REPRESENTATIONS OF SURFACE GROUPS: BACKGROUND MATERIAL FOR AIM WORKSHOP

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This document is an introduction to some topics of the Workshop "Representations of Surface Groups", March 19 - March 23, 2007 at the American Institute of Mathematics in Palo Alto. Its purpose is to frame the discussion of the workshop by presenting the prerequisite background, a common terminology and notation, and references for further study. The document is roughly divided into Part I discussing more the general background and into Part II which discusses several more specific aspects in the case of SL_2 as guiding example. In addition to this document, we will have a user-prepared online glossary of mathematical terms to help participants know what one another is talking about.

The workshop will concentrate on two general questions:

- Understand the topology, global structure (metrics, complex structures, geometry, symmetry) of the deformation space $\mathsf{Hom}(\pi, G)/G$ of representations of a surface group π in a Lie group G;
- Find invariants of representations $\pi \xrightarrow{\rho} G$ which detect special properties of ρ, such as when ρ is a discrete embedding.

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Part 1. General Background

1. What is a surface group representation?

1.1. **Surfaces and their fundamental groups.** Let $\Sigma = \Sigma_{g,k}$ be a compact connected surface of genus g with k holes (that is, Σ is the complement of k disjoint open discs in a closed orientable surface of genus g). For technical convenience, we choose an orientation and a smooth structure on Σ. By abusing basepoints, denote the fundamental group of Σ:

$$\pi := \pi_1(\Sigma).$$

The common decomposition of Σ of a $4g+k$-gon with $2g$ identifications of its sides leads to a presentation

$$\pi = \langle A_1, B_1, \dots, A_g, B_g, \dots, C_1, \dots, C_k \mid [A_1, B_1] \dots [A_g, B_g] C_1 \dots C_k = 1 \rangle \tag{1.1.1}$$

where $[A, B] := ABA^{-1}B^{-1}$. The fundamental group π is free $\Longleftrightarrow k \neq 0$. At first we only consider the case $k = 0$ (that is, $\partial\Sigma = \emptyset$).

1.2. **The deformation space.** Let G be a Lie group. Denote the set of representations $\pi \xrightarrow{\rho} G$ by $\mathsf{Hom}(\pi, G)$. Evaluation on a collection $\gamma_1, \dots, \gamma_N \in \pi$

$$\mathsf{Hom}(\pi, G) \longrightarrow G^N$$
$$\rho \longmapsto \begin{bmatrix} \rho(\gamma_1) \\ \vdots \\ \rho(\gamma_N) \end{bmatrix}$$

is an embeedding if $\gamma_1, \dots, \gamma_N$ generate π. Its image consists of N-tuples satisfying the defining relations of π satisfied by the generators $\gamma_1, \dots, \gamma_N$. If G is an algebraic group, this expresses $\mathsf{Hom}(\pi, G)$ as an algebraic subset of G^N. This algebraic structure is independent of the generating set. In particular $\mathsf{Hom}(\pi, G)$ inherits both the Zariski and the classical topology. By default we consider the classical topology unless otherwise noted.

In terms of the standard presentation (1.1), $\mathsf{Hom}(\pi, G)$ identifies with the subset of G^{2g+k} consisting of

$$(\alpha_1, \beta_1, \dots \alpha_g, \beta_g, \gamma_1, \dots, \gamma_k)$$

satisfying the single G-valued equation

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] \gamma_1 \dots \gamma_k = 1.$$

1.3. **Symmetries of $\mathsf{Hom}(\pi, G)$.** The natural action of $\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$ on $\mathsf{Hom}(\pi, G)$ defined by left- and right-composition

$$\mathsf{Aut}(\pi) \times \mathsf{Aut}(G) : \mathsf{Hom}(\pi, G) \longrightarrow \mathsf{Hom}(\pi, G)$$
$$(\varphi, \alpha) : \varphi \longmapsto \alpha \circ \rho \circ \varphi^{-1}$$

preserves the algebraic structure. We will mainly be concerned with the quotient

$$\mathsf{Hom}(\pi, G)/G := \mathsf{Hom}(\pi, G)/\big(\{1\} \times \mathsf{Inn}(G)\big)$$

upon which $\mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi)$ acts. In general the *mapping class group* $\pi_0(\mathsf{Diff}(\Sigma)$ embeds in $\mathsf{Out}(\pi)$, and if $k = 0$ (or $g = n = 1$), this embedding is an isomorphism. One motivation for this study is that the deformation spaces $\mathsf{Hom}(\pi, G)/G$ are natural objects upon which $\mathsf{Out}(\pi)$ acts.

1.4. **When G is abelian.** The simplest groups are commutative. Suppose that $k = 0$. When G is abelian, then the commmuators $[\alpha, \beta] = 1$ and the defining relation in (1.1) is vacuous. Thus

$$\mathsf{Hom}(\pi, G) \longleftrightarrow G^{2g}$$

Furthermore $\mathsf{Inn}(G)$ is trivial so

$$\mathsf{Hom}(\pi, G)/G \longleftrightarrow G^{2g}$$

as well.

Homological machinery applies. By the Hurewicz theorem and the universal coefficient theorem,

$$\mathsf{Hom}(\pi, G) \cong \mathsf{Hom}(\pi/[\pi, \pi], G) \cong \mathsf{Hom}(H_1(\Sigma), G) \cong H^1(\Sigma, G)$$

(or $H^1(\pi, G)$ if you prefer group cohomology). In particular when $G = \mathbb{R}$, then $\mathsf{Hom}(\pi, G)/G$ is the real vector space

$$H^1(\Sigma, \mathbb{R}) \cong \mathbb{R}^{2g}$$

which is naturally a *symplectic vector space* under the cup-product pairing

$$H^1(\Sigma, \mathbb{R}) \times H^1(\Sigma, \mathbb{R}) \longrightarrow H^2(\Sigma, \mathbb{R}) \cong \mathbb{R}.$$

(Note that the orientation of Σ is used to obtained a scalar-valued bilinear form.) The mapping class group action factors through the symplectic linear group:

$$\mathsf{Out}(\pi) \cong \pi_0(\mathsf{Diff}(\Sigma)) \longrightarrow \mathsf{Sp}(2g, \mathbb{Z}).$$

The other fundamental case occurs when $G = \mathsf{U}(1) \cong \mathbb{R}/\mathbb{Z} \approx S^1$. In this case $\mathsf{Hom}(\pi, G)/G$ is a $2g$-dimensional torus, which identifies with the quotient

$$H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$$

of the symplectic vector space $H^1(\Sigma, \mathbb{R})$ by its integer lattice $H^1(\Sigma, \mathbb{Z})$.

This quotient admits another intepretation when we fix a *complex structure* on Σ, i.e. an endomorphism

$$T_x\Sigma \xrightarrow{J} T_x\Sigma$$

of the tangent space $T_x\Sigma$ such that $J^2 = -\mathrm{Id}$. This makes Σ into a one-dimensional complex manifold, that is, a *Riemann surface,* which we denote

$$X = (\Sigma, J).$$

Over a Riemann surface $X = (\Sigma, J)$ diffeomorphic to Σ the space $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$ appears as the *Jacobian* $\mathsf{Jac}(X)$ of X, which parametrizes isomorphism classes of topologically trivial holomorphic line bundles over X (for more explanation see below). It is an *abelian variety* (a compact complex algebraic group), whose isomorphism class actually detects the isomorphism class of the Riemann surface X (Torelli's theorem). For the theory when $G = \mathbb{C}^*$ (the nonzero complex numbers), see the expository memoir [40]. This paper develops Higgs bundle theory over Riemann surfaces in the "trivial" cases of line bundles.

In these abelian cases $G = \mathbb{R}, S^1, \mathbb{C}^*$, the deformation space $\mathsf{Hom}(\pi, G)/G$ is a smooth manifold (in fact a Lie group).

1.5. **de Rham, Hodge and Dolbeault theory.** Going from a topological object like $H^1(\Sigma,\mathbb{R})/H^1(\Sigma,\mathbb{Z})$ to the holomorphic object $\mathsf{Jac}(X)$ plays an important role more generally. The *de Rham theory* identifies $H^*(\Sigma,\mathbb{R})$ with the *cosets* of closed differential forms by the subspace of exact differential forms. This requires a differentiable structure on Σ. *Hodge theory* identifies the de Rham cohomology with the space of harmonic *forms* (not just equivalence classes of forms). This requires a Riemannian metric on Σ. Briefly, a 1-form has an *energy,* which is just the square of its L^2-norm taken with respect to the metric. The unique form in a cohomology class c is then the closed form in c of minimum energy (the smallest, or closest to 0, one in the affine subspace defined by c).

The situation's even better for 1-forms on 2-manifolds. In this case the L^2-norm defined by the Riemannian metric depends only on the *conformal structure* defined by the Riemannian structure. In other words, it only depends on the measurement of angles, not lengths. In dimension two, the conformal structure is conveniently represented by the *almost complex structure,* the automorphism

$$T_x\Sigma \xrightarrow{J} T_x\Sigma$$

of the tangent space $T_x\Sigma$ defined by rotating vectors by $\pi/2$.

Thus a conformal structure makes Σ into a Riemann surface $X = (\Sigma, J)$.

The map on 1-forms induced by the complex structure is the *Hodge $\star$-operator* defined by:

$$\star\alpha := \alpha \circ J$$

where the 1-form

$$\alpha : T_x \longrightarrow \mathbb{R}$$

is a linear functional on tangent vectors. In terms of the conformal structure J, the energy of a 1-form α equals:

$$\|\alpha\|^2 = \int_X \alpha \wedge \star\alpha = \int_X \alpha \wedge (\alpha \circ J).$$

The *Euler-Lagrange equations* for minimum energy is just the *harmonic conditions:*

$$d\alpha = d \star \alpha = 0. \tag{1.5.1}$$

Sadly, J doesn't have eigenspaces. But the complex numbers come to the rescue! Complexifying the tangent space T_xX gives the *complex tangent space*

$$T_x^{\mathbb{C}}X := T_xX \otimes_{\mathbb{R}} \mathbb{C}$$

which is a two-dimensional complex vector space. J extends uniquely to a complex-linear map still satisfying $J^2 = -1$. It splits into two complex lines (the $+i$- and $-i$-eigenspaces respectively)

$$T^{\mathbb{C}}X := T^{(1,0)}X \oplus T^{(0,1)}X$$

where the $+i$-eigenspace $T^{(1,0)}X$ consists of *holomorphic tangent vectors* and the $-i$-eigenspace $T^{(0,1)}X$ consists of *anti-holomorphic tangent vectors.*

Suppose now that α is a *complex* 1-*form,* that is, a $\mathbb{C}$-linear functional

$$T_x^{\mathbb{C}}X \longrightarrow \mathbb{C}.$$

It uniquely decomposes as $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$ where $\alpha^{(1,0)}$ vanishes on $T^{(0,1)}X$ and $\alpha^{(0,1)}$ vanishes on $T^{(1,0)}X$. Harmonicity (1.5.1) is equivalent to the two conditions that

(1) $\alpha^{(1,0)}$ is a *holomorphic* 1-*form,* that is, when you write it locally in terms of a holomorphic coordinate z on X, that it looks like
$$f(z)dz$$
where f is a holomorphic function, and

(2) $\alpha^{(0,1)}$ is an *anti-holomorphic* 1-*form,* which in this case is equivalent to the condition that its complex conjugate $\overline{\alpha^{(0,1)}}$ is a holomorphic 1-form in the above sense.

What happens if α was real to begin with? Suppose that α takes the original *real tangent space* T_xX into $\mathbb{R}$? In that case
$$\overline{\alpha^{(0,1)}} = \alpha^{(1,0)},$$
so real de Rham cohomology classes $\longleftrightarrow$ harmonic (real) 1-forms $\longleftrightarrow$ holomorphic (necessarily $\mathbb{C}$-valued) 1-forms. Holomorphicity is easily expressed by decomposing the exterior derivative operator (de Rham differential)
$$d : \mathcal{A}^1(X, \mathbb{C}) \longrightarrow \mathcal{A}^2(X, \mathbb{C})$$
as
$$d = d^{(1,0)} + d^{(0,1)}$$
where $d^{(p,q)}(\alpha)$ is the (p,q)-component of $d\alpha$. More customary notation is $d' = d^{(1,0)}$ and $d'' = d^{(0,1)}$, and $d''f = 0$ coincides with the ordinary *Cauchy-Riemann equations.* The operator d'' will generalize to a *holomorphic structure* on a (smooth complex) vector bundle, whose solutions are defined to be the *holomorphic sections* of the bundle.

When describing these different points of view, Simpson calls the theory of representations of fundamental groups *Betti theory,* the theory of closed 1-forms (or more generally, flat connections, see §2) *de Rham theory,* the theory of harmonic representatives *Hodge theory* and the theory of holomorphic representatives *Dolbeault theory.* For a Riemann surface (i.e. a complex one-dimensional manifold) the GAGA principle implies that the holomomrphic objects (and hence representations of $\pi_1(X)$) can be studied using the tools of algebraic geometry. *Higgs bundles* are the holomorphic (and hence algebraic) objects corresponding to surface group representations in Lie groups.

Intepreting quite different mathematical objects as the "same" is best cast in the context of the notion of an *equivalence of categories.* A classification problem involves determining a class of restricted mathematical objects, and their equivalence classes under an equivalence relation. Often this equivalence relation is given by an operation of a group of transformations. The set of equivalence classes, *the set of moduli,* often admits additional structure (such as a topology, structure as a group or an algebraic variety). However, in many cases the quotient moduli space may contain much less information than the original group action. This may happen if the group action displays chaotic dynamics and the quotient is poorly separated. For this reason, it is better to consider a *deformation theory*

A *deformation theory* (or *transformation groupoid*) (S, G) consists of a category $\mathcal{C}$ defined by a group action as follows. Let $\alpha : G \times S \longrightarrow S$ be a left action of a group G on a set S. The *deformation theory* (S, G) consists of the category $\mathcal{C}$ whose objects form a set $\mathsf{Obj}(\mathcal{C}) = S$ with morphisms

$$x \xrightarrow{g} y.$$

corresponding to triples $(g, x, y) \in G \times S \times S$ such that $\alpha(g, x) = y$.

The identity element $e \in G$ determines, for each object $x \in S$ the identity morphism

$$x \xrightarrow{e} x.$$

The inverse of the morphism

$$x \xrightarrow{g} y$$

is

$$y \xrightarrow{g^{-1}} x$$

and the composition of morphisms

$$x \xrightarrow{g} y \xrightarrow{h} z$$

equals

$$x \xrightarrow{hg} z.$$

In particular every morphism is an isomorphism.

The *moduli set* corresponding to such a deformation theory is the set $\mathsf{Iso}(\mathcal{C})$ of isomorphism classes of objects. The *isotropy group* of an object $x \in \mathsf{Obj}(\mathcal{C})$ is the set $\mathsf{Mor}(x, x)$ consisting of morphisms $x \longrightarrow x$, which has the structure of a group. An *equivalence of categories* is a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that there exists a functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ and natural transformations from the compositions $F \circ H$ and $H \circ F$ to the identity functors of $\mathcal{B}$ and $\mathcal{A}$ respectively. (See Jacobson [51] or Gelfand-Manin [32], p.28 for discussion of this notion and Goldman-Millson [38] for an application closely related to this one.) An equivalence of categories induces a bijection $\mathsf{Iso}(\mathcal{A}) \longrightarrow \mathsf{Iso}(\mathcal{B})$, although in general $\mathsf{Obj}(\mathcal{A})$ and $\mathsf{Obj}(\mathcal{B})$ may be enormously different. For example, each groupoid arising from a group G operating on itself by left-multiplication is equivalent to the groupoid with one object and one morphism.

Equivalent deformation theories yield equivalent moduli sets. However the finer notion of equivalence has further implications— for example isotropy groups of corresponding points in the moduli spaces are isomorphic.

