THE ACTION OF THE MAPPING CLASS GROUP ON MAXIMAL REPRESENTATIONS

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ABSTRACT. Let Γ_g be the fundamental group of a closed oriented Riemann surface Σ_g, $g \geq 2$, and let G be a simple Lie group of Hermitian type. The Toledo invariant defines the subset of maximal representations $\mathrm{Rep}_{max}(\Gamma_g, G)$ in the representation variety $\mathrm{Rep}(\Gamma_g, G)$. $\mathrm{Rep}_{max}(\Gamma_g, G)$ is a union of connected components with similar properties as Teichmüller space $\mathcal{T}(\Sigma_g) = \mathrm{Rep}_{max}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$. We prove that the mapping class group $\mathbf{Mod}_{\Sigma_g}$ acts properly on $\mathrm{Rep}_{max}(\Gamma_g, G)$ when $G = \mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$, $\mathrm{SO}^*(4n)$, $\mathrm{Spin}(2, n)$.

CONTENTS

1. INTRODUCTION

Let Γ_g be the fundamental group of a closed oriented surface Σ_g of genus $g \geq 2$. Let G be a connected semisimple Lie group and $\mathrm{Hom}(\Gamma_g, G)$ the space of homomorphisms $\rho : \Gamma_g \to G$. The automorphism groups of Γ_g and G act on $\mathrm{Hom}(\Gamma_g, G)$ by

$$\begin{array}{rcl} \mathrm{Aut}(\Gamma_g) \times \mathrm{Aut}(G) \times \mathrm{Hom}(\Gamma_g, G) & \to & \mathrm{Hom}(\Gamma_g, G) \\ (\psi, \alpha, \rho) & \mapsto & \alpha \circ \rho \circ \psi^{-1} : \big(\gamma \mapsto \alpha(\rho(\psi^{-1}\gamma))\big) . \end{array}$$

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Considering homomorphisms only up to conjugation in G defines the representation variety

$$\mathrm{Rep}(\Gamma_g, G) := \mathrm{Hom}(\Gamma_g, G)/\operatorname{Inn}(G).$$

The above action induces an action of the group of outer automorphisms $\mathrm{Out}(\Gamma_g) := \mathrm{Aut}(\Gamma_g)/\operatorname{Inn}(\Gamma_g)$ of Γ_g on $\mathrm{Rep}(\Gamma_g, G)$:

$$\begin{array}{rcl} \mathrm{Out}(\Gamma_g) \times \mathrm{Rep}(\Gamma_g, G) & \rightarrow & \mathrm{Rep}(\Gamma_g, G) \\ (\psi, [\rho]) & \mapsto & [\psi\rho] := \left[\left(\gamma \mapsto \rho(\psi^{-1}\gamma)\right)\right]. \end{array}$$

Recall that $\mathrm{Out}(\Gamma_g)$ is isomorphic to $\pi_0(\mathrm{Diff}(\Sigma_g))$. The mapping class group $\mathbf{Mod}_{\Sigma_g}$ is the subgroup of $\mathrm{Out}(\Gamma_g)$ corresponding to orientation preserving diffeomorphisms of Σ_g. We refer to [16, 10] for a general introduction to mapping class groups and to [11] for a recent survey on dynamical properties of the action of $\mathrm{Out}(\Gamma_g)$ on representation varieties $\mathrm{Rep}(\Gamma_g, G)$.

This note is concerned with the action of the mapping class group on special connected components of $\mathrm{Rep}(\Gamma_g, G)$ when G is of Hermitian type. Recall that a connected semisimple Lie group G with finite center is said to be of Hermitian type if its associated symmetric space $\mathcal{X}$ is a Hermitian symmetric space. When G is of Hermitian type there exists a bounded continuous integer valued function

$$\mathrm{T} : \mathrm{Rep}(\Gamma_g, G) \rightarrow \mathbb{Z}$$

called the *Toledo invariant*.

The level set of the maximal possible modulus of T is the set of *maximal representations*

$$\mathrm{Rep}_{max}(\Gamma_g, G) \subset \mathrm{Rep}(\Gamma_g, G),$$

which is studied in [12, 13, 23, 15, 1, 14, 6, 3, 4, 21]. Since the Toledo invariant is locally constant, its level sets are unions of connected components.

Results of [12, 13, 6, 4] suggest that maximal representations provide a meaningful generalization of Teichmüller space when G is of Hermitian type [25]. This note supports this similarity by proving the following theorem

Theorem 1.1. *Let* $G = \mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$, $\mathrm{SO}^*(4n)$, $\mathrm{Spin}(2, n)$. *Then the action of* $\mathbf{Mod}_{\Sigma_g}$ *on* $\mathrm{Rep}_{max}(\Gamma_g, G)$ *is proper.*

The validity of Theorem 1.1 for all groups locally isomorphic to either $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$, $\mathrm{SO}^*(4n)$ or $\mathrm{Spin}(2, n)$ would follow from an affirmative answer to the following question:

QUESTION. If $G = \mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SU}(n,n)$, $\mathrm{SO}^*(4n)$ or $\mathrm{Spin}(2,n)$, $\overline{G}$ the adjoint form of G, and $\rho \in \mathrm{Rep}_{max}(\Gamma_g, \overline{G})$, does there exist a lift of ρ to G?

REMARK 1.2. Note that maximal representations factor through maximal subgroups of tube type [6, 4]. Therefore the only case which is not covered by the above theorem is the exceptional group $G = E_{7(-25)}$.

