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Group Theory/Geometry

Surface group representations with maximal Toledo invariant

Sur les représentations d'un groupe de surface compacte avec invariant de Toledo maximal

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Abstract

We study representations of compact surface groups on Hermitian symmetric spaces and characterize those with maximal Toledo invariant. ***To cite this article: M. Burger et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous étudions les représentations d'un groupe de surface compacte sur un espace symétrique hermitien et caractérisons celles avec invariant de Toledo maximal. ***Pour citer cet article : M. Burger et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Let $\Gamma = \pi_1(\Sigma_g)$ be the fundamental group of a compact oriented surface Σ_g of genus $g \geqslant 2$, and X a Hermitian symmetric space of noncompact type, equipped with its Bergman metric. The Toledo invariant τ_ρ of a representation $\rho : \Gamma \to \mathrm{Is(X)}^\circ$ is the integral over Σ_g of the pullback of the Kähler form ω_X of X by any smooth equivariant map $\widetilde{\Sigma_g} \to X$. Then $|\tau_\rho| \leqslant 2|\chi(\Sigma_g)|\pi r_X$, r_X being the rank of X [8,7]. The classical problem of characterizing representations with maximal Toledo invariant has been solved when X is of rank 1 [9,15] and partial results are available when X is associated to $\mathrm{SU}(p,q)$ [10,2].

Theorem 1.1. *Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Is(X)}^\circ$ be a representation with maximal Toledo invariant. Then*

(a) *the Zariski closure L of the image of ρ is reductive*;
(b) *the symmetric subspace $Y \subset X$ associated to L is isometric to a tube type domain*;

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(c) *the group $\pi_1(\Sigma_g)$ acts on Y properly discontinuously without fixed points.*

We give in Section 5 examples where Y is not holomorphically embedded into X. The theorem is optimal also in the following sense:

Proposition 1.2. *For any Hermitian symmetric space X of tube type and any $g \geqslant 2$ there exist representations $\rho : \pi_1(\Sigma_g) \to \mathrm{Is}^\circ(X)$ with maximal Toledo invariant and Zariski dense image.*

Surface group representations with maximal Toledo invariant provide therefore a class of geometrically meaningful Kleinian groups acting on higher rank Hermitian symmetric spaces.

The proof of the theorem relies heavily on [4,7,12,5] and [3]. For a comprehensive treatment of continuous bounded cohomology, we refer to [13].

2. Maximal representations with Zariski dense image

The Toledo invariant of a representation is the evaluation of a linear form on an appropriate bounded cohomology class. Namely, if $\rho : \pi_1(\Sigma_g) \to G$ is any homomorphism and $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G)$, we define $\tau(\rho, \kappa) := \langle \rho^*(\kappa), [\Sigma_g] \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing. Then $\tau_\rho = \tau(\rho, \kappa^{\mathrm{b}}_X)$, where $\kappa^{\mathrm{b}}_X \in \mathrm{H}^2_{\mathrm{cb}}(G)$ is the bounded Kähler class. Since $|\tau(\rho,\kappa)| \leqslant 2|\chi(\Sigma_g)| \|\rho^*(\kappa)\| \leqslant 2|\chi(\Sigma_g)| \|\kappa\|$, where $\chi(\Sigma_g)$ is the Euler characteristic of Σ_g and $\|\kappa\|$ is the Gromov norm of the class $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G)$, we say that ρ *is κ-maximal* if $\tau(\rho,\kappa) = 2|\chi(\Sigma_g)| \|\kappa\|$.

A totally geodesic embedding $t : \mathbb{D} \to X$ is *tight* if $t^*(\omega_X) = \|\kappa^{\mathrm{b}}_X\| / \|\kappa^{\mathrm{b}}_{\mathbb{D}}\| \omega_{\mathbb{D}}$. If t is holomorphic this is equivalent to saying that $\mathbb{D}$ is mapped diagonally into a maximal polydisc in X.