Often the sets $\mathsf{Obj}(\mathcal{A})$ admit additional algebraic or geometric structures, which induce additional structures on $\mathsf{Iso}(\mathcal{A})$. For the examples discussed here, these moduli sets are Lie groups, and the equivalences of deformation theories induces isomorphisms of (real) Lie groups.

Equivalent deformation theories may have different structures. An equivalence of a deformation theory $\mathcal{A}$ with another deformation theory may provide additional structures to $\mathsf{Iso}(\mathcal{A})$.

The following criterion is a useful tool for proving that a functor is an equivalence of categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an *equivalence* if and only if:

- **Surjective on isomorphism classes:** The induced map
$$F_* : \mathsf{Iso}(\mathcal{A}) \longrightarrow \mathsf{Iso}(\mathcal{B})$$
is surjective;
- **Full:** For $x, y \in \mathsf{Obj}(\mathcal{A})$, the map
$$F(x,y) : \mathsf{Mor}(x,y) \longrightarrow \mathsf{Mor}(F(x), F(y))$$
is surjective;
- **Faithful:** For $x, y \in \mathsf{Obj}(\mathcal{A})$, the map
$$F(x,y) : \mathsf{Mor}(x,y) \longrightarrow \mathsf{Mor}(F(x), F(y))$$
is injective.

1.6. **Singularities of $\mathsf{Hom}(\pi, G)/G$.** Let's return to the general structure of the deformation space. If G is not abelian, $\mathsf{Hom}(\pi, G)/G$ is not a manifold. Everything possible can go wrong:

- The algebraic set $\mathsf{Hom}(\pi, G)$ may be singular, so $\mathsf{Hom}(\pi, G)$ is not a manifold.
- The group G may not act properly, so $\mathsf{Hom}(\pi, G)/G$ may not be Hausdorff (except when G is compact);
- The group G may not act freely. Even where $\mathsf{Hom}(\pi, G)/G$ is a Hausdorff smooth manifold, the quotient $\mathsf{Hom}(\pi, G)/G$ may be only an orbifold.

Nonetheless we easily repair these spaces. If $k > 0$, then π is free of rank $N = 2g + k - 1$ and
$$\mathsf{Hom}(\pi, G) \leftrightarrow G^N$$
is a smooth manifold. When $k = 0$, then none of the standard generators in (1.1) can be eliminated, $\mathsf{Hom}(\pi, G)$ may indeed be singular.

At this point, restricting to *reductive groups* is handy. A subgroup $G \subset \mathsf{GL}(n, \mathbb{C})$ is *reductive* if it is *algebraic* (Zariski-closed), and the representation on $\mathbb{C}^n$ and all its tensor powers are *reductive* ("completely reducible", that is, a direct sum of irreducible representations). Equivalently, the *trace form* on its Lie algebra $\mathfrak{g} \subset \mathfrak{gL}(n, \mathbb{C})$
$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{C} \\ (a, b) &\longmapsto \mathsf{tr}(ab) \end{aligned}$$
is nondegenerate. All the classical linear groups are reductive, and every reductive group is a product of simple groups and abelian groups of diagonal matrices.

A representation is *reductive* if its Zariski-closure is reductive. that is, the smallest algebraic subgroup containing its image is reductive. For linear representations, this is equivalent to complete reducbility.

So suppose G is reductive. By computing the differential of the map
$$\begin{aligned} G^{2g} &\longrightarrow G \\ (\alpha_1, \dots, \beta_g) &\longmapsto [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g], \end{aligned}$$

the inverse function theorem implies that the smooth points of $\mathsf{Hom}(\pi, G)$ consist precisely of representations ρ with trivial *infinitesimal centralizer,* that is, every one-parameter subgroup of G which centralizes ρ lies in $\mathsf{Center}(G)$. For this calculation, see [34].

Next we discuss the $\mathsf{Inn}(G)$-action. The isotropy group of a point

$$h := (h_1, \dots, h_N) \in G^N$$

equals the common centralizer $Z(h)$ in $\mathsf{Inn}(G) \cong G/\mathsf{Center}(G)$ that is, all

$$[u] \in G/\mathsf{Center}(G)$$

such that $u = h_i u h_i^{-1}$. In particular $\mathsf{Inn}(G)$ acts freely on h if $Z(h)$ is trivial, and $\mathsf{Inn}(G)$ acts locally freely on h if $Z(h)$ is trivial, the same condition as smoothness above. The duality between smoothness of $\mathsf{Hom}(\pi, G)$ at ρ and the local freeness of the $\mathsf{Inn}(G)$-action at ρ is a nonlinear 2-dimensional Poincaré duality on Σ ([34]).

Whether $\mathsf{Hom}(\pi, G)/G$ is Hausdorff is more subtle. Continuing to assume G reductive, one can remove a somewhat larger subset of $\mathsf{Hom}(\pi, G)$ than the representations with trivial infinitesimal centralier to obtain a proper action. The precise condition is that the image $\rho(\pi)$ does not lie in a *parabolic subgroup* of G. For $G = \mathsf{GL}(n, \mathbb{C})$, this is equivalent to the *irreducibility* of the linear representation; the bad representations are the ones which can be put in simultaneous upper-triangular form.

For example, consider a representation ρ such that

$$\rho(\gamma) = \begin{bmatrix} \rho_{11}(\gamma) & \rho_{12}(\gamma) \\ 0 & \rho_{22}(\gamma) \end{bmatrix},$$

$\forall \gamma \in \pi$ and where $\rho_{12}(\gamma_0) \neq 0$ for some $\gamma_0 \in \pi$. Conjugating by the one-parameter subgroup

$$g_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

maps ρ to another representation by upper triangular matrices

$$g_\lambda(\rho)(\gamma) = \begin{bmatrix} \rho_{11}(\gamma) & \lambda^2 \rho_{12}(\gamma) \\ 0 & \rho_{22}(\gamma) \end{bmatrix}.$$

which as $\lambda \longrightarrow 0$ converges to a representation ρ_0 by diagonal matrices. Genericly ρ is inequivalent to ρ_0. Since the closure of the orbit ρ contains ρ_0, the point $[\rho]$ in the orbit space is not closed.

Let $\mathsf{Hom}(\pi, G)'$ denote the $\mathsf{Inn}(G)$-invariant subset of $\mathsf{Hom}(\pi, G)$ which do not lie in parabolic subgroups and have no extra symmetries under $\mathsf{Inn}(G)$. Then $\mathsf{Hom}(\pi, G)'$ is a smooth manifold, upon which $\mathsf{Inn}(G)$ acts properly and freely. The quotient map

$$\mathsf{Hom}(\pi, G)' \longrightarrow \mathsf{Hom}(\pi, G)'/G$$

is a smooth fibration over a smooth *Hausdorff* manifold.

2. Connection with connections

A differentiable structure on Σ lets us interpret representations $\pi_1(\Sigma) \xrightarrow{\rho}$ as smooth objects, namely *flat connections on G-bundles.* The two alternate approaches involve *principal G-bundles* or vector bundles. While principal bundles are more general, vector bundles are perhaps easier to develop. A good general reference on connections is Kobayashi-Nomizu [54]. Here we briefly summarize a few main points which we need later.

2.1. ***G*-vector bundles.** Suppose that $G \subset \mathsf{GL}(n, \mathbb{C})$ as above and Σ is a smooth manifold. A rank n complex vector bundle over Σ, denoted by $E \xrightarrow{\pi_E} \Sigma$ is a locally trivial fibration in which the fibers are copies of $\mathbb{C}^n$. An important special case is the *trivial bundle* where $E = \Sigma \times \mathbb{C}^n$ and π_E is Cartesian projection.

One way to describe a bundle is by means of a system of local trivializations glued together by a cocycle of transition functions:

- The *local trivializations* are pairs (U_α, ψ_α) where $U_\alpha \subset \Sigma$ are open sets and the ψ_α are homeomorphisms $U_\alpha \times \mathbb{C}^n \xrightarrow{\psi_\alpha} \pi^{-1}(U_\alpha)$,
- A *system of local trivializations* consists of a cover for Σ, $\{U_\alpha\}_{\alpha \in I}$, and a local trivialization for each α in the indexing set I, and
- a *cocycle of transition functions* is a collection of maps
$$U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} \mathsf{GL}(n, \mathbb{C})$$
with one such map for each pair (α, β) corresponding to a non-empty intersection $U_\alpha \cap U_\beta$ and such that $\forall x \in U_\alpha \cap U_\beta$, and $v \in \mathbb{C}^n$
$$\psi_\alpha(x, v) = \psi_\beta(x, g_{\alpha\beta} v).$$

This is a lot of baggage, so we usually just write E when the context is clear. Despite its cumbersomeness, the description in terms of local trivializations and transition functions provides a convenient way to describe geometric bundle structures which play a key role in our story. In particular:

2.1.1. To say that a rank n vector bundle $E \xrightarrow{\pi_E} \Sigma$ has structure group $G \subset \mathsf{GL}(n, \mathbb{C})$ means that one can find a system of local trivializations in which the transition functions all take their values in $G \subset \mathsf{GL}(n, \mathbb{C})$ (and the action $g_{\alpha\beta} v$ is thus via the representation defined by the inclusion $G \subset \mathsf{GL}(n, \mathbb{C})$)

2.1.2. If the base space is a smooth manifold and the transition functions are smooth maps, then the local trivial bundles $U_\alpha \times \mathbb{C}^n$ determine a smooth structure on the total space of the bundle. The projection map $E \xrightarrow{\pi_E} \Sigma$ is then a diffeomorphism and E is said to be a smooth bundle.

2.1.3. If the transition functions are not only smooth, but are locally constant, that is, if they satisfy the condition

$$dg_{\alpha\beta} = 0 \ , \tag{2.1.1}$$

where d is the exterior derivative, then the system of local trivializations is said to define a *flat structure* on E.

2.1.4. If the base manifold has a complex structure (making Σ a Riemann surface), and the structure group G is a complex Lie group, then there is another option for the transition functions: they can be *holomorphic maps*, that is, they can satisfy the condition

$$d'' g_{\alpha\beta} = 0 \ , \tag{2.1.2}$$

where d'' is the Cauchy Riemann operator described in Section 1.5. The local trivial bundles $U_\alpha \times \mathbb{C}^n$ are then complex coordinate patches on the total space of the bundle, and the projection map $E \xrightarrow{\pi_E} \Sigma$ is a holomorphic map between complex manifolds. In this case, the system of local trivializations defines a *holomorphic structure* on E.

2.2. **Gauge transformations.** If E, E' are two vector bundles over Σ, then a *gauge transformation* is a map $E \xrightarrow{\Phi} E'$ satisfying

- $\pi_{E'} \circ \Phi = \pi_E$;
- $\forall x \in \Sigma$, the restriction
$$\left(\pi_E\right)^{-1}(x) \xrightarrow{\Phi} \left(\pi_{E'}\right)^{-1}(x)$$
is a linear isomorphism, represented by an element of G.

For the trivial bundle, gauge transformations correspond to maps $\Sigma \longrightarrow G$.

If $G \subset G' \subset \mathsf{GL}(n, \mathbb{C})$ is a homomorphism of Lie groups, then every G-bundle is a G'-bundle. The G'-gauge group is larger than the G-gauge group. An important example occurs when G is the unitary group $\mathsf{U}(n) \subset \mathsf{GL}(n, C)$: the *unitary gauge group* preserves a Hermitian metric on $\mathsf{U}(n)$-vector bundle but is strictly smaller than the *linear gauge group* consisting of all linear automorphisms.

More generally, one may consider *bundle automorphisms* which cover a nontrivial map $\Sigma \longrightarrow \Sigma$, but still take fibers to fibers by transformations in G. A gauge tranformation is then a bundle automorphism covering the identity diffeomorphism of Σ.

2.3. **How does a representation determine a vector bundle?** To obtain a vector bundle from a representation ρ, first choose a universal covering space $\tilde{\Sigma} \longrightarrow \Sigma$ with deck group π. For example, $\tilde{\Sigma} = \tilde{\Sigma}(x_0)$ is the space of relative homotopy classes of based paths in Σ starting at a fixed basepoint x_0, and $\pi = \pi_1(\Sigma, x_0)$.

The action of π on $\tilde{\Sigma}$ lifts to an action by bundle automorphisms on the trivial $\mathbb{C}^n$-bundle $\tilde{\Sigma} \times \mathbb{C}^n \longrightarrow \tilde{\Sigma}$:

$$\begin{aligned} \pi \times \left(\tilde{\Sigma} \times \mathbb{C}^n\right) &\longrightarrow \left(\tilde{\Sigma} \times \mathbb{C}^n\right) \\ \gamma \ : \quad (\tilde{x}, v) &\longmapsto (\gamma\tilde{x}, \rho(\gamma)v) \end{aligned} \tag{2.3.1}$$

π acts properly and freely on $\tilde{\Sigma}$ and hence also on $\tilde{\Sigma} \times \mathbb{C}^n$; the resulting quotient map

$$\left(\tilde{\Sigma} \times \mathbb{C}^n\right)/\pi \longrightarrow \tilde{\Sigma}/\pi = \Sigma$$

is a rank n vector bundle over Σ. We denote this bundle by E_ρ and (for reasons which will be explained in Section 2.4) we call it the *flat bundle* with holonomy ρ.

Here is another way to describe this bundle: the universal cover $\tilde{\Sigma}$ can be regarded as a prinicipal π-bundle: the fibers of the covering are copies of $\pi = \pi_1(\Sigma)$ and π acts on the total space (that is, $\tilde{\Sigma}$) by deck transformations. For any Lie group G, a representation

$$\rho : \pi_1(X) \longrightarrow G \tag{2.3.2}$$

then defines an *associated G-bundle.* The total total space of this bundle is the analog of $\big(\tilde{\Sigma} \times \mathbb{C}^n\big)/\pi$, with $\mathbb{C}^n$ replaced by G. If we compose the homomorphism (2.3.2) with a representation of G, that is, with a homomorphism from G into $\mathsf{GL}(N, \mathbb{C})$ for some N, then the construction yields an *associated rank-N vector bundle.* If $G = \mathsf{GL}(n, \mathbb{C})$, and we can take the fundamental representation (that is, $N = n$), then we get the bundle E_ρ described above.

Exercise 2.3.1. *Find a coordinate atlas and a cocycle for the vector bundle E_ρ.*

2.4. **Flat bundles and covering spaces.** A vector bundle arising from a representation as above has very special properties. Firstly, it is trivial over the universal covering. Secondly, it has a system of local trivializations for which the transition functions (that is, the $g_{\alpha\beta}$) are (locally) constant. These are constructed from a system of local trivializations for the universal cover (viewed as a π_1-bundle over Σ). The transition functions for this are necessarily locally constant since as a topological space π_1 has the discrete topology. The resulting system of local trivializations thus defines a flat structure (as defined in Section 2.1.3) on the bundle.

The geometric significance, which can be seen from either point of view, is that the total space E_ρ has a foliation $\mathfrak{F}_\rho$ which is transverse to the fibration $E_\rho \longrightarrow \Sigma$, and such that the restriction of the fibration to each leaf is a covering space.

Exercise 2.4.1. *Describe this foliation in the above construction of E_ρ.*

Since at each point $p \in E_\rho$, the leaf of the foliation through p is transverse to the fiber containing p, it follows that the tangent space to the leaf is complementary (in the tangent space T_pE_ρ) to the vertical space at p (that is, the directions along the fiber). The tangents to the leaves thus define a *horizontal distribution* on E_ρ. Distributions of this sort provide one definition of a *connection* on a bundle (see section 2.5). Notice that, by construction, the distribution coming from $\mathfrak{F}_\rho$ is integrable. This is precisely what it means for a connection to be flat. Indeed, a *flat connection* is equivalent to a *flat structure* on a bundle .