We would like to remark that the study of maximal representations $\mathrm{Rep}_{max}(\Gamma_g, G) \subset \mathrm{Rep}(\Gamma_g, G)$ when G is of Hermitian type is related to the study of the Hitchin component $\mathrm{Rep}_H(\Gamma_g, G) \subset \mathrm{Rep}(\Gamma_g, G)$ for split real simple Lie groups G. François Labourie recently announced, as a consequence of his work on Anosov representations and crossratios [18, 20], that the mapping class group acts properly on $\mathrm{Rep}_H(\Gamma_g, \mathrm{SL}(n,\mathbb{R}))$. After finishing this note, we learned that he also has a proof for maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$ [19].

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2. Maximal Representations and Translation Lengths

2.1. **Maximal Representations.** For an introduction and overview the reader is referred to [3, 4]. Let G be a connected semisimple Lie group with finite center. Denote by $\mathcal{X} = G/K$, with $K < G$ a maximal compact subgroup, its associated symmetric space. G is said to be of *Hermitian type* if there exists a G-invariant complex structure J on $\mathcal{X}$. The composition of the Riemannian metric induced by the Killing form $\mathfrak{B}$ on $\mathcal{X}$ with the complex structure defines a Kähler form

$$\omega_{\mathcal{X}}(v,w) := \frac{1}{2}\mathfrak{B}(v, Jw)$$

which is a G-invariant closed differential two-form on $\mathcal{X}$.

Given a representation $\rho : \Gamma_g \to G$ consider the associated flat bundle E_ρ over Σ_g defined by

$$E_\rho := \Gamma_g \backslash (\widetilde{\Sigma}_g \times \mathcal{X}),$$

where Γ_g acts diagonally by deck transformations on $\widetilde{\Sigma}_g$ and via ρ on $\mathcal{X}$. As $\mathcal{X}$ is contractible, there exists a smooth section $f : \Sigma_g \to E_\rho$ which is unique up to homotopy. This section lifts to a smooth ρ-equivariant map $\tilde{f} : \widetilde{\Sigma}_g \to \widetilde{\Sigma}_g \times \mathcal{X} \to \mathcal{X}$. The pull back of $\omega_{\mathcal{X}}$ via $\tilde{f}$ is a Γ_g-invariant two-form $\tilde{f}^*\omega_{\mathcal{X}}$ on $\widetilde{\Sigma}_g$ which may be viewed as a two-form on the closed surface Σ_g. The *Toledo invariant* of ρ is

$$\mathrm{T}(\rho) := \frac{1}{2\pi} \int_{\Sigma_g} \tilde{f}^*\omega_{\mathcal{X}}.$$

The Toledo invariant is independent of the choice of the section f and defines a continuous function

$$\mathrm{T} : \mathrm{Hom}(\Gamma_g, G) \to \mathbb{Z}.$$

The map T is invariant under the action of $\mathrm{Inn}(G)$ and constant on connected components of the representation variety. The Toledo invariant satisfies a generalized Milnor-Wood inequality [8, 7]

$$|\mathrm{T}| \leq \frac{p_{\mathcal{X}}\, \mathrm{rk}_{\mathcal{X}}}{2} |\chi(\Sigma_g)|,$$

where $\mathrm{rk}_{\mathcal{X}}$ is the real rank of $\mathcal{X}$ and $p_{\mathcal{X}} \in \mathbb{N}$ is explicitly computable in terms of the root system.

Definition 2.1. A representation $\rho : \Gamma_g \to G$ is said to be *maximal* if

$$|\mathrm{T}(\rho)| = \frac{p_{\mathcal{X}}\, \mathrm{rk}_{\mathcal{X}}}{2} |\chi(\Sigma_g)|.$$

Remark 2.2. Changing the orientation of Σ_g switches the sign of T. We will restrict our attention to the case when ρ is maximal with $\mathrm{T}(\rho) > 0$.

We define the set of maximal representations

$$\mathrm{Rep}_{max}(\Gamma_g, G) := \{[\rho] \in \mathrm{Rep}(\Gamma_g, G) \mid \rho \text{ is a maximal representation}\},$$

which is a union of connected components of $\mathrm{Rep}(\Gamma_g, G)$. The set $\mathrm{Rep}_{max}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is the union of the two Teichmüller components of Σ_g [12].

The action of the group $\mathrm{Out}(\Gamma_g) := \mathrm{Aut}(\Gamma_g)/\,\mathrm{Inn}(\Gamma_g)$ of outer automorphism of Γ_g on $\mathrm{Rep}(\Gamma_g, G)$ given by

$$\begin{array}{rcl} \mathrm{Out}(\Gamma_g) \times \mathrm{Rep}(\Gamma_g, G) & \to & \mathrm{Rep}(\Gamma_g, G) \\ (\psi, [\rho]) & \mapsto & [\psi\rho] := [(\gamma \mapsto \rho(\psi\gamma))]. \end{array}$$

preserves $\mathrm{Rep}_{max}(\Gamma_g, G)$.

The mapping class group $\mathbf{Mod}_{\Sigma_g}$ preserves, and hence acts on the components of $\mathrm{Rep}_{max}(\Gamma_g, G)$ where $\mathrm{T} > 0$.

Remark 2.3. Note that whereas Teichmüller space, the set of quasi-fuchsian representations and Hitchin components are always contractible subsets of $\mathrm{Rep}(\Gamma_g, G)$, certain components of the set of maximal representations might have nontrivial topology [14, 2].

