint representation of G, where $h = \dim H$, and p_H to be a nonzero point in the one-dimensional subspace corresponding to the Lie subalgebra of H; see [DM6] for details (see also [EMS]).

The main ingredients in the proof are as follows. Let $\{T_i\}$ be the sequence as above and for each i let $\sigma_i = \{u_t x | 0 \le t \le T_i\}$. Firstly we prove that if A is an algebraic subvariety in a vector space V, then for any compact subset C of A and $\epsilon > 0$ there exists a (larger) compact subset D of A such that for any segment the proportion of time spent near C to that spent near D is at most ϵ; specifically, for any neighbourhood Φ of D there exists a neighbourhood Ψ of C such that for $y \notin \Phi$, any unipotent one-parameter subgroup $\{v_t\}$ of G and $T \ge 0$,

$$l(\{t \in [0,T] \mid v_t y \in \Psi\}) \le \epsilon\, l(\{t \in [0,T] \mid v_t y \in \Phi\}).$$

This depends on certain simple properties of polynomials and the fact that orbits of unipotent one-parameter subgroups in linear spaces are polynomial curves. Now consider V_H and any compact subset C of A_H and let D be the corresponding subset as above. Let $g \in G$ be such that $g\Gamma = x$. We apply the above assertion to the segments $\{u_t g\gamma p_H \mid 0 \le t \le T_i\}$, $\gamma \in \Gamma$. It turns out that there exists a neighbourhood Φ of D such that the points $g\gamma p_H$ are contained in Φ for at most two distinct γ, if we restrict to $g \in G$ such that $g\Gamma$ lies in a compact set disjoint from the 'self-intersection set' of $X(H,U)\Gamma/\Gamma$, namely the union of its proper subsets of the form $(X(H,U) \cap X(H,U)\alpha)\Gamma/\Gamma$, $\alpha \in \Gamma$. Using this and an inductive argument for the points on the self-intersection set we can combine the information about the individual segments and conclude that $\mu(\eta_H^{-1}(C)) < \epsilon$. Varying C and ϵ we get that $\mu(\eta_H^{-1}(A)\Gamma/\Gamma) = 0$, as desired.

It was noted earlier that Raghunathan's conjecture (Theorem 7.4) and even the weaker result Theorem 4.2 implies Oppenheim's conjecture. The former has the following stronger consequence in that direction; the particular case of it for $k = 2$ follows from Theorem 4.2 (see [DM1], Theorem 1).

7.5. Corollary: *Let B be a real nondegenerate symmetric bilinear form on $\mathbb{R}^n$, where $n \geq 3$, which is not a multiple of a rational form. Let $1 \leq k \leq n-1$ and $v_1, \ldots, v_k \in \mathbb{R}^n$. Then for any $\epsilon > 0$ there exist primitive integral vectors $x_1, \ldots, x_k$ such that*

$$|B(x_i, x_j) - B(v_i, v_j)| < \epsilon \quad \text{for all } i, j = 1, \ldots, k.$$

§ 8 Aftermath of Ratner's work

The classification of invariant measures of unipotent flows has yielded not only to a proof of Raghunathan's orbit-closure conjecture but a host of other interesting results as well. This section will be devoted to describing some of these.

Let me begin with some strengthenings of Theorem 7.3 on uniform distribution of orbits of unipotent one-parameter subgroups. Let G be a Lie group, Γ a lattice and $U = \{u_t\}$ be a unipotent one-parameter subgroup. As noted earlier, the theorem implies in particular that if $x \in G/\Gamma$ is a generic point then its orbit is uniformly distributed with respect to the G-invariant probability measure, say m. Thus, for any bounded continuous function φ on G/Γ

$$\frac{1}{T}\int_0^T \varphi(u_t x)\mathrm{d}t \to \int_{G/\Gamma} \varphi \mathrm{d}m \quad \text{as } i \to \infty$$

for any generic point x. A natural question is whether the convergence of the averages (for a fixed φ) is uniform over compact subsets of the set of generic points. Similarly one may ask what happens if we vary the unipotent one-parameter subgroup. These questions were considered in [DM6] and the results were applied to obtain lower estimates for the number of solutions in large enough sets, for quadratic inequalities as in Oppenheim's conjecture; see Corollary 8.3 below. One of our first results in this direction is the following:

8.1. Theorem: *Let G be a connected Lie group, Γ be a lattice in G and m be the G-invariant probability measure on G/Γ. Let $\{u_t^{(i)}\}$ be a sequence of unipotent one-parameter subgroups converging to a unipotent one-parameter subgroup $\{u_t\}$; that is $u_t^{(i)} \to u_t$ for all t, as $i \to \infty$. Let $\{x_i\}$ be a sequence in G/Γ converging to a point $x \in G/\Gamma$. Suppose that x is generic for the action of $\{u_t\}$. Let $\{T_i\}$ be a sequence in $\mathbb{R}^+$, $T_i \to \infty$. Then for any bounded continuous function φ on G/Γ,*

$$\frac{1}{T_i}\int_0^{T_i} \varphi(u_t^{(i)} x_i)\mathrm{d}t \to \int_{G/\Gamma} \varphi \mathrm{d}m \quad \text{as } i \to \infty.$$

(Marina Ratner has mentioned in [R6] that Marc Burger had pointed out to her in December 1990, which happens to be before we started our work on the above questions, that such a strengthening of her theorem can be derived applying her methods).

In proving the theorem we use Theorem 5.4 on classification of invariant measures but not Theorem 7.3 on uniform distribution. Our approach, which was indicated in the previous section in the special case of Theorem 7.3, is quite different from Ratner's approach.

We also proved another such 'uniform version' of uniform distribution in which we consider also the averages as on the left hand side for non-generic points as well, together with those for the generic points. The unipotent one-parameter subgroup is also allowed to vary over compact sets of such subgroups; the class of unipotent one-parameter subgroups of G is considered equipped with the topology of pointwise convergence, when considered as maps from $\mathbb{R}$ to G.

Let G and Γ be as above and let $\mathcal{H}$, $X(H,U)$ (for any subgroups H and U) be as defined in § 6. Recall that the set of generic points for the actions of a one-parameter subgroup $U = \{u_t\}$ consists of $\bigcup_{H\in\mathcal{H}} X(H,U)\Gamma/\Gamma$. The following result deals simultaneously with averages for generic as well as non-generic points outside certain *compact* subsets from *finitely* many $X(H,U)\Gamma/\Gamma$.

8.2. Theorem: *Let G, Γ and m be as before. Let $\mathcal{U}$ be a compact set of unipotent one-parameter subgroups of G. Let a bounded continuous function φ on G/Γ, a compact subset K of G/Γ and $\epsilon > 0$ be given. Then there exist finitely many subgroups $H_1, ..., H_k \in \mathcal{H}$ and a compact subset C of G such that the following holds: For any $U = \{u_t\} \in \mathcal{U}$ and any compact subset F of $K - \bigcup_{i=1}^k (C \cap X(H_i,U))\Gamma/\Gamma$ there exists a $T_0 \geq 0$ such that for all $x \in F$ and $T > T_0$,*

$$\left| \ \frac{1}{T}\int_0^T \varphi(u_t x)\,\mathrm{d}t - \int_{G/\Gamma} \varphi\,\mathrm{d}m \ \right| < \epsilon.$$

In [DM6] this was proved for $\mathcal{U}$ consisting of a single one-parameter subgroup; essentially the same proof goes through for a general compact set of one-parameter subgroups.

¿From Theorem 8.2 we deduced the following asymptotic lower estimates for the number of solutions of quadratic inequalities in regions of the form $\{v \in \mathbb{R}^n \mid \nu(v) \leq r\}$, as $r \to \infty$, where ν is a continuous 'homogeneous' function on $\mathbb{R}^n$; we call a function ν homogeneous if $\nu(tv) = t\nu(v)$ for all $t > 0$ and $v \in \mathbb{R}^n$. We use the notation $\#$ to indicate cardinality of a set and λ for the Lebesgue measure on $\mathbb{R}^n$.