By the path-lifting properties of covering spaces, paths in Σ can always be lifted to paths in the total space of E_ρ. The foliation $\mathfrak{F}_\rho$ picks out a distinguished class of such lifts. Specifically, given any point $\tilde{x}_0 \in E_\rho$ and any path $\gamma(t)$ starting at $\gamma(0) = \pi_E(\tilde{x}_0)$, $\exists!$ lifted path $\tilde{\gamma}(t)$ such that

- $\pi_E(\tilde{\gamma}(t)) = \gamma(t)$,
- $\tilde{\gamma}(0) = \tilde{x}_0$,and

- $\forall t$, the lifted path $\tilde{\gamma}(t)$ stays in the same leaf of $\mathfrak{F}_\rho$;

A system like this of path liftings defines a notion of *parallel transport* across fibers. In general, loops in Σ do not lift to closed loops. Moreover, the discrepancy at the endpoints can depends on the path. For the parallel transport defined by the foliation $\mathfrak{F}_\rho$, this *holonomy around loops* depends only on the relative homotopy class of the loop.

We note, finally, that the foliation defines a special class of sections of E_ρ, namely those sections which are graphs of "constant" maps into the fiber $\mathbb{C}^n$ with respect to the local trivializations ψ_α above. Precisely, let $A \subset \Sigma$ be a path-connected subset and suppose that

$$A \xrightarrow{s} (E_\rho)|_A := \pi_E^{-1}(A)$$

is a section over A. Then s is *parallel* $:\Longleftrightarrow$ it lies in a single leaf of $\mathfrak{F}_\rho$. Equivalently, s arises from the graph of a constant map

$$\begin{aligned} \tilde{A} &\longrightarrow \tilde{\Sigma} \times \mathbb{C}^n \\ \tilde{x} &\longmapsto (\tilde{x}, v_0) \end{aligned}$$

where $\tilde{A}$ is a path-component of $\pi_E^{-1}(A) \subset \tilde{\Sigma}$ and $v_0 \in \mathbb{C}^n$. This observation shows not only that parallel sections exist over A, but that there are lots of them. In fact, given a vector space basis for the fiber over any point in A, we can extend each basis element to a parallel section over A. The resulting sections will provide a basis for the fiber over each point in A, that is, they will constitute a local frame of parallel sections. Thus one may think of a flat bundle as a bundle together with a preferred class of local frames, viz. the local frames of *parallel sections.*

2.5. **Flat connections, curvature and parallel transport.** *Flat connections* are analytic objects (tensor fields) which are equivalent to representations of $\pi_1(\Sigma)$.

We have already alluded to one definition of a *connection* on a bundle E, namely a distribution on the total space of E which at each point is complementary to the vertical directions at that point. [1]. We briefly mention two other equivalent points of view and say something about how the various points of view are related.

2.5.1. A connection is a way of "connecting" the fibers of E. It does this by means of a parallel lifting of paths in Σ. For any path $\gamma(t)$, a connection picks out a class of of sections which will be parallel over γ. These provide the parallel lifts. In local coordinates it is a system of first-order differential equations in which $\tilde{x}_0$ is the initial condition. The system of differential equations extends the ordinary differential system defining the constant functions.

As such it is represented by a differential operator

$$\mathcal{A}^0(\Sigma, E_\rho) \xrightarrow{D} \mathcal{A}^1(\Sigma, E_\rho)$$

which extends the ordinary exterior derivative on functions

$$\mathcal{A}^0(\Sigma, \mathbb{C}) \xrightarrow{d} \mathcal{A}^1(\Sigma, \mathbb{C}).$$

[1]there is another condition which the distribution must satisfy, namely that it is compatible with the action of the structure group of the bundle. The interested reader is referred to *** for precise details

2.5.2. This provides the third description of a connection. The extension is enforced by the product identity

$$D(fs) = df \wedge s + fDs$$

so, in particular, a constant multiple of a parallel section remains parallel.

The *covariant derivative* of a section s with respect to a tangent vector $v \in T_x\Sigma$ equals the evaluation

$$D_v(s) := (Ds)_x(v)$$

of the linear functional $Ds_x \in \mathsf{Hom}(T_x\Sigma, E_x)$ on v. It measures the deviation of s from being parallel along paths $\gamma(t)$ with velocity vector $\gamma'(t) = v$. In particular section $s\big(\gamma(t)\big)$ over $\gamma(t)$ is parallel $\Longleftrightarrow$ the covariant derivative

$$D_{\gamma'(t)}s = 0.$$

Existence and uniqueness of parallel transport of a vector follow from existence of uniqueness of linear systems of differential equations.

Parallel sections passing through a point $p \in E$ can be used to lift tangent vectors at $\pi(p) \in \Sigma$ to tangent vectors at p. The resulting vectors define a subspace of horizontal directions in T_pE, that is, they define a horizontal distribution. This shows the relation to our first definition of a connection.

2.5.3. Connections can be computed in terms of local trivializations as follows. Suppose that $U \subset \Sigma$ is a single coordinate patch over which E has trivialization ψ. Let $e_1, \dots, e_n$ be the basis of sections corresponding to ψ and let x^1, x^2 be local coordinates on U. Then, writing

$$De_\alpha = \Gamma^\beta_{i\alpha} dx^i \otimes e_\beta,$$

we see that for an arbitrary section $s = f^\alpha e_\alpha$,

$$Ds \;=\; df^\alpha\, e_\alpha \;+\; f^\alpha \Gamma^\beta_{i\alpha}\, dx^i e_\beta,$$

which we write as

$$D = d + \Gamma$$

where $\Gamma \in \mathcal{A}^1(U, \mathsf{End}(E))$ is the matrix-valued 1-form

$$\Gamma = \Gamma^\beta_{i\alpha}\, dx^i$$

(where the indices $1 \leqslant \alpha, \beta \leqslant n$).

2.5.4. Parallel transport is independent of the path in its relative homotopy class if and only if the *curvature of* D vanishes. The curvature, F_D is defined locally (that is, over U) as

$$F_D = d\Gamma + \Gamma \wedge \Gamma \in \mathcal{A}^2(\Sigma, \mathsf{End}(E)).$$

Here $d\Gamma$ is the exterior derivative of the $\mathsf{End}(E)$-valued 1-form Γ with respect to the natural extension of D to $\mathsf{End}(E)$-valued differential forms and $\Gamma \wedge \Gamma$ is the $\mathsf{End}(E)$-valued 2-form obtained by matrix multiplication in $\mathsf{End}(E)$ and exterior multiplication of 1-forms. In terms of the Lie structure on $\mathsf{End}(E)$,

$$\Gamma \wedge \Gamma = \frac{1}{2}[\Gamma, \Gamma]$$

obtaining the general formula (valid for principal bundle connections)

$$F_D = d\Gamma + \frac{1}{2}[\Gamma, \Gamma].$$

Globally the curvature measure the failure of the horizontal distribution to be integrable. It also measure the failure of the covariant derivative to define a complex

$$\mathcal{A}^0(\Sigma, E_\rho) \xrightarrow{D} \mathcal{A}^1(\Sigma, E_\rho) \xrightarrow{D} \mathcal{A}^2(\Sigma, E_\rho) \xrightarrow{D} \cdots$$

that is, it measures the non-vanishing of D^2. In fact, $F_D = D^2$. For more details see [54, 40].

2.5.5. Gauge transformations g are sections of the subbundle of $\mathsf{Aut}(E)$ corresponding to G, and act on connections by

$$g : D \longmapsto g \circ D \circ g^{-1}$$

and curvature behaves like

$$F_{g \cdot D} = g \circ F_D \circ g^{-1}.$$

2.6. **Equivalent deformation theories.** Holonomy around loops based at x_0 defines a map

$$\{\text{Flat connections on } E\} \xrightarrow{\mathsf{hol}_{x_0}} \mathsf{Hom}(\pi_1(\Sigma, x_0), G)$$

respecting the natural equivalence relations. Namely, the group of G-gauge transformations on E evaluates to a transformation in G at the basepoint x_0 and hol_{x_0} is equivariant respecting this evaluation homomorphism. The corresponding map is an *equivalence of deformation theories* from the gauge transformations acting on flat connections to G acting on $\mathsf{Hom}(\pi, G)$. In particular this correspondence defines a natural homeomorphism between the quotient spaces $\mathsf{Hom}(\pi, G)/G$ and the space of gauge-equivalence classes of flat connections on G-bundles over Σ.

3. Compact representations and holomorphic structures

A good reference for this material is Kobayashi's book [53].

Surface group representations into compact groups are successfully understood in terms of *holomorphic vector bundles* over Riemann surfaces. It is important to distinguish between a *complex structure* on a real vector bundle, which is just an operation of $\sqrt{-1}$ on the fibers, and the deeper notion of *a holomorphic structure* which is a notion of holomorphic section, when the base is a complex manifold. Complex structures may exist on smooth vector bundles over arbitrary topological spaces, which holomorphic structures are only defined on smooth vector bundles with complex structure over complex manifolds. A *holomorphic vector bundle* over a complex manifold X is a complex vector bundle over X *with a holomorphic structure.*

3.1. **Holomorphic structures.** The starting point of this subject is the *Jacobi variety* $\mathsf{Jac}(X)$ of a Riemann surface X, which parametrizes topologically trivial holomorphic line bundles over X. This space identifies naturally with $\mathsf{Hom}(\pi, \mathsf{U}(1))$. If G is a compact Lie group, then the work of Narasimhan-Seshadri [70], extended by Ramanan, Ramanathan and and others, interpret representations $\pi_1(X) \xrightarrow{\rho} G$ as holomorphic vector bundles over X, satisfying a condition of *stability.*

Let X be a Riemann surface diffeomorphic to Σ and E a *smooth* complex vector bundle over X. That is, the transition cocycle consists of *smooth maps* $U_\alpha \cap U_\beta \longrightarrow \mathsf{GL}(n, \mathbb{C})$. With respect to such a structure, there is a well-defined notion of what it means for a section to be *smooth.* Smooth sections form a module over the ring of smooth functions on X. On a complex manifold, holomorphic functions form a subring of the ring of smooth functions.

A *holomorphic structure* on E is a way to specify a class of sections which will be holomorphic. This will be analogous to how a connection satisfies a class of sections to be parallel, or "locally constant." Of course, we would require that multiplying a holomorphic section defined over an open subset $U \subset X$ by a holomorphic function on U preserves holomorphicity. Just as connections allow us to extend the exterior derivative d on scalar-valued functions (and differential forms) to E-valued forms, a holomorphic structure will allow us to extend the Cauchy-Riemann operator d'' from scalar-valued forms to E-valued forms. Thus, a holomorphic structure on a complex vector bundle E is a linear operator

$$\mathcal{A}^k(X; E) \xrightarrow{D''} \mathcal{A}^{k+1}(X; E)$$

satisfying the product formula

$$D''(fs) = d''f \wedge s + fD''(s)$$

for $f \in \mathcal{A}^0(X, \mathbb{C})$ and $s \in \mathcal{A}^k(X; E)$. A section $s \in \mathcal{A}^0(X, E)$ is *holomorphic with respect to* D'' $\Leftrightarrow$ $D''s = 0$.

Just as the $(0,1)$ part of the exterior derivative d is the Cauchy-Riemann operator d'', the $(0,1)$-part of a flat connection D is a holomorphic structure.

3.2. **Hermitian metrics and connections.** A simple relationship exists between connections, Hermitian metrics and holomorphic structures. Let X be a complex manifold and E a smooth complex vector bundle over X. A *Hermitian metric* on E is a section h of the associated bundle of positive definite Hermitian forms on E — that is, at each point $x \in X$ is a positive definite Hermitian pairing

$$E_x \times E_x \xrightarrow{h_x} \mathbb{C}$$

which in terms of a local trivialization is given by the positive definite Hermitian matrix

$$h_{\alpha\bar{\beta}} := h_x(e_\alpha, e_\beta).$$

Let D be a connection on E and let h be a Hermitian structure on E. The following conditions are equivalent:

- The holonomy of D is unitary with respect to h;

- For each tangent vector $\xi \in T_x X$,

$$\xi h_x(s_1, s_2) = h_x\big(D_\xi(s_1), s_2\big) + h_x\big(s_1, D_\xi(s_2)\big).$$

- D is a $\mathsf{U}(n)$-connection with respect to the reduction of structure group to $\mathsf{U}(n)$ defined by h.

The following fundamental theorem is straightforward and elementary (see [53]):

Theorem 3.2.1. *For every Hermitian metric h and holomorphic structure D'', $\exists!$ connection D such that:*

- *D is unitary with respect to h;*
- *The $(0,1)$-part of D equals D''.*

The curvature F_D of D is a $(1,1)$-form and is necessarily closed (since $\dim_{\mathbb{C}}(X) = 1$). Its cohomology class (scaled by $-i/2\pi$) is a topological invariant of E, the *first Chern class* or the *degree* of E. In particular, E is topologically trivial (equivalent to the product bundle) $\Longleftrightarrow c_1(E) = 0 \Longleftrightarrow F_D$ is exact.

3.3. **Line bundles and divisors.** When $n = 1$, then E is a *line bundle.* Holomorphic line bundles are classified by the *Picard group,* $\mathsf{Pic}(X)$ and the most important invariant is the degree. Every holomorphic line bundle admits a meromorphic section s, any two nonzero meromorphic sections differ by multiplication by a meromorphic function. The line bundle is determined by s, and indeed just by the *divisor* $\mathsf{div}(s)$ of s, the formal integral linear combination of zeroes and poles of s, weighted by multiplicity. Two divisors determine the same line bundle if they differ by a *principal divisor,* that is, the divisor of a non-identically zero meromorphic function. The degree of the divisor

$$x = n_1[x_1] + n_2[x_2] + \cdots + n_l[x_l]$$

where $n_i \in \mathbb{Z}$ and $p_i \in X$ is the sum

$$\mathsf{deg}(x) := n_1 + n_2 + \cdots + n_l.$$

For a compact Riemann surface X of genus g, the components of $\mathsf{Pic}(X)$ are detected by the homomomorphism

$$\mathsf{Pic}(X) \xrightarrow{\mathsf{deg}} \mathbb{Z}$$

given by degree. The identity component $\mathsf{Jac}(X) = \mathsf{Ker}(\mathsf{deg})$ is a complex torus of complex dimension g.

For this classical theory, see the excellent texts of Farkas-Kra [27], Forster [28], Jost [52], Miranda [66], or Mumford [68].

The main classical result (see Goldman-Xia [40] for a detailed exposition) is that holomorphic line bundles over a closed Riemann surface admit a unique flat Hermitian metric, and hence

$$\mathsf{Jac}(X) \longleftrightarrow \mathsf{Hom}(\pi, \mathsf{U}(1)).$$

3.4. **Higher rank bundles and stability.** Holomorphic vector bundles of rank $n > 1$ are considerably more complicated. If E is a $\mathbb{C}^n$-bundle, then its *degree* is defined as the degree of its *determinant line bundle* $\mathsf{Det}(E) = \Lambda^n(E)$. Elementary obstruction theory implies that two $\mathbb{C}^n$-bundles over a surface Σ are isomorphic $\Longleftrightarrow$ they have the same degree.

Exercise 3.4.1. *Let E_ρ be a flat $\mathbb{C}^n$-bundle arising from a representation $\pi \xrightarrow{\rho} \mathsf{GL}(n, \mathbb{C})$. Then $\mathsf{deg}(E) = 0$.*

Now let X be a Riemann surface diffeomorphic to Σ. Unlike the case $n = 1$, the space of isomorphism classes of holomorphic $\mathbb{C}^n$-bundles for $n > 1$ fails to be Hausdorff. Here is a simple example. Let $\eta \in \Omega^1(X)$ be a nonzero holomorphic 1-form; being harmonic and nonzero its de Rham cohomology class in $H^1(X, \mathbb{C})$ is nonzero. The trivial $\mathbb{C}^2$-bundle over X admits a (trivial) holomorphic structure D_0'' by taking the two basic sections e_1, e_2 to be holomorphic:

$$D_0'' f^\alpha e_\alpha = d'' f^\alpha e_\alpha.$$

For each nonzero $t \in \mathbb{C}$ the new holomorphic structure D_t'' defined by:

$$D_t'' : \begin{cases} e_1 & \longmapsto 0 \\ e_2 & \longmapsto \bar{\eta} e_1 \end{cases}$$

is nontrivial. The gauge transformations defined by

$$g_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

map D_t'' to $D_{\lambda^2 t}''$. The equivalence class D_t'', for nonzero t, contains the equivalence class of D_0'' in its closure.