2.2. **Translation Lengths.** For a hyperbolization $h : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ define the *translation length* tr_h of $\gamma \in \Gamma_g$ as

$$\mathrm{tr}_h(\gamma) := \inf_{p \in \mathbb{D}} \mathrm{d}_{\mathbb{D}}(p, \gamma p).$$

For a representation $\rho : \Gamma_g \to G$ define similarly the translation length tr_ρ of $\gamma \in \Gamma_g$ as

$$\mathrm{tr}_\rho(\gamma) := \inf_{x \in \mathcal{X}_G} \mathrm{d}_{\mathcal{X}}(x, \rho(\gamma)x),$$

where $\mathrm{d}_{\mathcal{X}}$ is any left-invariant distance on the symmetric space associated to G.

Proposition 2.4. *Fix a hyperbolization h of Γ_g. Assume that for any maximal representation $\rho : \Gamma_g \to G$ there exists $A, B > 0$ such that*

$$A^{-1}\,\mathrm{tr}_h(\gamma) - B \leq \mathrm{tr}_\rho(\gamma) \leq A\,\mathrm{tr}_h(\gamma) + B \quad \text{for all } \gamma \in \Gamma_g. \tag{2.1}$$

Then $\mathbf{Mod}_{\Sigma_g}$ acts properly on $\mathrm{Rep}_{max}(\Gamma_g, G)$.

The Proposition relies on the fact that $\mathbf{Mod}_{\Sigma_g}$ acts properly discontinuous on Teichmüller space $\mathcal{T}(\Gamma_g)$, which is due to Fricke.

Lemma 2.5. [9, Proposition 5] *There exists a collection of simple closed curves $\{c_1, \cdots c_{9g-9}\}$ on Σ_g such that the map*

$$\begin{array}{rcl} \mathcal{T}(\Gamma_g) & \to & \mathbb{R}^{9g-9} \\ h & \mapsto & (\mathrm{tr}_h(\gamma_i))_{i=1,\cdots,9g-9}, \end{array}$$

where γ_i is the element of Γ_g corresponding to c_i, is injective and proper.

Remark 2.6. A family of such $9g-9$ curves is given by $3g-3$ curves α_i giving a pants decomposition, $3g-3$ curves β_i representing seems of the pants decomposition and the $3g-3$ curves given by the Dehn twists of β_i along α_i (see e.g. [10]).

Proof of Proposition 2.4. We argue by contradiction. Suppose that the action of $\mathbf{Mod}_{\Sigma_g}$ on $\mathrm{Rep}_{max}(\Gamma_g, G)$ is not proper. Then there exists a compact subset $C \subset \mathrm{Rep}_{max}(\Gamma_g, G)$ such that

$$\#\{\psi \in \mathbf{Mod}_{\Sigma_g} \mid \psi(C) \cap C\}$$

is infinite. Thus there exists an infinite sequence ψ_n in $\mathbf{Mod}_{\Sigma_g}$ and a representation $\rho \in \mathrm{Rep}_{max}(\Gamma_g, G)$ such that $\psi_n(\rho)$ converges to a representation $\rho_\infty \in \mathrm{Rep}_{max}(\Gamma_g, G)$. Since ψ_n acts properly on Teichmüller space $\mathcal{T}(\Gamma_g)$, the sequence of hyperbolizations $\psi_n h$ leaves every compact set of $\mathcal{T}(\Gamma_g)$. This implies that the sum of the translation lengths of the elements γ_i, $i = 1, \cdots, 9g-9$ tends to ∞:

$$\sum_{i=1}^{9g-9} \mathrm{tr}_{\psi_n h}(\gamma_i) \to \infty$$

By assumption (2.1)

$$A^{-1}\,\mathrm{tr}_h(\psi_n^{-1}\gamma_i) - B \leq \mathrm{tr}_\rho(\psi_n^{-1}\gamma_i),$$

hence

$$\sum_{i=1}^{9g-9} \mathrm{tr}_{\psi_n \rho}(\gamma_i) \to \infty.$$

This contradicts $\lim_{n\to\infty} \psi_n\rho = \rho_\infty$, since, by (2.1), the sum $\sum_{i=1}^{9g-9} \mathrm{tr}_{\rho_\infty}(\gamma_i)$ is bounded from above by $A\sum_{i=1}^{9g-9} \mathrm{tr}_h(\gamma_i) + B$. □

Note that the upper bound for the comparison of the translation lengths with respect to a hyperbolization h and a representation ρ is established quite easily

Lemma 2.7. *Fix a hyperbolization h. For every maximal representation $\rho : \Gamma_g \to G$ there exists $A, B \geq 0$ such that*

$$\mathrm{tr}_\rho(\gamma) \leq A\,\mathrm{tr}_h(\gamma) + B \quad \textit{for all } \gamma \in \Gamma_g.$$

Proof. Let $\mathcal{X}$ be the symmetric space associated to G. By [17, Proposition 2.6.1] there exists a ρ-equivariant (uniform) L-Lipschitz map $f : \mathbb{D} \to \mathcal{X}$. Let $p_0 \in \mathbb{D}$ such that $\mathrm{tr}_h(\gamma) = \mathrm{d}_\mathbb{D}(p_0, \gamma p_0)$, then

$$\begin{aligned}\mathrm{tr}_\rho(\gamma) &\leq \mathrm{d}_\mathcal{X}(f(p_0), \rho(\gamma)f(p_0)) = \mathrm{d}_\mathcal{X}(f(p_0), f(\gamma p_0)) \\ &\leq L\,\mathrm{d}_\mathbb{D}(p_0, \gamma p_0) = L\,\mathrm{tr}_h(\gamma).\end{aligned}$$

□

3. Maximal Representations into the Symplectic Group

The main objective of this section is to establish the following

Proposition 3.1. *For any hyperbolization h of Γ_g, there exist constants $A, B \geq 0$ such that*

$$A^{-1}\,\mathrm{tr}_h(\gamma) - B \leq \mathrm{tr}_\rho(\gamma) \leq A\,\mathrm{tr}_h(\gamma) + B$$

for all $\rho \in \mathrm{Rep}_{max}(\Gamma_g, \mathrm{Sp}(2n, \mathbb{R}))$ and all $\gamma \in \Gamma_g$.