8.3. Corollary: *Let $n \geq 3$, $1 \leq p < n$ and let $\mathcal{Q}(p,n)$ denote the set of all quadratic forms on $\mathbb{R}^n$ with discriminant ± 1 and signature $(p, n-p)$. Let $\mathcal{K}$ be a compact subset of $\mathcal{Q}(p,n)$ (in the topology of pointwise convergence). Let ν be a continuous homogeneous function on $\mathbb{R}^n$ such that $\nu(v) > 0$ for all $v \neq 0$. Then we have the following:*

i) for any interval I in $\mathbb{R}$ and $\theta > 0$ there exists a finite subset $\mathcal{E}$ of $\mathcal{K}$ such that each $Q \in \mathcal{E}$ is a scalar multiple of a rational form and for any $Q \in \mathcal{K} - \mathcal{E}$

$$\#\{z \in \mathbb{Z}^n \mid Q(z) \in I,\, \nu(z) \leq r\} \;\geq (1-\theta)\lambda(\{v \in \mathbb{R}^n \mid Q(v) \in I,\, \nu(v) \leq r\})$$

for all large r; further, for any compact subset C of $\mathcal{K} - \mathcal{E}$ there exists $r_0 \geq 0$ such that for all $Q \in C$ the inequality holds for all $r \geq r_0$.

ii) if $n \geq 5$, then for $\epsilon > 0$ there exist $c > 0$ and $r_0 \geq 0$ such that for all $Q \in \mathcal{K}$ and $r \geq r_0$

$$\#\{z \in \mathbb{Z}^n \mid \,|Q(z)| < \epsilon,\, \nu(z) \leq r\} \;\geq c\lambda(\{v \in \mathbb{R}^n \mid \,|Q(v)| < \epsilon,\, \nu(v) \leq r\}).$$

There exist nondegenerate rational quadratic forms in 4 variables with no nontrivial integral solutions. Since for a rational form, for sufficiently small $\epsilon > 0$ the set as on the left hand side consists of integral solutions, it follows that the condition that $n \geq 5$ is necessary for the conclusion in the second assertion to hold. On the other hand for $n \geq 5$ by Meyer's theorem any nondegenerate rational quadratic form in n variables has a nontrivial integral solution (see [Se]); the theorem is used in the proof of the second part of the corollary.

It can be verified that in terms of r the volume terms appearing on the right hand side of the inequalities are of the order of r^{n-2}. A particular case of interest is of course when ν is the euclidean norm, in which case the regions involved are balls of radius r. I may mention here that for a single quadratic form which is not a multiple of a rational form, an estimate as in (i) was obtained, for the case of balls, by S. Mozes and myself, with a (possibly small) positive constant in the place $1-\theta$ as above (unpublished).

In Theorem 8.1 we considered the trajectories of a sequence of points $\{x_i\}$ which converge to a generic point. Evidently it is not a necessary condition for the conclusion of the theorem to hold. The following theorem describes the general picture of what happens if we omit the condition.

8.4. Theorem: *Let $\{u_t^{(i)}\}$ be a sequence of unipotent one-parameter subgroups such that $u_t^{(i)} \to u_t$ for all t, let $U_i = \{u_t^{(i)}\}$ for all i and $U = \{u_t\}$. Let $\{x_i\}$ be a convergent sequence in G/Γ such that for any compact subset of Φ of G, $\{i \in \mathbb{N} \mid x_i \in (\Phi \cap X(H,U_i))\Gamma/\Gamma\}$ is finite. Let x be the* *limit of* $\{x_i\}$. *Then for any* $H \in \mathcal{H}$ *such that* $x \in X(H,U)\Gamma/\Gamma$ *there exists a sequence* $\{\tau_i\}$ *of positive real numbers such that the following holds: if* $\sigma_i = \{u_t^{(i)} x_i \,|\, 0 \le t \le T_i\}$, *where* $\{T_i\}$ *is a sequence in* $\mathbb{R}^+$, $T_i \to \infty$ *and the normalised linear measures on the segments* σ_i *converge to* μ *as* $i \to \infty$ *then*