The main point in the proof of the theorem is the following

Proposition 2.1. *Let X be an irreducible Hermitian symmetric space and $\rho : \Gamma \to \mathrm{Is}(X)^\circ$ a representation with maximal Toledo invariant and Zariski dense image. Then X is a symmetric space of tube type, on which $\rho(\Gamma)$ acts properly discontinuously without fixed points.*

We outline the main steps of the proof using results of [7] and following the methods developed in [4] and [12]. Let $\mathcal{D} \subset \mathbb{C}^n$ be the Harish-Chandra realization of X as a bounded symmetric domain with normalized Bergman kernel k. We have $k(x,y) = h(x,y)^{-2}$, where h is a polynomial in $x, \bar{y}$. Following [7] and [4] we say that $x, y \in \overline{\mathcal{D}}$ are transverse if $h(x,y) \neq 0$; then there is a unique continuous determination of the argument of $k(x,y)$ on the set of pairwise transverse points in $\overline{\mathcal{D}}$. Denoting by $\check{S}^{(3)}$ the set of triples of pairwise transverse points in the Shilov boundary $\check{S} \subset \partial\mathcal{D}$, the function $\check{\beta}_{\mathcal{D}}(x,y,z) := -(\arg k(x,y) + \arg k(y,z) + \arg k(z,x))$ is a well defined continuous alternating G-invariant cocycle on $\check{S}^{(3)}$, where $G = \mathrm{Is}(X)^\circ$. Define as in [4, §5], $Z_n := \{(x_1, x_2, \ldots, x_n) \in \check{S}\colon x_i, x_j \text{ are transverse for all } i \neq j\}$ and let $(\mathcal{B}^\infty_{\mathrm{alt}}(Z_n), d_n)$ be the complex of bounded alternating Borel functions on Z_n, endowed with the supremum norm. Using the formula for the symplectic area of a geodesic triangle in $\mathcal{D}$ given in [8,7], and arguing as in [4, Lemmas 5.1 and 5.2], the class $[\check{\beta}_{\mathcal{D}}]$ corresponds to κ^{b}_X under the canonical map $\mathrm{H}^\bullet(\mathcal{B}^\infty_{\mathrm{alt}}(Z_\bullet)^G) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G)$. Next, realize Γ as a cocompact lattice in $\mathrm{PSU}(1,1)$; by using that $\rho(\Gamma)$ is Zariski dense and that transversality in $\check{S}$ is given by a polynomial condition, we deduce as in [4, Proposition 6.2] the existence of a Γ-equivariant measurable map $\varphi : S^1 \to \check{S}$ such that for almost every $x, y \in S^1$, the points $\varphi(x), \varphi(y) \in \check{S}$ are transverse. As a consequence, $\varphi^*\check{\beta}_{\mathcal{D}}(x,y,z) := \check{\beta}_{\mathcal{D}}(\varphi(x), \varphi(y), \varphi(z))$ is a well defined measurable alternating Γ-invariant bounded cocycle on $(S^1)^3$, which corresponds [4, §7] to $\rho^*(\kappa^{\mathrm{b}}_X)$ under the isomorphism $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \simeq \mathcal{Z}L^\infty_{\mathrm{alt}}((S^1)^3)^\Gamma$.

As in [12, §3], we get that for almost every $x, y, z \in S^1$

$$\int_{\Gamma \backslash \mathrm{PSU}(1,1)} \check{\beta}_{\mathcal{D}}\big(\varphi(hx), \varphi(hy), \varphi(hz)\big)\,\mathrm{d}h = \frac{\tau_\rho}{2|\chi(\Sigma_g)|} \check{\beta}_{\mathbb{D}}(x,y,z). \tag{1}$$

If ρ is maximal, $\tau_\rho = 2|\chi(\Sigma_g)|\|\kappa_X^{\mathrm{b}}\|$, which together with $\|\check\beta_{\mathcal{D}}\|_\infty = \|\kappa_X^{\mathrm{b}}\|$ and (1) implies that for almost every $x, y, z \in S^1$, $\check\beta_{\mathcal{D}}(\varphi(x), \varphi(y), \varphi(z)) = \|\kappa_X^{\mathrm{b}}\|/\|\kappa_{\mathbb{D}}^{\mathrm{b}}\|\,\check\beta_{\mathbb{D}}(x, y, z)$. To conclude the proof of Proposition 2.1, fix x and y: by Fubini's theorem, for almost every z, $\check\beta_{\mathcal{D}}(\varphi(x), \varphi(y), \varphi(z)) = \pm\pi r_X$, hence by [7, Proof of Theorem 4.7] the essential image $\operatorname{Ess\,Im}\varphi$ of φ lies in the Shilov boundary of the tube type domain Y of X determined by $\varphi(x)$ and $\varphi(y)$. Since any two transverse points in $\operatorname{Ess\,Im}\varphi$ determine a tube type subdomain, which hence coincides with Y, it follows that Y is $\rho(\Gamma)$-invariant and, by Zariski density, also $G = \mathrm{Is}(X)^\circ$-invariant. Hence $X = Y$.