Proposition 3.1 in combination with Proposition 2.4 gives

COROLLARY 3.2. *The action of* $\mathbf{Mod}_{\Sigma_g}$ *on* $\mathrm{Rep}_{max}(\Gamma_g, \mathrm{Sp}(2n,\mathbb{R}))$ *is proper.*

That Theorem 1.1 can be deduced from Proposition 3.1 and Proposition 2.4 can be seen as follows - we refer the reader to [3, 5, 24] for more on tight homomorphisms and their properties. Satake [22, Ch. IV] investigated when a simple Lie group G of Hermitian type admits a homomorphism

$$\tau : G \to \mathrm{Sp}(2m,\mathbb{R}).$$

such that the induced homomorphism of Lie algebras

$$\pi : \mathfrak{g} \to \mathfrak{sp}(2m,\mathbb{R})$$

is a so called (H_2)-Lie algebra homomorphism. Examples of such are

$$\tau : \ \mathrm{SU}(n,n) \to \mathrm{Sp}(4n,\mathbb{R})$$
$$\tau : \ \mathrm{SO}^*(4n) \to \mathrm{Sp}(8n,\mathbb{R})$$
$$\tau : \ \mathrm{Spin}(2,n) \to \mathrm{Sp}(2m,\mathbb{R}), \text{ where } m \text{ depends on } n \mod 8.$$

In [24, 5] we prove that any such (H_2)-homomorphism τ is a tight homomorphism. This implies in particular that the composition of any maximal representation $\rho : \Gamma_g \to G$ for $G = \mathrm{SU}(n,n)$, $\mathrm{SO}^*(4n)$, $\mathrm{Spin}(2,n)$ with the homomorphism $\tau : G \to \mathrm{Sp}(2m,\mathbb{R})$ is a maximal representation $\rho_\tau := \tau \circ \rho : \Gamma_g \to \mathrm{Sp}(2m,\mathbb{R})$. By Proposition 3.1 the translation lengths $\mathrm{tr}_h(\gamma)$ and $\mathrm{tr}_{\rho_\tau}(\gamma)$ are comparable. Since the embedding $\mathcal{X}_G \to \mathcal{X}_{\mathrm{Sp}(2m,\mathbb{R})}$, defined by τ, is totally geodesic and the image $\rho_\tau(\Gamma_g)$ preserves $\mathcal{X}_G$, the same argument as in Lemma 3.9 below gives that $\mathrm{tr}_{\rho_\tau}(\gamma) = \mathrm{tr}_\rho(\gamma)$ for all $\gamma \in \Gamma_g$.

3.1. **The Symplectic Group.** For a $2n$-dimensional real vector space V with a nondegenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$, the symplectic group $\mathrm{Sp}(V)$ is defined as

$$\mathrm{Sp}(V) := \mathrm{Aut}(V, \langle \cdot,\cdot \rangle) := \{ g \in \mathrm{GL}(V) \,|\, \langle g\cdot, g\cdot \rangle = \langle \cdot, \cdot \rangle \}.$$

The symmetric space associated to $\mathrm{Sp}(V)$ is given by

$$\mathcal{X}_{\mathrm{Sp}} := \{ J \in \mathrm{GL}(V) \,|\, J^2 = -Id, \ \langle \cdot, J\cdot \rangle >> 0 \},$$

where $\langle \cdot, J\cdot \rangle >> 0$ indicates that $\langle \cdot, J\cdot \rangle$ is symmetric and positive definite. The action of $\mathrm{Sp}(V)$ on $\mathcal{X}_{\mathrm{Sp}}$ is by conjugation $g(J) = g^{-1}Jg$.

We specify a left invariant distance on $\mathcal{X}_{\mathrm{Sp}}$ as follows. Let $J_1, J_2 \in \mathcal{X}_{\mathrm{Sp}}$, the symmetric positive definite forms $\langle \cdot, J_i \cdot \rangle$ define a pair of Euclidean norms q_i on V. Denoting by $||Id||_{J_1,J_2}$ the norm of the identity map from (V, q_1) to (V, q_2) we define a distance on $\mathcal{X}_{\mathrm{Sp}}$ by

$$\mathrm{d}_{\mathrm{Sp}}(J_1, J_2) := \Big| \log ||Id||_{J_1,J_2} \Big| + \Big| \log ||Id||_{J_2,J_1} \Big|$$