i) if $\limsup (T_i/\tau_i) = \infty$ *then* $\mu(X(H,U)\Gamma/\Gamma) = 0$ *and*

ii) if $\limsup (T_i/\tau_i) < \infty$ *then there exists a curve* $\psi : ([0,1] - D) \to X(H,U)$, *where* D *is a finite subset of* $[0,1]$, *such that* $\eta_H \circ \psi$ *extends to a polynomial curve from* $[0,1]$ *to* V_H, $\operatorname{supp}\mu$ *meets* $\psi(t)N^0(H)\Gamma/\Gamma$ *for all* $t \in [0,1]-D$ *and is contained in their union; here* $N^0(H)$ *denotes the subgroup of the normaliser of* H *in* G *consisting of the elements* g *for which the map* $h \mapsto ghg^{-1}$ *preserves the Haar measure on* H.

An analogue of this result can also be proved for divergent sequences, that is, when $x_i \to \infty$ in X, for ∞ in the place of $X(H,U)\Gamma/\Gamma$.

Roughly speaking the theorem says that given the sequence $\{x_i\}$ if we take long enough segments then any limit of linear measures along the segments is G-invariant. If the segments are rather short (in the particular context of the sequences), though of lengths tending to infinity, then the limit gets distributed over a family of sets of the form $g_t H\Gamma/\Gamma$, for certain $H \in \mathcal{H}$. With some further analysis one can describe an ergodic decomposition of the limit.

The theorem also readily implies that for $\{u_t^{(i)}\}$ and $\{x_i\}$ as in the hypothesis there exists a sequence $\{\tau_i\}$ of positive real numbers such that the conclusion as in Theorem 8.1 holds for any sequence $\{T_i\}$ such that $T_i/\tau_i \to \infty$. On the other hand one can also conclude that the topological limit of sufficiently long orbit-segments contains generic points. Specifically, the following holds (cf. [DM6], Theorem 4).

8.5. Theorem: *Let* $\{u_t^{(i)}\}$ *and* $\{x_i\}$ *be as in Theorem 8.4. Then there exists a sequence* $\{t_i\}$ *in* $\mathbb{R}^+$ *such that* $\{u_{t_i} x_i\}$ *has a subsequence converging to a generic point with respect to the limit one-parameter subgroup. Further,* $\{t_i\}$ *may be chosen from any subset* R *of* $\mathbb{R}^+$ *for which there exists an* $\alpha > 0$ *such that* $l(R \cap [0,T]) \ge \alpha T$ *for all* $T \ge 0$.

The classification of invariant measures and the method sketched in the last section for analysing the limit measures has been used recently in many papers and several interesting results have been obtained: Mozes and Shah [MS] show that the set of probability measures on G/Γ which are invariant and ergodic under the action of some (not a fixed one) unipotent one-parameter subgroup is a closed subset of the space of probability measures. Shah [Sh2] applies the method to extend Ratner's uniform distribution theorem (Theorem 7.3) to polynomial trajectories. His result implies in particular the following assertion, which is in the spirit of H. Weyl's theorem of uniform distribution of polynomial trajectories (see [L]), in a more general context.

8.6. Theorem: *Let G be a closed subgroup of $SL(n, \mathbb{R})$ for some $n \geq 2$ and Γ be a lattice in G. Let $\phi : \mathbb{R} \to SL(n, \mathbb{R})$ be a polynomial map (all coordinate functions are polynomials) such that $\phi(G)$ is contained in G. Then the curve $\{\phi(t)\Gamma/\Gamma\}_{t>0}$ is uniformly distributed in G/Γ with respect to an algebraic measure.*

There are also more technical variations of this and multivariable versions (for polynomial maps from $\mathbb{R}^n$) proved in the paper. Using the multivariable case the author also generalises Theorem 7.3 to actions of higher-dimensional unipotent groups, answering in the affirmative a question raised by Ratner in [R4]. Incidentally, though in discussing distribution of orbits I have restricted to only cyclic or one-parameter subgroups it is possible to consider similar questions for actions of a larger class of groups, including all nilpotent Lie groups. The result of Shah pertains to such a question for actions on homogeneous spaces as above.

