Homework problem set 2

Submission deadline on 31 October 2022 at noon

Problem 1 (Almost flat world)**.** (i) Verify that under a conformal transformation $\tilde{h} = e^{2f}h$ the Ricci scalar (scalar curvature) of a Riemannian or pseudo-Riemannian manifold (M,h) transforms as

$$\tilde{R} = e^{-2f}R - 2(n-1)e^{-2f}\Delta f - (n-2)(n-1)e^{-2f}h^{-1}(df,df)$$

where n is the dimension of M, Δ is the Laplace-Beltrami operator on scalar functions, and h^{-1} is the induced metric on covectors.

Remember these formulas from General Relativity:

$$\begin{aligned}
\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2}h^{\lambda\kappa}\left(\partial_\mu h_{\nu\kappa} + \partial_\nu h_{\mu\kappa} - \partial_\kappa h_{\mu\nu}\right) \\
R_{\mu\nu} &= \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\mu \Gamma^{\lambda}{}_{\lambda\nu} + \Gamma^{\kappa}{}_{\mu\nu}\Gamma^{\lambda}{}_{\lambda\kappa} - \Gamma^{\kappa}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\kappa} \\
R &= h^{\mu\nu}R_{\mu\nu}\,.
\end{aligned}$$

Express the new $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ in terms of the old $\Gamma^{\lambda}{}_{\mu\nu}$ plus a new term, and then do the same for the Ricci curvature $\tilde{R}_{\mu\nu}$.

Hint: Calculations simplify a lot by working in Riemannian normal coordinates.[1] In these coordinates, the metric components are locally:

$$h_{\mu\nu} = \eta_{\mu\nu} + O(|x|^2)\,,$$

with $\eta_{\mu\nu}$ the local standard form of the metric. Therefore, at the origin of these coordinates (we can center them around $p \in M$), first derivatives of h and the Cristoffel symbols vanish. But be careful: derivatives of the Cristoffel symbols do not necessarily vanish at the origin. Moreover the Laplace-Beltrami operator simplifies to $\Delta f = h^{\mu\nu}\partial_\mu\partial_\nu f$

(ii) Deduce that when $n = 2$, in the neighborhood of any point on M, there exists a function f such that $\tilde{h}$ is flat. *Hint:* In $n = 2$, the Riemann tensor is completely determined by the Ricci scalar.

Problem 2 (Good vibrations)**.** We showed in class (assuming the result of problem 1) that given any classical string solution, there are worldsheet coordinates (τ,σ) in which

$$X(\tau,\sigma) = \frac{1}{2}\big(F(\tau+\sigma) + G(\tau-\sigma)\big)$$

where $F(u)$ and $G(v)$ are functions, valued in D-dimensional Minkowksi space, satisfying $F'^2 = G'^2 = 0$. We consider only closed strings in this problem, so that $X(\tau,\sigma+\sigma_1) = X(\tau,\sigma)$ is periodic in σ.

(i) Explain why, for a solution with a reasonable time-evolution, one may assume $X^0(\tau,\sigma) = \tau$ (i.e., flat fiducial gauge and static gauge are compatible). Express the general solution in terms of two periodic functions $f(u)$ and $g(v)$ valued in the $(D-2)$-dimensional unit sphere, and appropriate integration constants.

[1] If you're unfamilliar with this concept, you might want to consult a relativity textbook, e.g. Misner, Thorne, Wheeler §11.6, or Bartelman (https://heiup.uni-heidelberg.de/catalog/book/534) Chapter 3.2.2.

(ii) Write down the time evolution of a string which, at time $X^0 = 0$, forms a circle of radius R at rest in the X^1-X^2-plane. Calculate the mass of this solution.

(iii) Show that, for a generic solution in $D = 4$, there are points u_*, v_* in parameter space for which $f(u_*) = g(v_*)$. Show that around such points, the trace of the string in spacetime forms a cusp singularity moving (instantaneously) at the speed of light. *Hint:* An (ordinary) cusp in the x-y-plane can be defined (locally) as the set of solutions of the equation $x^3 = y^2$. You will find the cusp in parametrized form by expanding f and g around the singular point.