3.2. **Transverse Lagrangians and Causal Structure.** Let

$$\mathcal{L}(V) := \{L \subset V \mid \dim(L) = n,\ \langle \cdot, \cdot \rangle_{|L} = 0\}$$

be the space of Lagrangian subspaces of V. Two Lagrangian subspaces $L_+, L_- \in \mathcal{L}(V)$ are said to be transverse if $L_+ \cap L_- = \{0\}$. Any two transverse Lagrangian subspaces $L_+, L_- \in \mathcal{L}(V)$ define a symmetric subspace $\mathcal{Y}_{L_-,L_+} \subset \mathcal{X}_{\mathrm{Sp}}$ by

$$\mathcal{Y}_{L_-,L_+} := \{J \in \mathcal{X}_{\mathrm{Sp}} \mid J(L_\pm) = L_\mp\} \subset \mathcal{X}_{\mathrm{Sp}}.$$

Writing an element $g \in \mathrm{Sp}(V)$ in block decomposition $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the decomposition $V = L_- \oplus L_+$ defines a natural embedding $\mathrm{GL}(L_-) \to \mathrm{Sp}(V)$ given by

$$\begin{array}{ccc} \mathrm{GL}(L_-) & \to & \mathrm{Sp}(V) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & A^{T^{-1}} \end{pmatrix}, \end{array}$$

similarly for $\mathrm{GL}(L_+)$.

The subgroup $\mathrm{GL}(L_-)$ preserves the symmetric subspace $\mathcal{Y}_{L_-,L_+}$ and acts transitively on it.

Remark 3.3. The space of Lagrangians $\mathcal{L}(V)$ can be identified with the Shilov boundary $\check{S}_{\mathrm{Sp}}$ of $\mathcal{X}_{\mathrm{Sp}}$, and realized inside the visual boundary as the G-orbit of a specific maximal singular direction. Two Lagrangians are transverse if and only if the two corresponding points in the visual boundary can be joined by a maximal singular geodesic $\gamma_{L_\pm}$. The symmetric subspace $\mathcal{Y}_{L_-,L_+}$ is the parallel set of $\gamma_{L_\pm}$, i.e. the set of points on flats containing the geodesic $\gamma_{L_\pm}$; it is the noncompact symmetric space dual to $\mathcal{L}(V) \simeq \mathrm{U}(L_-)/\mathrm{O}(L_-)$.

For $J \in \mathcal{Y}_{L_-,L_+}$ the restriction of $\langle \cdot, J\cdot \rangle$ to L_- is a positive definite symmetric bilinear form on L_-, and conversely, fixing L_+, any positive definite symmetric bilinear form Z on L_- defines a complex structure $J \in \mathcal{Y}_{L_-,L_+}$. Therefore, the space

$$\mathcal{Y}_{L_-,s} := \{Z \mid Z \text{ positive bilinear form on } L_-\}.$$

with the action of $\mathrm{GL}(L_-)$ by

$$A(Z) := A^T Z A,$$

where we choose a scalar product on L_- (i.e. a base point) and realize a bilinear form on L_- as a symmetric $(n \times n)$ matrix Z, is $\mathrm{GL}(L_-)$-equivariantly isomorphic to $\mathcal{Y}_{L_-,L_+}$. We endow $\mathcal{Y}_{L_-,s}$ with the left-invariant distance induced by d_{Sp} via this isomorphism.

The space $\mathcal{Y}_{L_-,s}$ is endowed with a natural causal structure, given by the $\mathrm{GL}(L_-)$-invariant family of proper open cones

$$\Omega_Z := \{Z' \subset \mathcal{Y}_{L_-,s} \,|\, Z' - Z \text{ is positive definite } \}.$$

DEFINITION 3.4. A continuous map $f : [0,1] \to \mathcal{Y}_{L_-,s}$ is said to be *causal* if $f(t_2) \in \Omega_{f(t_1)}$ for all $0 \le t_1 < t_2 \le 1$.

A consequence of the proof of Lemma 8.10 in [3] is the following

LEMMA 3.5. *For all $Z \in \mathcal{Y}_{L_-,s}$, $Z' \in \Omega_Z$ and every causal curve $f : [0,1] \to \mathcal{Y}_{L_-,s}$ with $f(0) = Z$ and $f(1) = Z'$:*

$$\mathrm{length}(f) \le n\, \mathrm{d}_{\mathrm{Sp}}(Z, Z'),$$

where $n = \dim(L_-)$.

The claim basically follows from the last inequality in the proof of Lemma 8.10 in [3]. However, for the reader's convenience we give a direct proof here.

Proof. Since d_{Sp} is left invariant it is enough to prove the statement for $Z = Id_n \in \mathcal{Y}_{L_-,s}$. For any subdivision

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

let $f(t_i) = B_i^T B_i \in \mathcal{Y}_{L_-,s}$, and note that by causality

$$\det\left((B_i B_{i+1}^{-1})^T (B_i B_{i+1}^{-1})\right) < 1.$$

With $n = \dim(L_-)$, we have

$$\begin{aligned}
\mathrm{d}_{\mathrm{Sp}}(f(t_i), f(t_{i+1})) &= \mathrm{d}_{\mathrm{Sp}}(B_i^T B_i, B_{i+1}^T B_{i+1}) \\
&= \log\left[\lambda_{max}\left((B_{i+1}B_i^{-1})^T (B_{i+1}B_i^{-1})\right)\right] \\
&+ \log\left[\lambda_{min}\left((B_{i+1}B_i^{-1})^T (B_{i+1}B_i^{-1})\right)\right] \\
&\le \log\left[\det\left((B_{i+1}B_i^{-1})^T (B_{i+1}B_i^{-1})\right)\right] \\
&- n\log\left[\det\left((B_i B_{i+1}^{-1})^T (B_i B_{i+1}^{-1})\right)\right] \\
&\le n\log\left[\lambda_{max}(B_{i+1}^T B_{i+1})\right] \\
&- n\log\left[\lambda_{min}(B_i^T B_i)\right] \\
&+ n\log\left[\lambda_{min}(B_{i+1}^T B_{i+1})\right] \\
&- n\log\left[\lambda_{max}(B_i^T B_i)\right].
\end{aligned}$$