What makes this construction work is the existence of the "destabilizing" holomorphic subbundle (e_1) in which the deformation to a new structure takes values. If this subbundle were more twisted (having negative degree) such a construction would be impossible.

Exercise 3.4.2. *Generalize this example to show that if L is a holomorphic line bundle of degree $d \geqslant 0$ and X has genus $g \geqslant 2d - 1$, then $L \oplus L^{-1}$ is a holomorphic rank two bundle of degree 0 whose isomorphism class is in the closure of a different isomorphism class.*

This phenomenon can be avoided by restricting to holomorphic vector bundles which are *stable* in the following sense. Suppose that E is a holomorphic $\mathbb{C}^n$-bundle of degree 0. A *destabilizing subbundle* of E is a holomorphic subbundle of negative degree. Then E is *stable* $:\Longleftrightarrow$ E admits no destabilizing subbundle.

Theorem 3.4.3 (Narasimhan-Seshadri [70])**.** *If $\pi \xrightarrow{\rho} \mathsf{U}(n)$ is an irreducible representation, then E_ρ is a stable holomorphic vector bundle. Conversely every stable holomorphic $\mathbb{C}^n$-bundle admits a flat unitary connection with irreducible holonomy, and hence arises as E_ρ for an irreducible representation $\pi \xrightarrow{\rho} \mathsf{U}(n)$.*

The original proof in [70] uses Geometric Invariant Theory to identify two moduli spaces. A more direct gauge-theoretic proof was proved by Donaldson's thesis [19].

4. Fuchsian representations

Perhaps the most important class of surface group representations are *Fuchsian representations* in $G = \mathsf{PSL}(2,\mathbb{R})$. Such representations arise as holonomy of marked hyperbolic structures on Σ. Their equivalence classes comprise the *Fricke space* of Σ. By the *uniformization theorem,* Fricke space identifies with the *Teichmüller space* $\mathfrak{T}_\Sigma$, consisting of equivalence classes of marked *conformal structures* on Σ.

4.1. **Hyperbolic structures on surfaces.** A *hyperbolic structure on* Σ is a Riemannian metric on Σ having constant negative curvature, which we take to be -1. Such a metric is locally isometric to the hyperbolic plane H^2, and a hyperbolic structure can alternatively be described as a geometric structure locally modeled on H^2. That is, a hyperbolic surface M comes equipped with a coordinate atlas of charts mapping into H^2, such that the coordinate changes on overlapping patches are locally isometric. If $\tilde{M} \longrightarrow M$ is a universal covering space, then $\tilde{M}$ inherits a hyperbolic structure which is induced by a *developing map* $\tilde{M} \xrightarrow{\mathsf{dev}_M} \mathsf{H}^2$. All these maps are local isometries. The group $\pi_1(M)$ of deck transformations of $\tilde{M} \longrightarrow M$ acts on H^2 by isometries and dev is equivariant respecting this action: $\forall\gamma \in \pi_1(M)$, the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathsf{dev}_M} & \mathsf{H}^2 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\rho(\gamma)} \\ \tilde{M} & \xrightarrow[\mathsf{dev}_M]{} & \mathsf{H}^2. \end{array}$$

commutes. The correspondence $\gamma \longmapsto \rho(\gamma)$ is a homomorphism, $\pi_1(M) \xrightarrow{\mathsf{hol}_M} \mathsf{Isom}(\mathsf{H}^2)$, the *holonomy representation* of the hyperbolic surface M.

dev_M is unique up to composition with an isometry $\mathsf{H}^2 \xrightarrow{g} \mathsf{H}^2$, and the hol_M is unique up to composition of the inner automorphism $G \longrightarrow G$ defined by $g \in \mathsf{Isom}(\mathsf{H}^2)$.

If the hyperbolic structure is *complete,* that is, the Riemannian metric is geodesically complete, then the developing map is a *global isometry* $\tilde{M} \approx \mathsf{H}^2$. In that case the π-action on H^2 defined by the holonomy representation ρ is equivalent to the action by deck transformations, so ρ defines a proper free π-action on H^2 by isometries. Conversely if ρ defines a proper free iosometric π-action, then the quotient $M := \mathsf{H}^2/\rho(\pi)$ is a complete hyperbolic manifold with a preferred isomorphism

$$\pi_1(\Sigma) \xrightarrow{\rho} \rho(\pi) \subset G$$

which determines a preferred homotopy class of homotopy equivalences

$$\Sigma \longrightarrow M.$$

4.2. **Fricke space: marked hyperbolic structures.** A *marked hyperbolic structure on* Σ is defined as a pair (M, f) where M is a hyperbolic surface and f is a homotopy equivalence $\Sigma \longrightarrow M$. Two marked hyperbolic structures

$$\Sigma \xrightarrow{f} M, \qquad \Sigma \xrightarrow{f'} M'.$$

are *equivalent* $:\Longleftrightarrow \exists$ an isometry $M \xrightarrow{\varphi} M'$ such that

$$\begin{array}{ccc} & & M \\ & \overset{f}{\nearrow} & \Big\downarrow{\scriptstyle\varphi} \\ \Sigma & \xrightarrow[f']{} & M' \end{array}$$

homotopy-commutes, that is, $\varphi \circ f \simeq f'$. The *Fricke space* of Σ is the space of all such equivalence classes of marked hyperbolic structures on Σ. (When $\partial\Sigma \neq \emptyset$, the hyperbolic structures are assumed to have geodesic boundary or be complete with finite area; see Bers-Gardiner [2].) When $\partial\Sigma = \emptyset$, the Fricke space is diffeomorphic to an open cell of dimension $3|\chi(\Sigma)|$.

4.3. **Fuchsian components of $\mathsf{Hom}(\pi, G)/G$.** Using Poincaré's upper half plane model of H^2, the group $\mathsf{Isom}(\mathsf{H}^2)$ of isometries of H^2 identifies with $\mathsf{PGL}(2,\mathbb{R})$ (acting by real linear, or conjugate-linear, fractional transformations). Its identity component $\mathsf{Isom}^+(\mathsf{H}^2) = \mathsf{PSL}(2,\mathbb{R})$ consists of orientation-preserving isometries. The holonomy representation of an orientable hyperbolic surface maps to $\mathsf{PSL}(2,\mathbb{R})$. For the moment we consider $G = \mathsf{Isom}(\mathsf{H}^2) = \mathsf{PGL}(2,\mathbb{R})$.

To every equivalence class of marked hyperbolic structures is a well-defined equivalence class

$$[\rho] \in \mathsf{Hom}(\pi, G)/G.$$

A representation $\pi \xrightarrow{\rho} G$ is Fuchsian $:\Longleftrightarrow$ it arises as the holonomy of a hyperbolic structure on Σ. Equivalently, it satisfies the three conditions:

- ρ is injective;
- Its image $\rho(\pi)$ is a discrete subgroup of G;
- The quotient $G/\rho(\pi)$ is compact.

The first condition asserts that ρ is an *embedding,* and the second two conditions assert that $\rho(\pi)$ is a *cocompact lattice.* When $\partial\Sigma = \emptyset$, the third condition (compactness of $G/\rho(\pi)$) follows from the first two. In general, we say that ρ is a *discrete embedding* (or discrete and faithful) if ρ is an embedding with discrete image (the first two conditions).

Theorem 4.3.1. *Let* $G = \mathsf{Isom}(\mathsf{H}^2) = \mathsf{PGL}(2,\mathbb{R})$ *and* Σ *a closed connected surface with* $\chi(\Sigma) < 0$. *Fricke space, the subset of* $\mathsf{Hom}(\pi, G)/G$ *consisting of* G*-equivalence classes of Fuchsian representations, is a connected component of* $\mathsf{Hom}(\pi, G)/G$.

This result follows from three facts: openness of Fricke space (Weil [80]), closedness of Fricke space (Chuckrow [13]) and the connectedness of Fricke space (see for example, Jost [52],§4.3, Buser [12] or Thurston [76] for elementary proofs using Fenchel-Nielsen coordinates). The connectedness also follows from the uniformization theorem, together with the identification of $\mathfrak{T}_\Sigma$ with the vector space $H^0(X; K^2)$, see §4.5 below.)

When $G = \mathsf{Isom}^+(\mathsf{H}^2) = \mathsf{PSL}(2,\mathbb{R})$, the situation is slightly more complicated due to orientation. Assume that Σ is orientable, and fix an orientation on it. Also fix an orientation on H^2. Let $\Sigma \xrightarrow{f} M$ be marked hyperbolic structure on Σ. The orientation of M induces an orientation of $\tilde{M}$ which is invariant under $\pi_1(M)$. However, the developing map dev_M may or not preserve the (arbitrarily) chosen orientations of $\tilde{M}$ and H^2. Accordingly $\mathsf{Isom}^+(\mathsf{H}^2)$-equivalence classes of Fuchsian representations in G fall into two classes, which we call *orientation-preserving* and *orientation-reversing* respectively. These two classes are interchanged by inner automorphisms of orientation-reversing isometries of H^2.

Theorem 4.3.2. *Let $G = \mathsf{Isom}^+(\mathsf{H}^2) = \mathsf{PSL}(2,\mathbb{R})$ and Σ a closed connected oriented surface with $\chi(\Sigma) < 0$. The set of G-equivalence classes of Fuchsian representations forms two connected connected components of $\mathsf{Hom}(\pi, G)/G$. One component corresponds to orientation-prserving Fuchsian representations and the other to orientation-reversing Fuchsian representations.*

4.4. **Teichmüller space: Marked conformal structures.** The *Teichmüller space* $\mathfrak{T}_\Sigma$ of Σ is the deformation space of marked conformal structures on Σ.

A *marked conformal structure* on Σ is a pair (X, f) where X is a Riemann surface and f is a homotopy equivalence $\Sigma \longrightarrow X$. Marked conformal structures

$$\Sigma \xrightarrow{f} X, \qquad \Sigma \xrightarrow{f'} X'.$$

are *equivalent* $:\Longleftrightarrow \exists$ a biholomorphism $X \xrightarrow{\varphi} X'$ such that

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \Big\downarrow \varphi \\ \Sigma & \xrightarrow[f']{} & X' \end{array}$$

homotopy-commutes.

Theorem 4.4.1 (Uniformization)**.** *Let X be a Riemann surface with $\chi(X) < 0$. Then there exists a unique hyperbolic metric whose underlying conformal structure agrees with X.*

Hitchin [48] deduces this result from his general self-duality theorem, which also generalizes Donaldson's proof [19] of the Narasimhan-Seshadri [70] Theorem 3.4.3. This proof sets up a canonical Higgs bundle associated to a Riemann surface. Solving the self-duality equations produces a Hermitian metric which induces a constant curvature Riemannian metric which is conformal on X. We discuss these ideas in §6.

The Poincaré-Koebe Uniformization Theorem provides a canonical bijection between hyperbolic structures and conformal structures when $\chi(\Sigma) < 0$. Therefore it is tempting to confuse these two deformation spaces using this deep theorem.

However, for our purposes clarifying the two structures, and separating the Fricke space from the Teichmüller space is important. The theory of Higgs bundles heavily depends on a conformal structure X, although it has strong applications to the topological object associated to Σ (or just $\pi = \pi_1(\Sigma)$). Understanding the dependence of the Higgs bundle theory as the conformal structure varies over $\mathfrak{T}_\Sigma$ is an important question. For example, the mapping class group action does not respect the conformal structure and therefore involves the dependence of Higgs bundle theory on the Teichmüller parameter X. For this reason we have pedanticly chosen to call the deformation space of hyperbolic structures "Fricke space" rather than the more common "Teichmüller space."

An interesting example of our distinction is that each point of Teichmüller space, (represented by a Riemann surface X) defines the structure of a complex vector space on Fricke space (as $H^0(X, K^2)$). The same Fuchsian representation gives rise to quite different pairs (X, q) where $q \in H^0(X, K^2))$ is a holomorphic quadratic differential on X.

4.5. **Teichmüller space, Fricke space and harmonic maps.** Teichmüller described a complex structure, and a metric geometry, on $\mathfrak{T}_\Sigma$ using extremal quasiconformal mappings. Start with an initial marked Riemann surface $\Sigma \xrightarrow{f_0} X_0$ as our basepoint in $\mathfrak{T}_\Sigma$. Then any other point in $\mathfrak{T}_\Sigma$ corresponds to a map $X_0 \xrightarrow{\varphi} X$, which we may assume is a diffeomorphism. Teichmüller solved the problem of finding the *most conformal* homeomorphism homotopic to φ. Such a map is described by *holomorphic quadratic differentials* on the Riemann surfaces X_0 and X, and using these parameters, he parametrized $\mathfrak{T}_\Sigma$ as a bounded domain in a complex vector space. In particular the *complex structure* on $\mathfrak{T}_\Sigma$ is independent of the basepoint X_0. The *Teichmüller distance* between the corresponding points in $\mathfrak{T}_\Sigma$ is the *quasiconformal dilation* of the extremal map, that is, a quantitative measure of how far this most conformal differs from actually being conformal. For accounts of this theory, see the books of Nag, Gardiner, Hubbard.

However, another description of $\mathfrak{T}_\Sigma$, or rather Fricke space, is more relevant here. This is the parametrization of Fricke space by harmonic diffeomorphisms. For this discussion assume that Σ is a closed oriented surface with $\chi(\Sigma) < 0$.

Let X be a closed Riemann surface diffeomorphic to Σ, and consider a Fuchsian representation of $\pi = \pi_1(X)$. Equivalently, consider a homotopy class of homotopy equivalences

$$X \xrightarrow{f} M$$

where M is a hyperbolic surface. Since homotopy equivalences between closed surfaces are homotopic to diffeomorphisms, we may assume f is a diffeomorphism. The *energy density* of f is the exterior 2-form

$$\|df\|^2 \, dA \;=\; df \wedge \big(df \circ J\big)$$

where dA is the area form of a some metric in the conformal class and $\|df\|$ is the magnitude of the differential df, measured with respect to the same metric on X and the Riemannian metric on M. The second expression, particular to $\dim_{\mathbb{R}}(X) = 2$ depends only on the conformal structure. The *energy* of f is the integral

$$\int_X \|df\|^2 dA$$

and a critical point of this functional is a *harmonic map.*

Theorem 4.5.1 (Eels-Sampson [22],Schoen-Yau [62]). *Let X be a closed Riemann surface with $\chi(X) < 0$, let M be a hyperbolic surface, and $X \xrightarrow{f} M$ a homotopy-equivalence. Then f is homotopic to a unique harmonic map. Furthermore this harmonic map is a diffeomorphism.*

The existence and uniqueness are special cases of the much more general theory of Eels-Sampson [22] which guarantees existence under the assumptions that X is a closed Riemannian manifold and M is a nonpositively curved complete Riemannian manifold. They deduce uniqueness when M is assumed to have negative curvature. That the harmonic map is a diffeomorphism is due to Schoen-Yau [62].

If f is such a harmonic map, then its *Hopf differential* is a measure to which f fails to be conformal. Let g_M denote the metric tensor on M; complexify it to obtain a symmetric complex 2-tensor

$$T^{\mathbb{C}}M \times T^{\mathbb{C}}M \xrightarrow{g_M^{\mathbb{C}}} \mathbb{C}.$$

Pull it back to the Riemann surface to obtain a symmetric complex 2-tensor on X. Use the conformal structure of X to select its $(2,0)$-part $h(f)$, which is a smooth section of the holomorphic line bundle K^2. When f is harmonic $h(f)$ is holomorphic.

Theorem 4.5.2 (Tromba [79]). *The map*

$$\mathfrak{F}_X \longrightarrow H^0(X; K^2)$$

which associates to a marked hyperbolic surface $X \longrightarrow M$ the Hopf differential of the corresponding harmonic diffeomorphism is bijective.

