Bounded Cohomology and Representations of Surface Groups

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In this talk I report on joint work with M. Burger and A. Iozzi [1, 2].

Let Σ be a compact connected oriented surface of negative Euler characterstic $\chi(\Sigma) < 0$. The representation variety

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G,$$

where G is a semisimple Lie group with finite center and no compact factors, is isomorphic to the moduli space of flat G-bundles on Σ and, by choosing a complex structure on Σ, also to the moduli space of (polystable) G-Higgs bundles on Σ with vanishing Chern classes.

Our study of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is motivated by the well known fact that when Σ is a closed surface and $G = \mathrm{PSL}(2, \mathbf{R})$, the Teichmüller space of Σ embeds as a connected component into $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R})$. This connected component is homeomorphic to a ball and consists entirely of discrete and faithful representations.

There are two families of Lie groups G for which "higher Teichmüller spaces" in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ are known to exist.

If G is a split real form and Σ a closed surface Hitchin defined in [5] the Hitchin component $\mathrm{Hom}_{Hit}(\pi_1(\Sigma), G)/G \subset \mathrm{Hom}(\pi_1(\Sigma), G)/G$ and showed that it is homeomorphic to a ball. Recently, Labourie [6] showed that all representation in

$$\mathrm{Hom}_{Hit}(\pi_1(\Sigma), \mathrm{PSL}(n, \mathbf{R}))/\mathrm{PSL}(n, \mathbf{R})$$

are discrete, faithful and loxodromic. At the same time Fock and Goncharov introduced in [3] the set of positive representations

$$\mathrm{Hom}_{pos}(\pi_1(\Sigma), G)/G \subset \mathrm{Hom}(\pi_1(\Sigma), G)/G,$$

when G is a split real form, and showed that positive representations are discrete and faithful. For closed surface $\mathrm{Hom}_{Hit}(\pi_1(\Sigma), G)/G = \mathrm{Hom}_{pos}(\pi_1(\Sigma), G)/G$.

When G is a Lie group of Hermitian type, we define (using bounded cohomology) a continuous bounded function, the Toledo invariant

$$\tau : \mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathbf{R}.$$

The Toledo invariant satisfies a Milnor-Wood-type inequality

$$|\tau| \leq r_G |\chi(\Sigma)|,$$

where r_G denotes the real rank of G. The set of maximal representations

$$\mathrm{Hom}_{max}(\pi_1(\Sigma), G)/G \subset \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

is defined as the level set $\tau^{-1}(r_G|\chi(\Sigma)|)$.

When $G = \mathrm{PSL}(2, \mathbf{R})$, then τ equals the Euler number and Goldman showed in [4] that for closed surfaces $\mathrm{Hom}_{max}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R})$ is the image of the embedding of Teichmüller space.

We show that for every G of Hermitian type the set of maximal representations consists entirely of discrete and faithful representation. Further results and details of the proofs can be found in [2].

For $G = \mathrm{Sp}(2n, \mathbf{R})$, the unique Lie group which is both a split real form and of Hermitian type, we show that $\mathrm{Hom}_{Hit}(\pi_1(\Sigma), G)/G$ and $\mathrm{Hom}_{pos}(\pi_1(\Sigma), G)/G$ are proper subsets of $\mathrm{Hom}_{max}(\pi_1(\Sigma), G)/G$.

References

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