The set of ideas was applied by Eskin, Mozes and Shah [EMS] to get some notable results on the growth of the number of lattice points on certain subvarieties of linear spaces, within distance r from the origin, as $r \to \infty$. Their results generalise the results of Duke, Rudnik and Sarnak [DRS] and Eskin and McMullen [EM]; the results of these earlier papers apply essentially only to affine symmetric varieties, a limitation which has been overcome in [EMS]. This is achieved through the study of limits of certain sequences of probability measures on G/Γ, where G is a reductive Lie group and Γ is a lattice in G. Specifically, the sequences are of the form $\{g_i\mu\}$, where $\{g_i\}$ is a sequence in G and μ is an algebraic probability measure corresponding to some closed orbit of a reductive subgroup H (which may not necessarily contain any nontrivial unipotent element). In the case when H is a maximal connected reductive subgroup of G intersecting Γ in a lattice and $\{g_i\}$ is such that $\{g_iH\}$ is a divergent sequence in G/H (namely, has no convergent subsequence) then $\{g_i\mu\}$ converges to the G-invariant probability measure on G/Γ. Appropriate generalisations are also obtained in the case when H is not necessarily maximal, but I will not go into the details. I will also not go into the details of the application to the counting problem as above in its generality, but content myself by describing an interesting particular case; the reader is referred to [BR] for some consequences of the result.

8.7. Theorem: *Let p be a monic polynomial of degree n with integer coefficients and irreducible over $\mathbb{Q}$. Then the number of integral $n \times n$ matrices with p as the characteristic polynomial and Hilbert-Schmidt norm less than T is asymptotic to $c_pT^{n(n-1)/2}$, where c_p is a constant.*

§ 9 Miscellanea

1. *Other applications :*

In the earlier sections in discussing applications I concentrated largely on diophantine approximation and related questions. The ideas and results have found applications in various in other contexts also. Let me mention here some of the problems, without going in to the details (but giving suitable references wherever possible), to which the results are applied. No attempt is made to be exhaustive in respect of the applications and I am only mentioning the results which have come to my notice, just to give a flavour of the variety of possibilities in this regard.

Classification of transformations and flows upto isomorphism is one of the central problems in ergodic theory. Similarly it is of interest to understand factors of such systems, joinings etc. Ratner's theorem on invariant measures enables one to understand these issues satisfactorily in the context of the unipotent flows (or translations by unipotent elements) on homogeneous spaces of finite volume. On the other hand Ratner's work on the classification of invariant measures and Raghunathan's conjecture draws quite considerably from her earlier work on these problems. The reader is referred to [R2] and [Wi2] for details on the problems and the results on them.

In the course of his work on the structure of lattices in semisimple Lie groups Margulis showed that if G is a semisimple group of $\mathbb{R}$-rank at least 2, P is parabolic subgroup of G and Γ is a lattice in G then all measurable factors of the Γ-action on G/P are (up to isomorphism) the actions on G/Q, where Q is a parabolic subgroup containing P (see [Mar5], [Z2]). He raised the question whether a similar assertion is true for topological factors of the action. The question was answered in the affirmative in [D11]; certain partial results were obtained earlier in [Z1] and [Sp]. The arguments are based on results about orbit closures of certain subgroups. A more general result in this direction has been proved by Shah in [Sh3], where he considers, given a connected Lie group L, a closed semisimple subgroup G of L, a lattice Λ in L and a parabolic subgroup P of G the factors of the G-actions on $L/\Lambda \times G/P$ which can be factored further to get the projection factor on to G/P; under certain appropriate additional conditions it is shown that the factor is of the form $L/\Lambda \times G/Q$ for a parabolic subgroup Q of G. In the case when $L = G$ this corresponds to studying the factors of the Γ-action on G/P.