Using harmonic maps, the Fricke space of a Riemann surface X identifies with the complex vector space of holomorphic quadratic differentials on X. In particular the choice of X imparts a complex structure to Fricke space. The uniformization theorem identifies the Fricke space of X with the Teichmüller space $\mathfrak{T}_\Sigma$, which already has a complex structure biholomorphic to a bounded domain. Hence the complex structures are different. Indeed, the complex structures on $\mathfrak{F}_X$ corresponding to different Riemann surfaces X heavily depends on X.

4.6. **Dynamics of Fuchsian action on the circle.** Closely related to surface group representations is the fundamental action of a closed surface group π on S^1. Let Σ be a closed connected surface with $\chi(\Sigma) < 0$ and let $\pi = \pi_1(\Sigma)$. A choice of hyperbolic structure on Σ defines a Fuchsian representation of π in $\mathsf{PSL}(2,\mathbb{R})$, and $\mathsf{PSL}(2,\mathbb{R})$ acts by orientation-preserving projective transformations of $\mathbb{RP}^1 \approx S^1$. The projective line appears as the ideal boundary $\partial\mathsf{H}^2$, defined synthetically as equivalence classes of asymptotic geodesic rays. Thus a point in $\mathfrak{T}_\Sigma$ determines a projective action of π on S^1.

This action has many beautiful properties. It is *minimal,* that is, every orbit is dense, and it is *structurally stable,* in that every small C^1 perturbation of it is topologically conjugate to it. It intimately relates to the geodesic flow of M as follows. The orientationd of Σ induces an orientation on S^1. Positively oriented ordered distinct triples of points in $S^1 = \partial\mathsf{H}^2$ correspond to unit tangent vectors on H^2. Suppose that (v_+, v_0, v_-) is such a triple, then v_+, v_- are the endpoints at infinity of a unique oriented geodesic

ℓ. Let $p \in \ell$ the unique point on ℓ closest to v_0. Then the unit vector $v = v(v_+, v_0, v_-)$ corresponding to (v_+, v_0, v_-) is the unit tangent vector at p which is tangent to the geodesic ℓ pointing towards v_+.

Although the action of π on S^1 is minimal, the action on the invariant open subset of positively oriented triples $(S^1)^{3+} \subset S^1 \times S^1 \times S^1$ is discrete. Its quotient is the unit tangent bundle UM of the hyperbolic surface.

Exercise 4.6.1. *Identify the stable manifolds of the geodesic flow in terms of this parametrization by S^1. Show that the foliation of UM defined by the trajectories of the geodesic flow depends only on the topology of M and not its hyperbolic structure. In contrast show that two hyperbolic surfaces whose geodesic flows are conjugate by a diffeomorphism must be isometric.*

Exercise 4.6.2. *Show that for different points in $\mathfrak{F}_\Sigma$, these actions are topologically conjugate. (In fact the conjugating homeomorphism is Hölder continuous.) Show that two such actions are smoothly conjugate $\Longleftrightarrow$ they are conjugate in $\mathsf{PSL}(2, \mathbb{R})$, that is, they represent the same point in $\mathfrak{F}_\Sigma$.*

This action can be constructed directly just from π. Namely, giving π the word metric coming from its Cayley graph, S^1 can be obtained as equivalence classes of *quasi-geodesic rays* in π, that is, Lipschitz maps of $\mathbb{N}$ into π. If $f : X \longrightarrow M$ is the harmonic diffeomorphism corresponding to a conformal structure X and a point in $\mathfrak{F}_X$, then lifting f to the universal covering yields a harmonic diffeomorphism

$$\tilde{X} \xrightarrow{\tilde{f}} \tilde{M} \approx \mathsf{H}^2$$

which extends continuously to a ρ-equivariant homeomorphism $S^1 \to S^1$.

In section 8.1 we make use of these structures when we discuss very briefly the notion of Anosov structures, introduced by Labourie in [57], and its relation to Fuchsian representations. Anosov structures play an important role in Labourie's work on Hitchin components, but also for maximal representations.

4.7. **The Euler class.** The Fricke space $\mathfrak{F}_\Sigma$ can also be recovered by looking at the Euler class of a representation $\pi \xrightarrow{\rho} \mathsf{PSL}_2(\mathbb{R})$. Since this point of view then leads to the more general consideration of maximal representations, it is discussed in Section 7 in Part II.

5. $\mathsf{SL}(n, \mathbb{C})$-Higgs bundles

In this section we describe briefly some basic features of Higgs bundles, and why they are relevant to the subject of this workshop, namely fundamental group representations in a reductive Lie group G.

As we shall see, a Higgs bundle is a bundle with both a complex (holomorphic) structure and also a flat structure. The flat structure is the link to fundamental group representations; the complex structure is responsible for new geometric features on the moduli space of representations (that is, for non-abelian Hodge theory), and also for the new tools which Higgs bundles bring to the study of the representations. We consider representations $\pi = \pi_1(\Sigma) \xrightarrow{\rho} G$.

We now describe in three steps how to get from such a representation to the notion a Higgs bundle.

5.1. **From representations to flat bundles: universal covers and associated bundles.** We have already seen, in sections 2.3 and 2.4 how a representation defines a G-bundle with a flat connection. Conversely, given a principal G-bundle $E_G \longrightarrow \Sigma$ with a flat connection, the holonomy around loops in Σ defines a representation of $\pi_1(\Sigma)$ in the structure group of the bundle, that is, it defines a representation (2.3.2). This correspondence defines an equivalence of deformation theories between flat G-bundles over Σ with G-gauge transformations and $\mathsf{Hom}(\pi, G)$ with inner automorphisms of G. In particular the quotients spaces $\mathsf{Hom}(\pi, G)/G$ and the space of isomorphism classes of flat G-bundles over Σ.

5.2. **From flat bundles to harmonic bundles: what reductivity means.** Assume that G is a reductive Lie group. In general, in order for the space $\mathsf{Hom}(\pi_1(\Sigma), G)/G$ to have good topological properties, it is necessary to exclude the G-orbits of non-reductive representations in $\mathsf{Hom}(\pi_1(\Sigma), G)$. We thus define the *moduli space of representations* of $\pi_1(\Sigma)$ in G to be the orbit space

$$\mathcal{R}(G) = \mathsf{Hom}^+(\pi_1(\Sigma), G)/G$$

of reductive representations. With the quotient topology, $\mathcal{R}(G)$ has the structure of an algebraic variety.

For the sake of definiteness, we explain the case $G = \mathsf{SL}(n, \mathbb{C})$. For an arbitrary reductive Lie group, the reader may consult the survey paper [5]. Let $\mathbf{E}$ be a smooth $\mathsf{SL}(n, \mathbb{C})$-bundle over Σ. Let D be an $\mathsf{SL}(n, \mathbb{C})$ connection on $\mathbf{E}$ and let F_D be its curvature. If D is flat then $F_D = 0$. By the correspondence described above, we get an identification

$$\mathcal{R} \equiv \mathcal{R}(\mathsf{SL}(n, C)) \cong \{\text{Reductive } \mathsf{SL}(n, \mathbb{C})\text{-connections } D \ : \ F_D = 0\}/\mathscr{G}^c, \tag{5.2.1}$$

where, by definition, a flat connection is reductive if the corresponding representation of $\pi_1(\Sigma)$ in $\mathsf{SL}(n, \mathbb{C})$ is reductive, and $\mathscr{G}^c$ is the group of automorphisms of $\mathbf{E}$ — the *gauge group*[2].

Now consider Hermitian metrics on $\mathbf{E}$. A metric on $\mathbf{E}$ is a smoothly varying choice of Hermitian inner product on the fibers of the $\mathbb{C}^n$ vector bundle naturally associated to $\mathbf{E}$. On each fiber it allows one to define a unitary basis. Moreover, these fiber-wise unitary bases fit together to form local unitary frames for $\mathbf{E}$. The transition functions which relate overlapping unitary frames must be unitary transformations. Thus a choice of Hermitian metric is equivalent to a reduction of structure group from $\mathsf{GL}(n, \mathbb{C})$ to $\mathsf{U}(n)$ (or, if orientation is respected, from $\mathsf{SL}(n, \mathbb{C})$ to $\mathsf{SU}(n)$). This can equivalently be described as a choice of section for the bundle $\mathbf{E} \times_{\mathsf{SL}(n,\mathbb{C})} \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n)$, i.e the associated bundle with fiber $\mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n)$.

This point of view is particularly useful if $\mathbf{E}$ has a flat structure, that is, if $\mathbf{E} = E_\rho$ for some representation $\pi \xrightarrow{\rho} \mathsf{SL}(n, \mathbb{C})$ (as in §2.3). Then

$$\mathbf{E} \times_{\mathsf{SL}(n,\mathbb{C})} \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n) \ = \ \big(\tilde{\Sigma} \times \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n)\big)/\rho(\pi) \tag{5.2.2}$$

[2]Thus orbits of $\mathscr{G}^c$ correspond to isomorphism classes of flat bundles

and a section h of $\mathbf{E} \times_{\mathsf{SL}(n,\mathbb{C})} \mathsf{SL}(n,\mathbb{C})/\mathsf{SU}(n)$ lifts to the graph of π-equivariant map

$$\widetilde{\Sigma} \xrightarrow{\tilde{h}} \mathsf{SL}(n,\mathbb{C})/\mathsf{SU}(n) \ . \tag{5.2.3}$$

A choice of bundle metric, while independent of a choice of connection, nevertheless interacts with connections in the following way. We have seen (in section 2.5.3) that with respect to a local trivialization of $\mathbf{E}$, a connection D is described by

$$D = d + \Gamma$$

where Γ is a $\mathfrak{sl}(n,\mathbb{C})$-valued 1-form. The metric defines a decomposition

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n) \oplus \mathsf{Sym}(n) \tag{5.2.4}$$

where $\mathsf{Sym}(n)$ denotes the vector space symmetric $n \times n$ matrices. We thus obtain a decomposition (which depends on the metric)

$$\Gamma = A + \Psi$$

We denote the metric by h, and the resulting $\mathsf{SU}(n)$ bundle by $\mathbf{E}_h$. It is not hard to see that the local expressions $d + A$ define an $\mathsf{SU}(n)$-connection on $\mathbf{E}_h$, while Ψ defines a 1-form with values in $\mathrm{ad}(\mathbf{E}_h)$. Thus every $\mathsf{SL}(n,\mathbb{C})$-connection D on $\mathbf{E}$ decomposes uniquely as

$$D = d_A + \Psi, \tag{5.2.5}$$

where d_A is an $\mathsf{SU}(n)$-connection on $\mathbf{E}_h$ and $\Psi \in \Omega^1(\Sigma, \mathrm{ad}(\mathbf{E}_h))$.

Let F_A be the curvature of d_A. The flatness of D, that is, the condition $F_D = 0$, is then equivalent to the conditions

$$\begin{aligned} F_A + \tfrac{1}{2}[\Psi,\Psi] &= 0 \\ d_A\Psi &= 0. \end{aligned} \tag{5.2.6}$$

If D is flat, then the metric is given by a π_1-equivariant map as in (5.2.3). There is a good notion of harmonicity for such maps. For this, we choose a Riemannian metric on Σ. This lifts to a Riemannian metric on $\widetilde{\Sigma}$, and the the target space $\mathsf{SL}(n,\mathbb{C})/\mathsf{SU}(n)$ is a Riemannian symmetric space. The one-form Ψ can be identified with the differential of $\tilde{h}$, and d_A with the pull-back of the Levi-Civita connection on $\mathsf{SL}(n,\mathbb{C})/\mathsf{SU}(n)$. Hence, the harmonicity condition of $\tilde{h}$ expressed in terms of D and its decomposition (5.2.5) becomes

$$d_A^*\Psi = 0, \tag{5.2.7}$$

where d_A^*, the adjoint of the covariant derivative operator, is defined using the metric h and the metric on Σ. Actually, this condition only depends on the conformal class of the Riemannian metric of Σ.

As a set of conditions for the pair (d_A, Ψ), the equations (5.2.6) and (5.2.7) are invariant under the action of $\mathscr{G}$, the gauge group of $\mathbf{E}_h$. A theorem of Donaldson [20] in rank 2 and Corlette for arbitrary rank [17] says the following.

Theorem 5.2.1. *There is a homeomorphism*

$$\big\{\text{Reductive } G\text{-connections } D \text{ with } F_D = 0\big\}/\mathscr{G}^c \tag{5.2.8}$$

$$\cong \big\{(d_A, \Psi) \text{ satisfying (5.2.6) and (5.2.7)}\big\}/\mathscr{G}. \tag{5.2.9}$$

Theorem 5.2.1 asserts that the $\mathscr{G}^c$-orbit of every reductive flat $\mathsf{SL}(n,\mathbb{C})$-connection, say D_0, contains a flat $\mathsf{SL}(n,\mathbb{C})$-connection $D = g(D_0)$ such that if we write $D = d_A + \Psi$, the additional condition $d_A^*\Psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h' = g(h)$ of $\mathbf{E}$ to a $\mathsf{SU}(n)$-bundle obtained by the action of $g \in \mathscr{G}^c$ on h. Equation $d_A^*\Psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$-equivariant map $\widetilde{X} \to \mathsf{SL}(n,\mathbb{C})/\mathsf{SU}(n)$ corresponding to the new reduction of structure group h'.

Define a *harmonic* $\mathsf{SL}(n,\mathbb{C})$*-bundle* to be a flat $\mathsf{SL}(n,\mathbb{C})$-bundle $(\mathbf{E}, D)$ with a harmonic reduction of structure group to $\mathsf{SU}(n)$, that is, with a reduction h to $\mathbf{E}_h$ such that the induced decomposition $D = d_A + \Psi$ yields a pair (d_A, Ψ) satisfying (5.2.6) and (5.2.7).

5.3. **From harmonic bundles to Higgs bundles: the holomorphic picture.** On Σ a conformal structure and a complex structure are equivalent and we will show now that harmonic bundles can be described in holomorphic terms; the result is a stable $\mathsf{SL}(n,\mathbb{C})$-Higgs-bundle.

Let X be the Riemann surface defined by Σ with its complex structure. Using the complex structure on X, we can decompose the covariant derivative d_A into $(0,1)$ (that is, anti-holomorphic) and $(1,0)$ (that is, holomorphic) parts, that is, we can write

$$d_A = \bar\partial_A + d_A^{(1,0)}.$$

Furthermore, the section $\Psi \in \Omega^1(X, \mathrm{ad}(\mathbf{E}_h))$ can be written as

$$\Psi = \Phi - \tau(\Phi),$$

where $\Phi \in \Omega^{1,0}(X, \mathrm{ad}(\mathbf{E}))$ and τ is the conjugation on $\mathfrak{sl}(n,\mathbb{C})$ defining its compact real form, i.e $\tau(\Phi) = -\Phi^*$. The defining equations for the harmonic bundle, that is, (5.2.6) and (5.2.7), then become

$$\begin{aligned} &\bar\partial_A^2 = 0 = [\Phi, \Phi] \\ &\bar\partial_A \Phi = 0 \\ &F_A - [\Phi, \tau(\Phi)] = 0. \end{aligned} \tag{5.3.1}$$

The conditions on the first line are automatic since the complex dimension of X is one. In any case, the condition $\bar\partial_A^2 = 0$ says that the operator $\bar\partial_A$ defines a holomorphic structure on the $\mathsf{SL}(n,\mathbb{C})$-bundle $\mathbf{E}$. The condition $\bar\partial_A\Phi = 0$ then says that Φ is a *holomorphic section* of $\mathrm{ad}(\mathbf{E})\otimes K_X$, where K_X is the canonical bundle on X. This leads directly to the notion of a ($\mathsf{SL}(n,\mathbb{C})$-)*Higgs bundle*, namely

Definition 5.3.1. *An* $\mathsf{SL}(n,\mathbb{C})$*-Higgs bundle over* X *is a pair* $(\mathcal{E},\Phi)$ *where* $\mathcal{E}$ *is a holomorphic principal* $\mathsf{SL}(n,\mathbb{C})$*-bundle and* Φ *is a holomorphic section of* $\mathrm{ad}(\mathcal{E})\otimes K_X$*.*

For those who prefer to work with vector bundles, rather than principal bundles, we can replace this definition with the following:

Definition 5.3.2. *An* $\mathsf{SL}(n,\mathbb{C})$*-Higgs bundle over* X *is a pair* (E,Φ) *where* E *is a rank* n *holomorphic vector bundle with trivial determinant and* $\Phi\in H^0(X,\mathrm{ad}(E)\otimes K_X)$*, where* $\mathrm{ad}(E)$ *is the adjoint* $\mathfrak{sl}(n,\mathbb{C})$*-bundle (that is,* Φ *is a traceless endomomorphism* $\Phi: E\to E\otimes K_X$*).*

Such Higgs-bundles where introduced by Hitchin in [49][3].