Summing over the subdivision we obtain

$$\begin{aligned}\text{length}(f) &\leq \sum_{i+1}^{m} \mathrm{d}_{\mathrm{Sp}}(f(t_i), f(t_{i+1})) \\ &\leq n\,[\log\lambda_{max}(f(1)) + \log\lambda_{min}(f(1))] \\ &- n\,[\log\lambda_{max}(f(0)) - \log\lambda_{min}(f(0))] \\ &= n\,[\log\lambda_{max}(f(1)) + \log\lambda_{min}(f(1))] \\ &= n\,\mathrm{d}_{\mathrm{Sp}}(f(0), f(1))\end{aligned}$$

as needed. □

3.3. **Quasi-isometric Embedding.** Let $\rho : \Gamma_g \to \mathrm{Sp}(V)$ be a maximal representation. The choice of a hyperbolization h of Γ_g defines a natural action of Γ_g on $S^1 = \partial\mathbb{D}$.

LEMMA 3.6. [3, Corollary 6.3] *There exists a ρ-equivariant continuous map $\varphi : S^1 \to \mathcal{L}(V)$ such that distinct points $x, y \in S^1$ are mapped to transverse Lagrangians $\varphi(x), \varphi(y) \in \mathcal{L}(V)$.*

A triple $(L_-, L_0, L_+) \in \mathcal{L}(V)^3$ of pairwise transverse Lagrangians gives rise to a complex structure

$$J_{L_0} = \begin{pmatrix} 0 & -T_0^+ \\ T_0^- & 0 \end{pmatrix} \quad \text{on} \quad V = L_- \oplus L_+,$$

where $T_0^{\pm} : L_{\pm} \to L_{\mp}$ is the unique linear map such that $L_0 = \mathrm{graph}(T_0^{\pm})$. A triple (L_-, L_0, L_+) of pairwise transverse Lagrangians is *maximal* if the symmetric bilinear form $\langle \cdot, J_{L_0} \cdot \rangle$ is positive definite, that is if $J_{L_0} \in \mathcal{Y}_{L_-,L_+} \subset \mathcal{X}_{\mathrm{Sp}}$. We denote by $\mathcal{L}(V)^{3+}$ the space of maximal pairwise transverse triples in $\mathcal{L}(V)$.

Under the identification of the unit tangent bundle of the Poincaré-disc $T^1\mathbb{D} \simeq (S^1)^{3+}$ with positively oriented triples in S^1, the map φ gives rise to a ρ-equivariant map (Equation (8.9) in [3])

$$\begin{aligned} J : T^1\mathbb{D} \cong (S^1)^{3+} \to \mathcal{L}(V)^{3+} &\;\to\; \mathcal{X}_{\mathrm{Sp}} \\ u = (u_-, u_0, u_+) \mapsto (\varphi(u_-), \varphi(u_0), \varphi(u_+)) &\;\mapsto\; J(u), \end{aligned}$$

where $J(u)$ is the complex structure defined by the maximal triple

$$(\varphi(u_-), \varphi(u_0), \varphi(u_+)) \in \mathcal{L}(V)^{3+}.$$

Let g_t be the lift of the geodesic flow on $T^1\Sigma_g$ to $T^1\mathbb{D}$. Then for all t the image of $g_t u = (u_-, u_t, u_+)$ under J is contained in the symmetric subspace $\mathcal{Y}_{\varphi(u_-),\varphi(u_+)} \subset \mathcal{X}_{\mathrm{Sp}}$ associated to the two transverse Lagrangians $\varphi(u_-), \varphi(u_+)$.

LEMMA 3.7. [3, Equation 8.8] *Let $\gamma \in \Gamma_g \backslash \{Id\}$ and $p \in \mathbb{D}$ a point. Denote by $u \in T^1\mathbb{D}$ the unit tangent vector at p of the geodesic connecting p and γp. Then there exists a constant $A' > 0$ such that*

$$A'^{-1}\, \mathrm{d}_{\mathbb{D}}(\gamma p, p) \leq \mathrm{d}_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)).$$

REMARK 3.8. Note that the statement of Lemma 3.7 implies that the action of $\mathbf{Mod}_{\Sigma_g}$ on the connected components of maximal Toledo invariant in $\mathrm{Hom}(\Gamma_g, G)$ is proper, but is not sufficient to deduce Theorem 1.1. The inequality in Lemma 3.7 is with respect to specific points in $\mathcal{X}_{\mathrm{Sp}}$, but to compare the translation lengths we have to take infima on both sides of the inequality. There is in general no direct way to compare the translation length of $\rho(\gamma)$ with the displacement length of $\rho(\gamma)$ with respect to a specific point $x \in \mathcal{X}_{\mathrm{Sp}}$. In our situation we will make use of the causal structure on $\mathcal{Y}_{L_-,s}$ to compare the translation length of $\rho(\gamma)$ with the displacement length $\mathrm{d}_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u))$.

