Homework problem set 1

Submission deadline on 24 October 2022 at noon

Aufgabe 1 (Little things)**.**

(a) We denote by $\mathscr{L} = SO^+(1,d)$ the (proper, orthochronous, $d+1$-dimensional) Lorentz group and for $p = (p_\mu) \in \mathbb{R}^{1,d}$ by

$$W_p = \{\Lambda \in \mathscr{L} | \Lambda(p) = p\}$$

the stabilizer of p.

Show that W_p is constant (as an abstract group) on the orbit of p under $\mathscr{L}$, which comes in one of four types, depending on the invariant mass $m^2 = p^2 = p_\mu p^\mu$:

1. $m^2 > 0$ (time-like hyperboloid / massive particle)
2. $m^2 = 0, p \neq 0$: (light-cone / massless particle)
3. $m^2 = p = 0$ (origin / vacuum)
4. $m^2 < 0$ (space-like hyperboloid / tachyonic particle)

and determine W_p (then called Wigner's Little Group) in each of these four cases.
Hint: Bring p in a standard form by using the action of $\mathscr{L}$. Do not worry (too much) about global issues. The second is the only tricky case.

(b) Wigner introduced W_p for the study of the *irreducible unitary representations* of the Poincaré group $\mathscr{P}$, the affine version of $\mathscr{L}$. Recall from your elementary courses that $\mathscr{P} = \mathbb{R}^{1,d} \rtimes \mathscr{L}$ is the semi-direct product of $\mathscr{L}$ by the translations, i.e., an element $(a,\Lambda) \in \mathscr{P}$ acts on $x \in \mathbb{R}^{1,d}$ by $x \mapsto \Lambda(x) + a$. Then, if $(\mathscr{H}, U)$ is a Hilbert space carrying a unitary representation of the Poincaré group, we may diagonalize translations and decompose (formally)

$$\mathscr{H} = \bigoplus_p \mathscr{H}_p$$

into joint eigenspaces. In other words, $U(a)|\psi\rangle = e^{ia\cdot p}|\psi\rangle$ for all $|\psi\rangle \in \mathscr{H}_p$.

Show that $\mathscr{H}_p$ carries a natural representation of W_p and that if $\mathscr{H}$ is irreducible, then so is $\mathscr{H}_p$. It is a fundamental fact that only finite-dimensional possibilities are realized by Nature.
Hint: In order to show that $\mathscr{H}$ carries a representation of W_p you need to show that $U((0,\Lambda))|\psi\rangle \in \mathscr{H}_p$. For the irreducibility first check that there is a $m^2 \in \mathbb{R}$ such that for all $\mathscr{H}_p$ either $p^2 = m^2$ or $\mathscr{H}_p = 0$.

Aufgabe 2 (Upside down)**.** We consider a system of two uncoupled equal-period harmonic oscillators, with familiar creation/annihilation operators $a, a^\dagger$ and $b, b^\dagger$, and Hamiltonian

$$H_+ = a^\dagger a + b^\dagger b + const.$$

(a) Show that real and imaginary parts of $J = a^\dagger b$ are symmetries of the system and give them a physical interpretation, for instance by rewriting them in terms of positions and momenta. Also review the spectrum of H_+, including the degeneracy of states.

(b) Replace H_+ with

$$H_- = a^\dagger a - b^\dagger b + const.,$$

discuss its symmetries, and show that its spectrum is infinitely degenerate and unbounded below. Why is that a problem?

(c) By switching (the role of) b and $b^\dagger$, one may restore a unique ground state for H_- and recover a more conventional spectrum. Why is this not a good idea?