The holomorphic vector bundle E can be viewed as a smooth vector bundle together with a $\bar\partial$-operator $\bar\partial_E$. If we fix the underlying smooth bundle, then the Higgs bundles correspond to pairs $(\bar\partial_E,\Phi)$. We will use this description interchangeably with the notation (E,Φ). Now given a metric h on the smooth bundle, there is a bijective correspondence between operators $\bar\partial_E$ and connections d_A. The last condition in (5.3.1) can thus be read as a condition for a metric on the Higgs bundle $(\bar\partial_E,\Phi)$. Here's where things get interesting: In ([49]) Hitchin showed how to construct a moduli space of such Higgs bundles. This moduli space is meant to be a nice geometric object whose points parameterize isomorphism classes of Higgs bundles.

As is well known from the corresponding construction for plain holomorphic bundles, the construction of such a moduli space requires the exclusion of certain 'bad' classes from the set of all isomorphism classes. The precise definition of 'bad' is inspired by the concept of stability in Geometric Invariant Theory [69, 71]

To define stability we will relax the condition of E having trivial determinant and consider Higgs bundles (E,Φ) consisting of a holomorphic vector bundle E and $\Phi\in H^0(X,\mathsf{End}(E)\otimes K_X)$. For simplicity we will consider only the case in which $\mathsf{deg}E=0$.

Definition 5.3.3. *Let* (E,Φ) *be a Higgs bundle such that* $\mathsf{deg}E=0$*. Then* (E,Φ) *is stable* $:\Longleftrightarrow\ \forall$ Φ*-invariant subbundle* $E'\subset E$*,*

$$deg(E')<0. \tag{5.3.2}$$

Here Φ*-invariant means that* $\Phi(E')\subseteq E'\otimes K_X$*. The Higgs bundle is* semistable $:\Longleftrightarrow$ *(5.3.2) holds as a weak inequality, and is* polystable $:\Longleftrightarrow$ *it decomposes as a direct sum of stable Higgs bundles of degree* 0*.*

Note that if (E,Φ) is a polystable $\mathsf{SL}(n,\mathbb{C})$-Higgs bundle, the vector bundles in the various summands need not have trivial determinant. Let $\mathcal{M}$ be *the moduli space* $\mathcal{M}$ *of polystable* $\mathsf{SL}(n,\mathbb{C})$*-Higgs bundles.* In [49] Hitchin gave an analytic construction of $\mathcal{M}$ in the rank 2 case [4]. In this situation, $\mathcal{M}$ is an irreducible quasiprojective algebraic

[3] Actually in [49] Hitchin doesn't use the term 'Higgs bundle', but does introduce the name 'Higgs field' for Φ; the name 'Higgs bundle' was introduced by Simpson.

[4] A GIT construction was given later by Nitsure for arbitrary rank [72].

variety of complex dimension $6g - 6$, where g is the genus of X, which is assumed to satisfy $g \geqslant 2$. The set of stable points defines a dense open smooth subvariety. Hitchin (for rank 2) [48] and Simpson (for arbitrary rank) [73] proved the following crucial - for us - theorem.

Theorem 5.3.4. *An $\mathsf{SL}(n,\mathbb{C})$-Higgs bundle (E,Φ) is polystable if and only if E admits a Hermitian metric with respect to which the corresponding pair (d_A,Φ) satisfies $F_A - [\Phi,\tau(\Phi)] = 0$. This metric is unique up to $\mathsf{SU}(n)$-gauge transformations.*

It follows from this that

Theorem 5.3.5. *There is a homeomorphism*

$$\mathcal{M} \cong \{(d_A,\Phi) \ \textit{satisfying} \ (5.3.1)\}/\mathscr{G}. \tag{5.3.3}$$

To complete the circle we just need the following.

Proposition 5.3.6. *The correspondence $(d_A,\Phi) \mapsto (d_A,\Psi := \Phi - \tau(\Phi))$ defines a homeomorphism*

$$\{(d_A,\Phi) \ \textit{satisfying} \ (5.3.1)\}/\mathscr{G} \cong \{(d_A,\Psi) \ \textit{satisfying} \ (5.2.7)\}/\mathscr{G}. \tag{5.3.4}$$

(5.2.1), (5.2.8)), (5.3.3)), and (5.3.4) together imply the following fundamental result:

Theorem 5.3.7. *The varieties $\mathcal{M}$ homeomorphic.*

Remark 5.3.8. Notice that the complex structures of $\mathcal{M}$ and $\mathcal{R}$ are different. The complex structure of $\mathcal{M}$ is induced by the complex structure of X, while that of $\mathcal{R}$ is induced by the complex structure of $\mathsf{SL}(n,\mathbb{C})$.

We have thus seen that $\mathsf{SL}(n,\mathbb{C})$-Higgs bundles are the objects of the *Dolbeault theory* corresponding to the Betti theory of $\mathsf{SL}(n,\mathbb{C})$-representations and the de Rham theory of flat $\mathsf{SL}(n,\mathbb{C})$-connections. The Dolbeault theory relates to the de Rham theory via *Hodge theory* of harmonic objects — Hermitian metrics arising as solutions to the *self-duality equations.*

Part 2. SL_2 as guiding example

6. Higgs bundles and $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbb{R})$ representations of $\pi_1(X)$

In this section we restrict to the case $n = 2$ and describe some more specific features and applications of Higgs bundles, which will also generalize to a bigger class of Lie groups G (see [3, 4, 5, 29, 30, 31, 41, 48, 49, 83]).

6.1. $\mathsf{SL}(2,\mathbb{C})$-Higgs bundles and real forms.

In this section $\mathcal{R} = \mathcal{R}(\mathsf{SL}(2,\mathbb{C})$ and $\mathcal{M}$ is the moduli space of polystable $\mathsf{SL}(2,\mathbb{C})$-Higgs bundles.

The Narasimhan-Seshadri Theorem 3.4.3 identifies the moduli space of representations of $\pi_1(X)$ in $\mathsf{SU}(2)$ with the moduli space of polystable rank 2 vector bundles with trivial determinant. This defines a subvariety of $\mathcal{M}$ consisting of Higgs bundles with $\Phi = 0$.

Our goal now is to identify also the moduli space of $\mathsf{SL}(2,\mathbb{R})$ representations $\pi_1(X)$, that is,

$$\mathcal{R}(\mathsf{SL}(2,\mathbb{R})) := \mathsf{Hom}^+(\pi_1(X), \mathsf{SL}(2,\mathbb{R}))/\mathsf{SL}(2,\mathbb{R}),$$

in terms of Higgs bundles. In fact, we will see that both moduli spaces of $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbb{R})$ representations appear as fixed points of a certain antiholomorphic involution in $\mathcal{R}$.

Before doing that we recall that there is a topological invariant that one can attach to an element in $\mathcal{R}(\mathsf{SL}(2,\mathbb{R}))$. This is the Euler class $d \in \mathbb{Z}$ of the corresponding flat $\mathsf{SL}(2,\mathbb{R})$-bundle. We can then define the subvarieties

$$\mathcal{R}_d := \{\rho \in \mathcal{R}(\mathsf{SL}(2,\mathbb{R})) \quad : \quad \text{with Euler class } d\}.$$

By Milnor [65], $\mathcal{R}_d = \emptyset$ unless $|d| \leqslant g-1$.

It turns out that the conjugations with respect to both real forms, $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbb{R})$, of $\mathsf{SL}(2,\mathbb{C})$ are inner equivalent and hence they induce the same antiholomorphic involution $\bar{\iota} : \mathcal{R} \to \mathcal{R}$, where we recall that the complex structure of $\mathcal{R}$ is that naturally induced by the complex structure of $\mathsf{SL}(2,\mathbb{C})$. To be more precise, at the level of Lie algebras, the conjugation with respect to the real form $\mathfrak{su}(2)$ is given by the $\mathbb{C}$-antilinear involution

$$\begin{aligned} \tau : \mathfrak{sl}(2,\mathbb{C}) &\to \mathfrak{sl}(2,\mathbb{C}) \\ A &\mapsto -\overline{A}^t, \end{aligned}$$

while the conjugation with respect to the real form $\mathfrak{sl}(2,\mathbb{R})$ is given by the $\mathbb{C}$-antilinear involution

$$\begin{aligned} \sigma : \mathfrak{sl}(2,\mathbb{C}) &\to \mathfrak{sl}(2,\mathbb{C}) \\ A &\mapsto \overline{A}. \end{aligned}$$

Now,

$$\sigma(A) = J\tau(A)J^{-1}$$

for $J \in \mathfrak{sl}(2,\mathbb{R})$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is simply because for every $A \in \mathfrak{sl}(2,\mathbb{R})$, one has that

$$JA = -A^tJ. \tag{6.1.1}$$

Under the correspondence $\mathcal{M} \cong \mathcal{R}$, the antiholomorphic involution of $\mathcal{R}$ defined by τ and σ becomes the holomorphic involution $\mathcal{M} \xrightarrow{\iota} \mathcal{M}$ given by

$$(E,\Phi) \mapsto (E,-\Phi), \tag{6.1.2}$$

where we recall that the complex structure of $\mathcal{M}$ is that induced by the complex structure of X. This follows basically from the fact that the flat $\mathsf{SL}(2,\mathbb{C})$-connection D corresponding to $(\bar{\partial}_E, \Phi)$ under Theorem 5.3.7 is given by

$$D = \bar{\partial}_E + \tau(\bar{\partial}_E) + \Phi - \tau(\Phi),$$

and then

$$\tau(D) = \tau(\bar{\partial}_E) + \bar{\partial}_E + \tau(\Phi) - \Phi,$$

$(E, -\Phi)$. Notice also that $\tau(D) \cong \sigma(D)$.

Proposition 6.1.1. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\iota} & \mathcal{M} \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{R} & \xrightarrow{\bar{\iota}} & \mathcal{R}, \end{array}$$

where the vertical arrows correspond to the homomorphism between $\mathcal{M}$ and $\mathcal{R}$.

The fixed points of ι are of two types:

Type 1: $(E, \Phi) \in \mathcal{M}$ such that $\Phi = 0$.

Type 2: $(E, \Phi) \in \mathcal{M}$ such that $E = L \oplus L^{-1}$, where L is a holomorphic line bundle and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where $\beta \in H^0(X, L^2K_X)$ and $\gamma \in H^0(X, L^{-2}K_X)$. This is clear since the gauge transformation $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ brings (E, Φ) to $(E, -\Phi)$.

The subvariety of fixed points of the first type is identified with the moduli space of polystable rank 2 vector bundles with trivial determinant, and corresponds to the moduli of representations of $\pi_1(X)$ in $\mathsf{SU}(2)$, as well-know by the already mentioned theorem of Narasimhan and Seshadri [70],

As for the second type, the degree of L allows us to define the moduli space $\mathcal{M}_d$ of Higgs bundles as above with fixed $\mathsf{deg} L = d$. Here the natural gauge transformations are $\mathbb{C}^*$-transformations, that is, those of L. Allowing $\mathsf{SL}(2, \mathbb{C})$-transformation naturally identifies $\mathcal{M}_d$ and $\mathcal{M}_{-d}$ inside $\mathcal{M}$. Now, the semistability condition gives a constraint on the possible degrees that L may have, namely, we must have

$$|d| \leqslant g - 1. \tag{6.1.3}$$

This can be seen very easily. Assume that $d \geqslant 0$ (similar argument for $d \leqslant 0$). Suppose that $d > g - 1$. Then $\gamma = 0$ and L is a Φ-invariant line subbundle of E. By the semistability of (E, Φ) we must have that $d \leqslant 0$ which gives a contradiction.

We thus have the following.

Theorem 6.1.2. *$\mathcal{R}_d$ is homeomorphic to $\mathcal{M}_d$.*

The moduli space of representations of $\pi_1(X)$ in $\mathsf{SL}(2, \mathbb{R})$ was studied by Goldman [33, 35], who showed that for d satisfying $|d| = g - 1$ there are 2^{2g} isomorphic connected components that can be identified with Teichmüller space, and showed that for d such that $|d| < g - 1$ there is only one component. In [49] this was also proved by Hitchin, who also gave a very explicit description of each component.

¿From the description of $\mathcal{M}_d$ as the set of equivalence classes (L, β, γ) where L is a line bundle of degree d and

$$\beta \in H^0(X, L^2K_X), \qquad \gamma \in H^0(X, L^{-2}K_X).$$

$\mathcal{M}_d$ is the total space of a holomorphic complex vector bundle of rank $g+2|d|-1$ over a 2^{2g}-fold covering of the $2g-2-2|d|$-symmetric power

$$\mathsf{Sym}^{2g-2-2|d|}X$$

of X. To see this, assume that $d \geqslant 0$ (the case $d \leqslant 0$ is similar), then the pair (L,γ) defines an element in $\mathsf{Sym}^{2g-2-2d}(X)$, given by the zeros of γ. But if L_0 is a line bundle of degree 0 of order two, that is, $L_0^2 = \mathcal{O}$, then the element (LL_0,γ) defines also the same divisor in $\mathsf{Sym}^{2g-2-2d}(X)$. Hence the set of pairs (L,γ) gives a point in the 2^{2g}-fold covering of $\mathsf{Sym}^{2g-2-2d}(X)$. The section β now gives the fiber of the vector bundle. Notice Riemann-Roch $\Longrightarrow H^1(X, L^2K_X) = 0$.

¿From this proposition we deduce:

- $\dim \mathcal{M}_d = 3g-3$,
- $\mathcal{M}_d$ is connected if $|d| < g-1$,
- $\mathcal{M}_d$ has 2^{2g} connected components if $|d| = g-1$, each isomorphic to $\mathbb{C}^{3g-3}$ (the fiber of a rank $3g-3$ vector bundle over a 2^{2g}-fold covering of a point!). This is clear since if $\mathsf{deg} L = g-1$, the line bundle $L^{-2}\otimes K_X$ is of zero degree and hence has a section (unique up to multiplicaton by a scalar) if and only if $L^{-2}\otimes K_X \cong \mathcal{O}_X$, i.e. if L is a square root of K_X. For each of the 2^{2g} choices of square root $L = K_X^{1/2}$, one has a connected component which is parametrized by $\beta \in H^0(X, K_X^2)$. As explained in Section 6.3, each of these components is diffeomorphic to Teichmüller space.

6.2. **Uniformization and Riemann surfaces.** Let ρ be an orientation-preserving Fuchsian representation. The developing map defines a diffeomorphism

$$\Sigma \longrightarrow M := \mathsf{H}^2/\rho(\pi)$$

where M is a compatibly oriented hyperbolic surface. Forgetting the metric but remembering the conformal structure on M is a Riemann surface X for which the corresponding map $X \longrightarrow M$ is biholomorphic.

Theorem 6.2.1 (Uniformization). *Let X be a compact Riemann surface. Then $\exists!$ hyperbolic structure M compatible with the conformal structure on X.*

In particular $\exists$ a Fuchsian representation ρ for which the corresponding map $X \longrightarrow \mathsf{H}^2/\rho(\pi)$ is biholomorphic.