3.4. **Translation Length and Displacement Length.** Fix a hyperbolization $h : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$. Let $\gamma \in \Gamma_g \backslash \{Id\}$. Denote by $\gamma^+, \gamma^- \in S^1$ the attracting, respectively repelling, fixed point of γ and by $L_\pm \in \mathcal{L}(V)$ the images of $\gamma^\pm$ under the ρ-equivariant boundary map $\varphi : S^1 \to \mathcal{L}(V)$. Let $\mathcal{Y}_{L_-,L_+} \subset \mathcal{X}_{\mathrm{Sp}}$ be the symmetric subspace associated to L_+, L_-.

LEMMA 3.9. *The translation length of $\rho(\gamma)$ is attained on $\mathcal{Y}_{L_-,L_+}$:*

$$\mathrm{tr}_\rho(\gamma) = \inf_{J' \in \mathcal{Y}_{L_-,L_+}} \mathrm{d}_{\mathrm{Sp}}(J', \rho(\gamma)J').$$

Proof. The symmetric subspace $\mathcal{Y}_{L_-,L_+}$ is a totally geodesic submanifold of $\mathcal{X}_{\mathrm{Sp}}$. Denote by $\mathrm{pr}_{\mathcal{Y}} : \mathcal{X}_{\mathrm{Sp}} \to \mathcal{Y}_{L_-,L_+}$ the nearest point projection onto $\mathcal{Y}_{L_-,L_+}$. The projection $\mathrm{pr}_{\mathcal{Y}}$ is distance decreasing. Since the element $\rho(\gamma)$ stabilizes $L_\pm$, the projection $\mathrm{pr}_{\mathcal{Y}}$ onto $\mathcal{Y}_{L_-,L_+}$ is $\rho(\gamma)$-equivariant. Thus, for every $J \in \mathcal{X}_{\mathrm{Sp}}$

$$\mathrm{d}_{\mathrm{Sp}}\left(J, \rho(\gamma)J\right) \geq \mathrm{d}_{\mathrm{Sp}}\left(\mathrm{pr}_{\mathcal{Y}}(J), \mathrm{pr}_{\mathcal{Y}}(\rho(\gamma)J)\right) = \mathrm{d}_{\mathrm{Sp}}\left(\mathrm{pr}_{\mathcal{Y}}(J), \rho(\gamma)\, \mathrm{pr}_{\mathcal{Y}}(J)\right).$$

In particular

$$\inf_{J \in \mathcal{X}_{\mathrm{Sp}}} \mathrm{d}_{\mathrm{Sp}}(J, \rho(g)J) = \inf_{J' \in \mathcal{Y}_{L_-,L_+}} \mathrm{d}_{\mathrm{Sp}}(J', \rho(g)J').$$

□

LEMMA 3.10. *Let $p_0 \in \mathbb{D}$ be some point lying on the geodesic c_γ connecting γ^- to γ^+. Let $u = (\gamma^-, u_0, \gamma^+) \in T^1\mathbb{D}$ be the unit vector tangent to c_γ at p_0. Then*

(1) $\rho(\gamma)J(u) \in \Omega_{J(u)}$.

(2) *For any* $Z \in Y_{L_-,s}$ *there exists an* N *such that for all* $m \geq N$*:* $\rho(\gamma)^m J(u) \in \Omega_Z$.

Proof. (1) The map $u \mapsto J(u)$ is ρ-equivariant, thus $\rho(\gamma)J(u) = J(v) \in \mathcal{Y}_{L_-,s}$, with $v = (\gamma^-, \gamma u_0, \gamma^+)$. Since γ^+ is the attracting fixed point of γ, the triple $(u_0, \gamma u_0, \gamma^+)$ is positively oriented, and $(\varphi(u_0), \varphi(\gamma u_0), \varphi(\gamma^+))$ is a maximal triple. Then, by [3, Lemma 8.2], $\rho(\gamma)J(u) - J(u) = J(v) - J(u)$ is positive definite, thus $\rho(\gamma)J(u) \in \Omega_{J(u)}$.
(2) Let μ be the maximal eigenvalue of $Z \in \mathcal{Y}_{L_-,s}$. It suffices to show that there exists some N such that the minimal eigenvalue of $\rho(\gamma)^N J(u) \in \mathcal{Y}_{L_-,s}$ is bigger than μ. Then $\rho(\gamma)^N J(u) \in \Omega_Z$ and, with statement (1), we have that $\rho(\gamma)^m J(u) \in \Omega_{\rho(\gamma)^N J(u)} \subset \Omega_Z$ for all $m > N$. Note that $\gamma^m u_0 \to \gamma^+$ as $m \to \infty$ and, since φ is continuous, $\rho(\gamma)^m \varphi(u_0) \to \varphi(\gamma^+)$. Moreover $\rho(\gamma)^{i+1} J(u) - \rho(\gamma)^i J(u)$ is positive definite for all i. Hence, the eigenvalues of $\rho(\gamma)^i J(u)$ grow monotonically towards ∞. In particular, there exists N such that the minimal eigenvalue of $\rho(\gamma)^N J(u)$ is bigger than μ. □

Combining Lemma 3.10 with Lemma 3.5 we obtain

Lemma 3.11. *There exist constants* $A'', B'' > 0$ *such that for every* $Z \in \mathcal{Y}_{L_-,s}$

$$\mathrm{d}_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) \leq A'' \, \mathrm{d}_{\mathrm{Sp}}(Z, \rho(\gamma)Z) + B''.$$