A. N. Starkov has applied Ratner's theorem on invariant measures (Theorem 5.4) to prove a conjecture of B. Marcus that any mixing flow on a finite-volume homogeneous space is mixing of all degrees (see [St6]). The idea of proving mixing via the study of invariant measures was earlier employed by S. Mozes [Moz1], who proved that for a Lie group G such that $\mathrm{Ad}\, G$

is closed and the center of G is finite, any mixing action on a Lebesgue space is mixing of all degrees.

R. Zimmer has applied Ratner's invariant measures theorem in [Z5], to obtain interesting information on the fundamental groups of compact manifolds with an action of a semisimple Lie group of $\mathbb{R}$-rank greater than one. He also used it in [Z4] to show that for certain homogeneous spaces G/H, where G is a real algebraic group and H is an algebraic subgroup, there are no lattices; that is, there exist no discrete subgroups Γ of G such that $\Gamma\backslash G/H$ is compact. E. Glasner and B. Weiss [GW] have used Ratner's description of joinings of the horocycle flows to give an example of a simple weakly mixing transformation with nonunique prime factors.

2. *Asymptotics of the number of integral solutions*

It is natural to ask whether the expressions for the lower estimates as in Corollary 8.3 are actually assymptotic to the corresponding numbers on the left hand side. It was shown by P. Sarnak that there exist quadratic forms in 3 variables for which this is not true. Recently A. Eskin, G. A. Margulis and S. Mozes have shown the answer to be in the affirmative for all nondegenerate indefinite quadratic forms with signatures different from $(2,1)$ and $(2,2)$, in particular for all nondegenerate indefinite quadratic forms in 5 or more variables; for forms with signatures $(2,1)$ and $(2,2)$ the authors obtain upper estimates which are $\log r$ times larger than the lower estimates as in Corollary 8.3; (at the time of this writing the results are in the process of being written).

3. *Extensions*

The reader would have noticed that in the results on orbit closures and invariant measures there has always been a condition involving unipotence. It should be noted that even in the case of affine automorphisms of tori the dynamical behaviour is quite different when the automorphism component has an eigenvalue of absolute value other than 1 (see [DGS] and [F1]); however if all eigenvalues are of absolute value 1 then the behaviour is similar to the unipotent case (see [DMu]). Ratner's work has been followed up in this respect by A. N. Starkov (see [St4] and [St7]), to get a wider perspective in the case of flows on finite-volume homogeneous spaces, namely on G/C, where G a Lie group and $C \in \mathcal{F}(G)$, defined by one-parameter subgroups $\{g_t\}$. He shows that if $\{g_t\}$ is quasiunipotent, namely for all t all (complex) eigenvalues of $\mathrm{Ad}\, g_t$ are of absolute value 1, then all orbit closures are smooth manifolds and all finite ergodic invariant measures consist of smooth measures on these manifolds; they need not however be homogeneous spaces when the one-parameter subgroup is not unipotent. On the other hand it is shown that if $\{g_t\}$ is not quasiunipotent then there exist orbit-closures which are not smooth manifolds.

In another direction, one can also look for other subgroups, not necessarily generated by unipotent elements, for which the conclusion as in Theorems 5.4 or 7.4 hold. Certain interesting examples of such subgroups have been given by Ratner (see [R6], Theorem 9) and Mozes (see [Moz2]).

Before concluding I would like to mention that analogues of many of the results for actions of unipotent subgroups and in particular the conjecture of Raghunathan have been proved in p-adic and S-arithmetic cases as well (see [MT] and [R7]). The analogue of Oppenheim's conjecture in the p-adic and S-arithmetic cases was proved earlier by A. Borel and Gopal Prasad (see [BP1], [BP2], [B1], and [B2]).

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