We shall use both the upper-half plane model H^2 and the unit disc model $\mathbb{D}$, both subdomains of $\mathbb{P}(\mathbb{C}^2)$ which are projectively equivalent. The automorphism group $\mathsf{Aut}(\mathsf{H}^2)$ identifies with the projective linear group $\mathsf{PSL}(2,\mathbb{R})$ and $\mathsf{Aut}(\mathbb{D})$, it identifies with the projective unitary group $\mathsf{PU}(1,1)$.

6.3. **Uniformization and vector bundles.** We interpret this construction in terms of holomorphic vector bundles. This requires a lifting of the representation

$$\pi \xrightarrow{\tilde{\rho}} \mathsf{SL}(2,\mathbb{C}),$$

which is equivalent to the choice of a *spin structure* on X. The obstruction to lifting the projective representation ρ to a linear representation $\tilde{\rho}$ is the *second Stiefel-Whitney* *class*

$$w_2(X) \in H^2(X;\mathbb{Z})$$

which vanishes since it is the $\mathbb{Z}/2$-reduction of the Euler class of X. Since $\chi(\Sigma) = 2-2g$ is even, $w_2 = 0$ and ρ lifts.

Let E_ρ be the flat $\mathbb{C}^2$-bundle over X with holonomy ρ. For each $x \in X$, $\varphi(x)$ is a line in $\mathbb{C}^2$, and the map $\tilde{X} \xrightarrow{\varphi} \mathbb{D}$ defines a holomorphic line bundle $L \subset E$.

The *canonical line bundle* K_X is the holomorphic line bundle defined by the cotangent bundle of X. We claim that $L^2 = K_X$.

To this end, let $\ell \subset \mathbb{C}^2$ be a line which determines a point, also denoted ℓ, of $\mathbb{P}(\mathbb{C}^2)$. The tangent space naturally identifies

$$T_\ell \mathbb{P}(\mathbb{C}^2) \xrightarrow{\cong} \mathsf{Hom}(\ell, \mathbb{C}^2/\ell).$$

The $\mathsf{SL}(2,\mathbb{C})$-invariant symplectic structure on $\mathbb{C}^2$ defines an natural isomorphism

$$\mathbb{C}^2/\ell \xrightarrow{\cong} \ell^*$$

with the vector space ℓ^* dual to ℓ. Since

$$\tilde{X} \xrightarrow{\varphi} \mathbb{P}(\mathbb{C}^2)$$

is a local biholomorphism, its differential identifies the tangent line bundle K_X^{-1} with

$$\mathsf{Hom}(L, L^{-1}) \cong L^{-2}.$$

Thus $L^2 \cong K_X$,

$$\mathsf{deg}(L) = 1 - g < 0.$$

and E is *unstable,* destabilized by L.

One approach to obtaining a class of stable objects is to replace E by a stable Higgs bundle (see Section 5). The holomorphic $\mathbb{C}^2$-bundle E is a *nontrivial extension*

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \longrightarrow 0$$

and we replace E by the direct sum $V = L \oplus L^{-1}$ with an auxiliary Higgs field. In this particular case, $L^2 \cong K_X$ and the isomorphism defines a holomorphic section Φ of

$$K_X \otimes \mathsf{Hom}(L, L^{-1}) \subset K_X \otimes \mathsf{End}(V)$$

which is everywhere nonzero.

This section Φ is the $\mathsf{Hom}(L, L^{-1}) \subset \mathsf{End}(V)$-valued holomorphic 1-form corresponding to the differential of φ.

The above Higgs bundle is stable since $\mathsf{deg}(V) = 0$ and the only Φ-invariant holomorphic subbundle is the image $L^{-1} \subset V$ of Φ which has degree $1-g$.

6.3.1. *Hermitian metrics and uniformization.* The uniformization $\tilde{X} \xrightarrow{\varphi} \mathbb{D}$ defines the extra structure of a *Hermitian metric* on the vector bundle V as follows. $\mathbb{D}$ admits an $\mathsf{Aut}(\mathbb{D})$-invariant Riemannian metric $g_{\mathbb{D}}$ of curvature -1 (a *hyperbolic metric,* which pulls back to a π-invariant hyperbolic metric $\varphi^* g_{\mathbb{D}}$ on $\tilde{X}$, and hence a hyperbolic metric on X. This Riemannian metric on X is compatible with the conformal structure on X and therefore defines a Hermitian metric on the holomorphic line bundle K^{-1} corresponding to the tangent bundle TX. Furthermore this Hermitian structure defines Hermitian metrics on all bundles associated to K_X, such as L and V. In particular the Hermitian metric h_V on V plays a crucial role in the Higgs bundle machinery as follows.

6.3.2. *Two decompositions.* The original flat vector bundle E has a holomorphic structure D'' arising from the flat connection D. However, the holomorphic vector bundle underlying the Higgs bundle corresponding to E is quite different. Namely, let D_h be the unique Hermitian connection preserving h_V. Its $(0,1)$-part D''_h is a holomorphic structure.

Thus we have used the Hermitian metric h to decompose the original flat connection into a skew-Hermitian part D_h, which is the Hermitian connection, and a Hermitian (self-adjoint) part Ψ, which is an $\mathsf{End}(E)$-valued 1-form (in fact its values are endomorphisms which are self-adjoint with respect to h):

$$D = D_h + \Psi.$$

The complex structure on X provides further decompositions

$$\begin{aligned} D_h &= D'_h + D''_h \\ \Psi &= \Phi + \Phi^* \end{aligned}$$

where Φ — the Higgs field — is the $(1,0)$-part of Ψ and the $(0,1)$-part of Ψ equals the adjoint of Φ with respect to h.

We continue to consider the holomorphic $\mathbb{C}^2$-bundle $V = L \oplus L^{-1}$. The vector bundle $\mathsf{End}(V)$ decomposes as a direct sum

$$\begin{aligned} \mathsf{End}(V) = \mathsf{Hom}(L, L) &\oplus \mathsf{Hom}(L, L^{-1}) \\ &\oplus \mathsf{Hom}(L^{-1}, L) \oplus \mathsf{Hom}(L^{-1}, L^{-1}). \end{aligned}$$

A Higgs field $\Phi \in \Omega^1(X, \mathsf{End}(V)$ decomposes as a 2×2 matrix of holomorphic 1-forms

$$\begin{bmatrix} \Phi_{11} & \Phi_{11} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

where Φ_{11} is a holomorphic section of $K_X \otimes \mathsf{Hom}(L, L) \cong K_X$, where Φ_{12} is a holomorphic section of $K_X \otimes \mathsf{Hom}(L, L^{-1}) \cong \mathbb{C}$, where Φ_{21} is a holomorphic section of $K_X \otimes \mathsf{Hom}(L^{-1}, L) \cong K_X^2$, and where Φ_{22} is a holomorphic section of $K_X \otimes \mathsf{Hom}(L^{-1}, L^{-1}) \cong K_X$. For simplicity we assume that the holomorphic 1-forms

$$\Phi_{11} = \Phi_{22} = 0.$$

If Φ_{12} vanishes somewhere, then it vanishes identically, and L destabilies Higgs bundle (V, Φ). Hence we assume that Φ_{12} is everywhere nonzero. Thus the Higgs field is completely determined by precisely by Φ_{21}, which corresponds to a *holomorphic quadratic differential,* that is, a holomorphic section of K_X^2.

7. Invariants of representations - the Euler number

When π is the fundamental group of a closed oriented surface (that that is, $k = 0$), one source of invariants of representations $\pi \longrightarrow G$ are *characteristic classes.* The characteristic classes of the flat principal G-bundle over Σ determine invariants of $\pi \to$

G. When G is a connected Lie group, the only invariants lie in $H^2(\Sigma, \pi_1(G))$, and they are given by an obstruction map

$$\mathsf{Hom}(\pi, G) \xrightarrow{o_2} H^2(\Sigma, \pi_1(G)) \cong \pi_1(G).$$

In particular, nonzero characteristic invariants exist only if $\pi_1(G)$ is nontrivial. The fundamental group of $G = \mathsf{PSL}(2,\mathbb{R})$ is infinite cyclic; the corresponding obstruction o_2 is the *Euler number* of the associated $\mathbb{RP}^1$-bundle over Σ:

$$\mathsf{Hom}(\pi, G) \xrightarrow{e} \mathbb{Z}.$$

7.1. **Central Extensions.** The Euler number is the obstruction to trivialize the $\mathbb{RP}^1$-bundle over Σ associated to $\pi \to \mathsf{PSL}(2,\mathbb{R})$. Trivializing the $\mathbb{RP}^1$-bundle over Σ is essentially the same as lifting the structure group from $\mathsf{PSL}(2,\mathbb{R})$ to the universal cover $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$. Consider the central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathsf{PSL}(2,\mathbb{R})} \to \mathsf{PSL}(2,\mathbb{R}) \to 1. \tag{7.1.1}$$

When we present π by its standard presentation (1.1) and try to lift the representation $\pi \to \mathsf{PSL}(2,\mathbb{R})$ to a representation $\pi \to \widetilde{\mathsf{PSL}(2,\mathbb{R})}$. Thus we consider commutators $[\tilde{g}, \tilde{h}]$ in $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ of the lifts of two elements $g, h \in \mathsf{PSL}(2,\mathbb{R})$ to $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ Since the extension (7.1.1) is central, the commutator $[\tilde{g}, \tilde{h}]$ does only depend on $g, h \in \mathsf{PSL}(2,\mathbb{R})$, producing a *lifted commutator map*

$$\mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R}) \xrightarrow{[\cdot,\cdot]^\sim} \widetilde{\mathsf{PSL}(2,\mathbb{R})}.$$

The Euler number of $\pi \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{R})$ equals

$$\begin{aligned} e(\rho) = [\rho(A_1), \rho(B_1)]^\sim \dots [\rho(A_g), \rho(B_g)]^\sim \\ \in \mathbb{Z} = \pi_1(\mathsf{PSL}(2,\mathbb{R})) \end{aligned}$$

and satisfies the *Milnor-Wood-inequality* [65, 82]:

$$|e(\rho)| \leqslant 2g - 2.$$

To understand the space of homomorphisms $\mathsf{Hom}(\pi, G)$ it is often useful to decompose $\mathsf{Hom}(\pi, G)$ into simpler pieces $\mathsf{Hom}(\pi^{(i)}, G)$, where $\pi^{(i)}$ are the fundamental groups of the subsurfaces Σ_i in a decomposition $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_l$, and we would like that the Euler class, similar to the Euler characteristic, is additive under this decomposition. Given a representation $\pi \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{R})$ with restrictions

$$\pi^{(i)} \xrightarrow{\pi^i} \mathsf{PSL}(2,\mathbb{R})$$

the total Euler class is *additive:*

$$e(\rho) = e(\rho_1) + \dots + e(\rho_l).$$

For this we first have to understand the Euler number for a representation $\pi \to PSL(2,\mathbb{R})$ when π is the fundamental group of a surface with nonempty boundary, that is, if $k > 0$. In this case π is isomorphic to a free group of rank $2g+k-1$ and in particular, there are no invariants coming from characteristic classes, since $H^2(\Sigma, \pi_1(G))$.

One way around this is to consider the relative Euler number of the the $\mathbb{RP}^1$-bundle over Σ, relative with respect to fixed trivializations along the boundary. This requires to prescribe some conditions for the holonomy around the boundary. ([35]). We will describe another definition/formular for the Euler number involving "translation numbers" (and rotation numbers).

7.2. **Quasimorphism/Translation number.** $\mathsf{PSL}(2,\mathbb{R})$ acts on $S^1 = \mathbb{RP}^1$ by rotations. This action lifts to an action of $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ on $\mathbb{R} = \widetilde{S^1}$ by translation. Let us recall how the translation number of an element $\tilde{g} \in \widetilde{\mathsf{PSL}(2,\mathbb{R})}$ and the rotation number of an element $g \in \mathsf{PSL}(2,\mathbb{R})$ can be recovered from the central extension (7.1.1). Central extensions of G by $\mathbb{Z}$ are classified by second cohomology classes $[c] \in H^2(G,\mathbb{Z})$. In particular, the central extension (7.1.1) is given by a cocycle:

$$\begin{aligned} \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R}) &\xrightarrow{c} \mathbb{Z} \\ (g,h) \qquad &\longmapsto c(g,h). \end{aligned}$$

This means that we can identify $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ as a set with $\mathsf{PSL}(2,\mathbb{R}) \times \mathbb{Z}$, where the group multiplication is defined by

$$(g,m) \cdot (h,n) := (gh, m+n+c(g,h)).$$

The associativity of this group multiplication is precisely the cocycle condition for c.

An important feature is now that our central extension (7.1.1) is indeed given by a *bounded* cocycle, that is,

$$\sup_{g,h \in \mathsf{PSL}(2,\mathbb{R})} |c(g,h)| = D$$

for some $D \in \mathbb{N}$. This implies that the function which is the projection onto the second factor

$$\begin{aligned} \widetilde{\mathsf{PSL}(2,\mathbb{R})} = \mathsf{PSL}(2,\mathbb{R}) \times \mathbb{Z} &\xrightarrow{f} \mathbb{Z} \\ \tilde{g} = (g,m) \qquad &\longmapsto m, \end{aligned}$$

is a *quasimorphism*, that is,

$$\sup_{g,h \in \mathsf{PSL}(2,\mathbb{R})} |f(g,h) - f(g) - f(h)| = \sup_{g,h \in \mathsf{PSL}(2,\mathbb{R})} |c(g,h)| = D.$$

In particular the homogeneization

$$\begin{aligned} \widetilde{\mathsf{rot}} : \widetilde{\mathsf{PSL}(2,\mathbb{R})} &\to \mathbb{R} \\ g &\mapsto \widetilde{\mathsf{rot}}(g) := \lim_{n\to\infty} \frac{f(g^n)}{n} \end{aligned}$$

is well defined. Moreover $\widetilde{\mathsf{rot}}$ is a continuous quasimorphism, which is a homomorphism on cyclic subgroups and is invariant under conjugation. Let $\tilde{g} \in \widetilde{\mathsf{PSL}(2,\mathbb{R})}$, then $\widetilde{\mathsf{rot}}(\tilde{g})$ is the translations number of $\tilde{g}$.

Note that $\widetilde{\mathsf{rot}}$ is actually equivariant with respect to the action of $\mathbb{Z}$ by decktransformations on $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ and by translation on $\mathbb{R}$, so that it induces a continuous map, the *rotation number function:*

$$\mathsf{PSL}(2,\mathbb{R}) \xrightarrow{\mathsf{rot}} S^1 = \mathbb{R}/\mathbb{Z}.$$

7.3. **The Euler number revisited.** Now we have all our tools to define the Euler number of a representation $\rho : \pi \to \mathsf{PSL}(2,\mathbb{R})$, when π is the fundamental group of a surface with boundary. We assume that $k > 0$, so that π is isomorphic to the free group. There is no obstruction to lift the representation ρ to $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$, so let us just choose a lift

$$\widetilde{\rho} : \pi \to \widetilde{\mathsf{PSL}(2,\mathbb{R})}. \tag{7.3.1}$$

We again consider π in its standard presentation given in (1.1).

Definition 7.3.1. *The Euler number of the representation ρ is*

$$e(\rho) := -\sum_{i=1}^{n} \widetilde{\mathsf{rot}}(\widetilde{\rho}(c_i)) \in \mathbb{R}. \tag{7.3.2}$$

The reader may check that this definition is independent of the lift chosen in (7.3.1).

Let us emphasize that since we use the presentation of π, the Euler number is not an invariant of a the free group isomorphic to π, but depends on how this free group is realized as fundamental group of a surface.

7.4. **Properties of the Euler number.** Now that we have a definition of the Euler number for a representation of any surface group π let us state some properties, proofs of which can be found in [65, 33, 9].