The image $\rho(\Gamma)$ is discrete: for $r_X = 1$ this follows from [9] (or by [12]); for $r_X \geqslant 2$, G has at least three open orbits in $\check S^3$ since X is of tube type [6, Theorem 4.3, Lemma 5.3], while $\operatorname{Ess\,Im}(\varphi^3)$ is contained in the closure of two open orbits in $\check S^3$, namely $\{(x, y, z) \in \check S^{(3)}\colon\ \check\beta_{\mathcal{D}}(x, y, z) = \pm\pi r_X\}$. Hence $\operatorname{Ess\,Im}(\varphi^3) \neq \check S^3$, which implies that $\rho(\Gamma)$ is not dense, and thus discrete.

The cocycle $\check\beta_{\mathcal{D}}$ can be used to equip the essential graph F of φ with a cyclic ordering. Arguing as in [12, Lemma 5.6] we conclude that if $(x_1, \eta), (x_2, \eta) \in F$ then $x_1 = x_2$, hence ρ is faithful.

3. Proof of Theorem 1.1

Let $L := \overline{\rho(\Gamma)}^{\,\mathrm{Z}}(\mathbb{R})$ be the real points of the Zariski closure of $\rho(\Gamma)$. By passing to a finite index subgroup of Γ we may assume that L is connected. Since the radical of L is amenable, the projection $p : L \to M$ of L to its semisimple part M induces a canonical isometric isomorphism in bounded cohomology, $\mathrm{H}^2_{\mathrm{cb}}(L) \simeq \mathrm{H}^2_{\mathrm{cb}}(M)$ (see [5, Corollary 4.2.4]), with respect to which the class $\kappa_X^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(L)$ defines a class $\mathsf{k} \in \mathrm{H}^2_{\mathrm{cb}}(M)$. Let $M' = M_1 \times \cdots \times M_\ell$ be the product of the simple factors of M such that $\mathsf{k}_i := \mathsf{k}|_{M_i} \neq 0$ and let $\rho_i := \mathrm{pr}_i \circ p \circ \rho : \Gamma \to M_i$, where $\mathrm{pr}_i : M \to M_i$ is the projection, $i = 1, \dots, \ell$. From $2|\chi(\Sigma_g)|\|\mathsf{k}\| = \tau(\rho, \mathsf{k}) = \sum_{i=1}^l \tau(\rho_i, \mathsf{k}_i) \leqslant 2|\chi(\Sigma_g)| \sum_{i=1}^l \|\mathsf{k}_i\|$ and $\|\mathsf{k}\| = \sum_{i=1}^l \|\mathsf{k}_i\|$, it follows that for all i, the representations ρ_i are k_i-maximal. Hence Proposition 2.1 implies that the Hermitian symmetric space Y associated to M' is of tube type.

Let $H < L$ be a connected semisimple subgroup which is isogenous to M' via p, and let $Z \subset X$ be a subsymmetric space associated to H, such that the induced equivariant map $\psi : Y \to Z$ satisfies $\psi^*(\kappa_X^{\mathrm{b}}|_Z) = \sum \mathsf{k}_i$. For any triple of points in the Shilov boundary of Y for which $\check\beta_Y$ is defined and maximal, we get by [7, Theorem 4.7] a holomorphic tight embedding $t : \mathbb{D} \to Y$. The map $T = \psi \circ t : \mathbb{D} \to X$ associated to a homomorphism $\pi : \mathrm{SU}(1,1) \to G$ satisfies $T^*(\omega_X|_Z) = \|\kappa_X^{\mathrm{b}}\|/\|\kappa_{\mathbb{D}}^{\mathrm{b}}\|\,\omega_{\mathbb{D}}$. Since up to scaling T is an isometry and the Euclidean metric on X as a bounded symmetric domain is dominated by the Riemannian metric and T is tight, the map T extends to a π-equivariant map of the boundary $T : \partial\mathbb{D} \to \partial\mathcal{D}$ with $T(\partial\mathbb{D}) \subset \check S$. Let C be the centralizer of $\pi(\mathrm{SU}(1,1))$ in G.