Proof. Fix $Z \in \mathcal{Y}_{L_-,s}$. Choose, by Lemma 3.10, N big enough such that $Z' := \rho(\gamma)^N J(u) \in \Omega_Z$. By Lemma 3.10 there are causal, distance realizing curves f_Z from Z to Z' and f_i from $\rho(\gamma)^i Z'$ to $\rho(\gamma)^{i+1} Z'$ for all $0 \leq i$. For every $k \geq 0$ the concatenation $f = f_{k-1} * \cdots * f_0 * f_Z$ is a causal curve from Z to $\rho(\gamma)^k Z' = \rho(\gamma)^{N+k} J(u)$. Thus applying Lemma 3.5 we get that for every $k \geq 0$

$$\begin{aligned}
& \mathrm{d}_{\mathrm{Sp}}(Z, Z') + k \, \mathrm{d}_{\mathrm{Sp}}(Z', \rho(\gamma)Z') \\
= \;& \mathrm{d}_{\mathrm{Sp}}(Z, Z') + \sum_{i=0}^{k-1} \mathrm{d}_{\mathrm{Sp}} \left(\rho(\gamma)^i Z', \rho(\gamma)^{i+1} Z'\right) \\
= \;& \mathrm{length}(f) \\
\leq \;& n \, \mathrm{d}_{\mathrm{Sp}}(Z, \rho(\gamma)^k Z') \\
\leq \;& n \left[\mathrm{d}_{\mathrm{Sp}}(Z, \rho(\gamma)^k Z) + \mathrm{d}_{\mathrm{Sp}}(\rho(\gamma)^k Z, \rho(\gamma)^k Z')\right] \\
\leq \;& n \left[\sum_{i=0}^{k-1} \mathrm{d}_{\mathrm{Sp}}(\rho(\gamma)^i Z, \rho(\gamma)^{i+1} Z) + \mathrm{d}_{\mathrm{Sp}}(Z', Z)\right] \\
= \;& n \, \mathrm{d}_{\mathrm{Sp}}(Z', Z) + nk \, \mathrm{d}_{\mathrm{Sp}}(Z, \rho(\gamma)Z).
\end{aligned}$$

In particular

$$d_{\mathrm{Sp}}(Z', \rho(\gamma)Z') \leq n\, d_{\mathrm{Sp}}(Z, \rho(\gamma)Z) + \frac{n-1}{k}\, d_{\mathrm{Sp}}(Z', Z).$$

Thus, for $A'' = n$ and $B'' > 0$ fixed we can choose k big enough such that $\frac{n-1}{k}\, d_{\mathrm{Sp}}(Z', Z) \leq B''$ to get

$$d_{\mathrm{Sp}}(Z', \rho(\gamma)Z') \leq A''\, d_{\mathrm{Sp}}(Z, \rho(\gamma)Z) + B''.$$

Since d_{Sp} is left invariant, this implies

$$\begin{aligned} d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) &\leq d_{\mathrm{Sp}}(\rho(\gamma)^N J(u), \rho(\gamma)^{N+1} J(u)) \\ &= d_{\mathrm{Sp}}(Z', \rho(\gamma)Z') \\ &\leq A''\, d_{\mathrm{Sp}}(Z, \rho(\gamma)Z) + B'', \end{aligned}$$

hence the claim. □

Lemma 3.12. *There exist constants $A'', B'' > 0$ depending only on $\dim(V)$ such that for all $u \in T^1\mathbb{D}$ and $\gamma \in \Gamma_g$*

$$d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) \leq A''\, \mathrm{tr}_\rho(\gamma) + 2B''.$$

Proof. Fix some $\epsilon > 0$. By Lemma 3.9 there exists $Z_0 \in \mathcal{Y}_{L_-,s}$ such that

$$d_{\mathrm{Sp}}(Z_0, \rho(\gamma)Z_0) \leq \inf_{X \in \mathcal{Y}_{L_-,s}} d_{\mathrm{Sp}}(X, \rho(\gamma)X) + \epsilon = \mathrm{tr}_\rho(\gamma) + \epsilon.$$

Therefore Lemma 3.11 implies

$$d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) \leq A''\, d_{\mathrm{Sp}}(Z_0, \rho(\gamma)Z_0) + B'' \leq A''\, \mathrm{tr}_\rho(\gamma) + A''\epsilon + B''.$$

Since this holds for all $\epsilon > 0$ we get that

$$d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) \leq A''\, \mathrm{tr}_\rho(\gamma) + 2B''.$$

□

Proof of Proposition 3.1. Let ρ be a maximal representation of Γ_g into $\mathrm{Sp}(V)$ and let $p \in \mathbb{D}$ be such that $\mathrm{tr}_h(\gamma) = d_{\mathbb{D}}(p, \gamma p)$;, then p lies on the unique geodesic c_γ connecting γ^- to γ^+.

Let $u = (\gamma^-, u_0, \gamma^+) \in T^1\mathbb{D}$ be the unit tangent vector to c_γ at p and $J(u) \in \mathcal{Y}_{L_-,L_+}$ the image of u under the mapping $J : T^1\mathbb{D} \to \mathcal{X}_{\mathrm{Sp}}$. By Lemma 3.7 there exists a constant A' such that

$$A'^{-1}\, \mathrm{tr}_h(\gamma) = A'^{-1}\, d_{\mathbb{D}}(p, \gamma p) \leq d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)).$$

Applying Lemma 3.12 to this, there exist constants $A'', B'' > 0$ such that

$$\mathrm{tr}_h(\gamma) \leq A'\, d_{\mathrm{Sp}}(J(u), \rho(\gamma)J(u)) \leq A''A'\, \mathrm{tr}_\rho(\gamma) + 2A'B''.$$

This in combination with Lemma 2.7 finishes the proof. □

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