Proposition 7.4.1.
1. *The Euler number $e : \mathsf{Hom}(\pi, \mathsf{PSL}(2,\mathbb{R})) \to \mathbb{R}$ is a continuous real valued function.*
2. *The Euler number is additive under admissible decompositions of surfaces: $e(\rho) = e(\rho_1) + e(\rho_2)$ (see above discussion).*
3. *The Euler number is bounded $|e(\rho)| \leqslant (2g-2+k)$.*
4. *The Euler number satisfies the congruence property $e(\rho) \equiv -\sum_{i=1}^{k} \mathsf{rot}(\rho(c_i))$ mod $\mathbb{Z}$.*

7.5. **Representations with maximal Euler number.** We will now focus on Property (3), by making the following definition.

Definition 7.5.1. *A representation $\rho : \pi(\Sigma_{g,k}) \to \mathsf{PSL}(2,\mathbb{R})$ is said to be* maximal *if $e(\rho) = 2g-2+k$.*

With these representations of maximal Euler number we recover uniformizations:

Theorem 7.5.2. *Let $\rho : \pi \to \mathsf{PSL}(2,\mathbb{R})$ be maximal. Then ρ is the holonomy homomorphism of a complete hyperbolic structure on Σ. In particular, ρ is discrete and faithful.*

This result was proven by Goldman [33, 35] under the assumption of some boundary condition; without any boundary condition it is proved by Burger and Iozzi [6].

An important step to prove this theorem is to transfer the maximality of the Euler number of the representation ρ to a a more geometric "maximality". The theorem is basically proven if we show that given a maximal representation ρ there exists a semi continuous ρ-equivariant *orientation-preserving* map $S^1 \to S^1 = \mathbb{RP}^1$, where for sake of simplicity π acts on the left side by identifying $\partial\pi$ with an oriented circle S^1.

Indeed the existence of such a map completely characterizes maximal representations.

Theorem 7.5.3. *Let $\pi \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{R})$ be a homomorphism. Then ρ is maximal if and only if there exists a semi-continuous* orientation-preserving *ρ-equivariant map*

$$\partial\pi = S^1 \xrightarrow{\xi} S^1,$$

where $\partial\pi$ is the boundary of $\partial\pi$ with its natural action by π.

To give an idea how the maximality of the Euler number is related to orientation preserving maps, we have to discuss a little bit the geometric origin of the cocycle c which defined the central extension (7.1.1) which was so essential in our definition of the Euler number.

7.6. **The orientation cocycle.** Let $\mathbb{D} \subset \mathbb{C}$ be the Poincaré disk and consider the boundary $\partial\mathbb{D} = S^1$ with a fixed orientation.

Let us define the *orientation cocycle*

$$S^1 \times S^1 \times S^1 \xrightarrow{\beta} \mathbb{Z},$$

which associates 1 to a positively oriented triple, -1 to a negatively oriented triple and 0 two a triple on non-pairwise distinct points.

Remark 7.6.1.
(1) Note that the $\beta(x,y,z)$ is equal to $\frac{1}{2\pi}$ times the volume of the ideal triangle in $\mathbb{D}$ spanned by x, y and z. Indeed the orientation cocycle, and can be obtained by extending the volume function, which assigns to an ordered triple $(x,y,z) \in \mathbb{D}^3$ the volume of the unique geodesic triangle they determine.
(2) An orientation preserving map from $S^1 \to S^1$ is a map which send every positively oriented triple to a positively oriented triple.

The action of $\mathsf{PSL}(2,\mathbb{R})$ on $\mathbb{D}$ extends continuously to an action by orientation preserving homeomorphism on the boundary $\partial\mathbb{D} = S^1$. Fixing a point $z_0 \in S^1$ the orientation cocycle gives rise to a cocycle

$$\begin{array}{ccc} \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R}) & \xrightarrow{c'} & \mathbb{Z} \\ (g,h) & \longmapsto & \beta(z_0, gz_0, ghz_0), \end{array}$$

which is obviously bounded. This cocycle is essentially the cocycle c corresponding to the central extension in (7.1.1).

Relating these different description of this bounded cocycle c' then allows one to apply techniques from bounded cohomology, for which we refer the reader to in [11, 7, 50] (the techniques nedded for maximal representation are also reviewed in [9] and [8]). This techniques enable us to derive from the maximality of the Euler number that there exists a ρ-equivariant measurable map $S^1 \xrightarrow{\varphi} S^1$ such that for almost all positively oriented triples $(x,y,z) \in S^1 \times S^1 \times S^1$ the following integral formula holds

$$\int_{\pi\backslash\mathsf{PSL}(2,\mathbb{R})} \beta(\varphi(\overline{g}x), \varphi(\overline{g}y), \varphi(\overline{g}z) d\mu(\overline{g}) = 1.$$

Since $\sup_{S^1\times S^1\times S^1}|\beta| = 1$. This implies that the ρ-equivariant orientation preserving map $S^1 \to S^1$ exists "measurably". Then there is still some work to do to promote this to a semi-continuous map.

7.7. **Generalization - Representations of maximal Toledo invariant.** What we described for the Euler number and maximal representation generalizes to representation $\rho : \pi \to G$ when G is a simple Lie group with $\pi_1(G) \cong \mathbb{Z}$, which is the case when G is the isometry group of a Hermitian symmetric space. To learn about the general case we refer the reader to [9] or to [8], a survey about maximal representations $\rho : \pi \to \mathsf{Sp}(2n, \mathbb{R})$ into the symplectic group.

Let us just remark that these general maximal representation are discrete and faithful. Moreover, there is a characterization of maximal representations similar to the characterization in Theorem 7.5.3.

To define a generalization of a positively oriented triples in S^1, the orientation cocycle on S^1 is replace by the *Maslov cocycle* on the *Shilov boundary* $\check{S}$ of the Hermitian symmetric space X associated to G. The Maslov cocycle is a function

$$\check{S} \times \check{S} \times \check{S} \xrightarrow{\beta} \mathbb{Z}$$

which can also be realized as extension of the integral of the Kähler form over geodesic triangles in X. The Maslov cocycle satisfies

$$\sup_{\check{S}^3} |\beta| \leqslant \mathsf{rank}_{\mathbb{R}}(X),$$

where $\mathsf{rank}_R(X)$ denotes the $\mathbb{R}$-rank of X. So that we can call a triple $(x, y, z) \in \check{S}^3$ *maximal* if $\beta(x, y, z) = \mathsf{rank}_{\mathbb{R}}(X)$. When $\check{S} = S^1$ the Maslov cocycle is precisely the orientation cocycle and maximal triples are positively oriented triples. For more details on the Maslov cocycle the reader is refered to [15, 14]

8. Hitchin components and positive or convex representations

In this section we describe the moduli space $\mathcal{T}(\Sigma)$ of complete hyperbolic structures from two other perspectives which relate to the study of Hitchin components and positive representation.

8.1. **Anosov representations and Hitchin components.** The geodesic flow of a hyperbolic metric on a closed surface Σ is Anosov. This Anosov property can be used to describe $\mathcal{T}(\Sigma)$.

Let us assume that π is the fundamental group of a closed surface and $\rho : \pi \to \mathsf{PSL}(2, \mathbb{R})$ be a representation. We choose a hyperbolic metric on Σ_g such that the universal covering $\widetilde{\Sigma_g}$ is identified with $\mathbb{D}$.

The geodesic flow $\tilde{g}_t$ on the unit tangent bundle $T^1\mathbb{D}$ gives rise to a flow $\tilde{g}_t{}^\rho$ on the total space of the flat $\mathbb{R}^2$-bundle $\widetilde{E}^\rho := T^1\mathbb{D} \times \mathbb{R}^2$ over $T^1\mathbb{D}$, where $\tilde{g}_t{}^\rho(u, x) = (\tilde{g}_t u, x)$, commuting with the diagonal π-action given by $\gamma(u, x) := \big(\gamma u, \rho(\gamma)x\big)$. Hence $\tilde{g}_t{}^\rho$ descends to a flow g_t^ρ on the quotient $E^\rho := \Gamma\backslash(T^1\mathbb{D} \times \mathbb{R})$ which is a flat $\mathbb{R}^2$-bundle over the unit tangent bundle $T^1\Sigma$. Note that this is indeed the flat bundle associated to the representation $\pi_1(T^1\Sigma) \to \pi \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{R})$.

The projection

$$p : E^\rho \to T^1\Sigma_g \tag{8.1.1}$$

is then equivariant with respect to the g_t^ρ-action on E^ρ and to the action of the geodesic flow g_t on $T^1\Sigma_g$.

Definition 8.1.1. *The representation $\rho : \Gamma_g \to \mathsf{PSL}(2,\mathbb{R})$ is said to be an* Anosov representation *if there exists a g_t^ρ-invariant continuous splitting of E^ρ into two line bundles $E^\rho = L^+ \oplus L^-$ and constants $A, B > 0$ such that*

$$\|g_t^\rho \xi\| \leqslant e^{-At}\|\xi\| \textit{ for all } \xi \in L^+$$

and

$$\|g_{-t}^\rho \xi\| \leqslant e^{-At}\|\xi\| \textit{ for all } \xi \in L^- ,$$

for all $t > 0$.

If ρ is an Anosov representation we get a a continuous, non-constant, ρ-equivariant curve $\xi : \partial\pi \to S^1$, where $\partial\pi$ is identified with the space of stable (or unstable) leaves of the geodesic flow g_t on $T^1\Sigma_g$.

The relation between Anosov representations and holonomy representations of hyperbolic structures is given by the following fact.

Proposition 8.1.2. *Let $\rho : \pi \to \mathsf{PSL}(2,\mathbb{R})$ be a representation. Suppose that there exists a continuous, non-constant, ρ-equivariant curve $\xi : \partial\pi \to S^1$ then ρ is a discrete and faithful representation.*

Proof. Note that the action of a non-trivial normal subgroup $\Delta \lhd \pi$ on $\partial\pi$ is minimal (that is, every orbit is dense). If $\mathsf{Ker}\rho$ were nontrivial, the curve ξ^1 were constant, which contradicts the hypothesis. Thus ρ is faithful. Moreover since π acts minimally on $(\partial\pi \times \partial\pi) -$ diagonal, ξ is also injective, and $\rho(\gamma)$-is split over $\mathbb{R}$. Proper Lie subgroups of $\mathsf{PSL}(2,\mathbb{R})$ are virtually solvable, so the faithful representation ρ has Zariski dense image. A Zariski dense subgroup of a simple Lie group is either dense or discrete. Since $\rho(\pi)$ does not intersect the open set of elliptic elements nontrivially it is not dense. Hence $\rho(\pi)$ is discrete. □

8.1.1. *Definition of the Hitchin component.* The Hitchin component in $\mathrm{Rep}(\pi, G)$ was defined by Hitchin [49] for all split real Lie groups, that is,. $G = \mathsf{PSL}(n,\mathbb{R})$, $\mathsf{Sp}(2n,\mathbb{R})$, $\mathsf{SO}(n, n+1)$, $\mathsf{SO}(n,n)$ and expectional cases, where he also showed - using Higgs bundle methods - that $\mathrm{Rep}_{Hit}(\pi, G)$ is homeomorphic to a ball of dimension $-\chi(\Sigma)\dim(G)$.

Remember that maximal representations were defined when G is the isometry group of a Hermitian symmetric space The only group for which both, Hitchin components and maximal representations are defined is $\mathsf{Sp}(2n,\mathbb{R})$.

Let us focus on $G = \mathsf{PSL}(n,\mathbb{R})$. Then the Hitchin components $\mathrm{Rep}_{Hit}(\pi, \mathsf{PSL}_n(\mathbb{R})) \subset \mathrm{Rep}(\pi, \mathsf{PSL}_n(\mathbb{R}))$ is the connected component of $\mathrm{Rep}(\pi, \mathsf{PSL}_n(\mathbb{R}))$ containing a representation

$$\rho_{irr} \circ \iota : \pi \to \mathsf{PSL}_n(\mathbb{R}),$$

obtained as the composition of the inclusion $\iota : \pi \to \mathsf{PSL}_2(\mathbb{R})$ as a uniform lattice with the n-dimensional irreducible representation $\rho_{irr} : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_n(\mathbb{R})$.

Introducing the notion of Anosov representations $\rho : \pi \to \mathsf{PSL}(n,\mathbb{R})$ in [57], Labourie showed that all representations in the Hitchin component are discrete, faithful and semisimple.

8.1.2. *Characterization via convex curves.* Combining work of Labourie [57] and Guichard [45], representations in the Hitchin component are characterized by *convex curves* into projective space. A curve

$$\xi : \partial\pi \to \mathbb{RP}^{n-1}$$

is said to be convex [5] if for every n-tuple $(t_1, \dots, t_n)$ of pairwise distinct points $t_i \in \partial\pi$ the lines $\xi(t_i)$ are in direct sum

$$\bigoplus_{i=1}^{n} \xi(t_i) = \mathbb{R}^n.$$

Convex curves $\xi : \partial\pi \to \mathbb{RP}^2$ are precisely injective curves which parametrize the boundary of a strictly convex domain in $\mathbb{RP}^2$.

Theorem 8.1.3 (Labourie [57], Guichard [45])**.** *Let $\rho : \pi \to \mathsf{PSL}_n(\mathbb{R}))$ be a representation. Then ρ is in Hitchin component $\mathrm{Rep}_{Hit}(\pi, \mathsf{PSL}_n(\mathbb{R}))$ if and only if there exists a ρ-equivariant convex curve $\xi : \partial\pi \to \mathbb{RP}^{n-1}$.*

This characterization is very similar to the characterization of maximal representations. The similarity becomes even more apparent if convex curves are related to positive curves into flag varieties, mentioned below.

8.2. **Shear coordinates and positive representations.** Assume for simplicity that we have a surface Σ with exactly one puncture and let us choose a triangulation of Σ with the only vertex being the puncture. Given a complete finite volume hyperbolic structure on Σ we can lift the triangulation to an ideal triangulation of the universal cover $\widetilde{\Sigma} \simeq \mathbb{D}$. To every edge e of the triangulation we can associate now a positive real number. Let $x_1, x_2, x_3, x_4 \in \mathbb{RP}^1$ be the four vertices of the ideal triangles sharing the edge e. We associate to e the exponential of the cross ratio of x_1, x_2, x_3, x_4. This defines *shear coordinates* on $\mathcal{T}(\Sigma)$. Given a positive real number for every edge of our triangulation of Σ, we can construct a hyperbolic structure, by first identifying every triangle with the unique ideal hyperbolic triangle and then gluing any ideal triangles, adjacent in e in such a way that the crossratio of the four endpoints will give rise to the number associate to the edge e.

[5] Note that in [57, 44, 45] convex curves are called hyperconvex curves.

8.2.1. *Positive representations.* Fock and Goncharov generalized this idea of shear coordinates to define the set of *positive representations* of π into any real split simple Lie group G. For $G = \mathsf{PSL}(n, \mathbb{R})$ these generalized shear coordinates can be describe in a nice geometric way.

We refer the reader to [25] for a review of shear coordinates from their point of view, to [26] for a geometric description of the generalized shear coordinates when $G = \mathsf{SL}(3, \mathbb{R})$ and to [24] for the general case.

The Fock-Goncharov coordinates give rise to more structures on the set of positive representations related to cluster algebras, and admit a natural quantization.

8.2.2. *Characterizations via positive curves.* Let us emphasize that parallel to the above discussion for maximal representations and Hitchin components, positive represesentations of Fock and Goncharov can be described similar as the above in terms of "positve maps". Without giving any definition we mention that for example for $G = \mathsf{PSL}(n, \mathbb{R})$ there exists a notion of positive triples of flags in $\mathcal{F}lag(\mathbb{R}^n)$.

Theorem 8.2.1 (Fock-Goncharov [24]). *Let $\rho : \pi \to \mathsf{PSL}(n, \mathbb{R})$ be a representation. Then ρ is a positive representation if and only if there exists a (semi-continuous) curve $\xi : \partial\pi = S^1 \to \mathcal{F}lag(\mathbb{R}^n)$ sending positively oriented triples in S^1 to positive triples of flags. In particular, every positive representation is discrete and faithful.*

9. Remarks

There are many topics, developments, generalizations and questions which we could not touch in this short introduction. You are invited to check the list of references for further topics and to discuss during the workshop ....

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