Lemma 3.1. *Let γ be a geodesic in $T(\mathbb{D})$ connecting two points $x, y \in T(\partial\mathbb{D}) \subset \check S$. Then for all $g \in C$ the geodesic $g\gamma$ connects the same points x, y.*

Proof. One can realize the Shilov boundary, which is represented as G/Q, as the equivalence classes of asymptotic maximal singular Weyl chamber walls of type Q. There are natural projections $G/Q' \to G/Q$ for all parabolic subgroups $Q' \subset Q \subset G$, where G/Q' can also be realized as the equivalence classes of asymptotic Weyl chamber (walls) of type Q'. The geodesic γ connects $x, y \in \check S$ and hence lies in a Weyl chamber (wall) of type Q' for some $Q' \subset Q$. The geodesic $g\gamma$ lies in a Weyl chamber (wall) of the same type. Since $g \in C$, the distance between γ and $g\gamma$ is uniformly bounded, it follows that they determine the same point in G/Q', hence in $G/Q = \check S$. □

By the above lemma, any three distinct points $x, y, z \in T(\partial\mathbb{D})$ are fixed by C, hence C fixes the barycenter of x, y, z and is therefore compact.

If L were not reductive, by [1] it would be contained in a proper parabolic subgroup P of G. But then the center of an appropriate Levi component of P would be contained in C and noncompact, which is a contradiction since C is compact. Therefore L is reductive and hence $\rho(\Gamma)$ acts on Y. By Proposition 2.1 the action is properly discontinuous without fixed points.

4. Maximal Zariski dense representations into a tube type domain

For the construction of a representation as in Proposition 1.2, realize the fundamental group as an amalgamated product over a separating geodesic, $\Gamma = A *_{\langle\gamma\rangle} B$. Choose a hyperbolization of Γ, $\pi : \Gamma \to \mathrm{PSU}(1,1)$ and use the diagonal embedding $\Delta : \mathrm{PSU}(1,1) \to \mathrm{PSU}(1,1)^r$ to define hyperbolizations $\rho_i := pr_i \circ \Delta \circ \pi|_A : A \to \mathrm{PSU}(1,1)$ and $\omega_i := pr_i \circ \Delta \circ \pi|_B : B \to \mathrm{PSU}(1,1)$. Let $t_A, t_B : \mathrm{PSU}(1,1)^r \to G$ be two different embeddings, which coincide on $\Delta(\mathrm{PSU}(1,1))$. Choose now two one-parameter families of deformations ρ_i^t, ω_i^t, such that the ρ_i^t's, $i = 1, \dots, r$, respectively the ω_i^t's, are pairwise not conjugated for all t and $\rho_i^t(\gamma) = \rho_i(\gamma)$, respectively $\omega_i^t(\gamma) = \omega_i(\gamma)$, for all t. The representations of A, respectively B, given by $\rho^t(a) = t_A(\rho_1^t(a), \dots, \rho_r^t(a))$, respectively $\omega^t = t_B(\omega_1^t(a), \dots, \omega_r^t(a))$, have Zariski dense image in $t_A(\mathrm{PSU}(1,1)^r)$, respectively $t_B(\mathrm{PSU}(1,1)^r)$, and define a representation $\pi^t : \Gamma \to G$ by the universal property of amalgamated products. By construction π^t has maximal Toledo invariant, hence the Zariski closure of its image is reductive and of maximal rank, since it contains the image of t_A. The symmetric space corresponding to its semisimple part is of tube type and holomorphically embedded into X. Using the characterizations of holomorphic embeddings in [14,11], one can choose t_A, t_B in such a way that the group generated by its images coincides with G.

5. Nonholomorphic tight embeddings

The complex irreducible representation π_p of $\mathrm{SU}(1,1)$ of dimension $2p$ admits an invariant hermitian form unique up to scaling, which is of signature (p, p). The corresponding homomorphism $\pi_p : \mathrm{SU}(1,1) \to \mathrm{SU}(p,p)$ gives rise to a tight embedding $\mathbb{D} \to X_{p,p}$ into the Hermitian symmetric space associated to $\mathrm{SU}(p,p)$, which is holomorphic if and only if $p = 1$. For $p \geqslant 2$ this gives rise to representations of surface groups on $X_{p,p}$ with maximal Toledo invariant, and preserving a nonholomorphically tight embedded disc.

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