Lecture 5 String observables

Note:
- As we'll be able to verify more precisely in a bit, the constraints

$$(\dot{X} \pm X')^2 = 0$$

that we impose as condition of "physical string motion" on the 2-d wave equation $\partial_+ \partial_- X_\mu = 0$, amount to setting to zero the infinite number of charges that are conserved (on eoms) as a consequence of "conformal invariance". In physics parlance, this is often referred to as "gauging conformal symmetry".
- In N-G string, of the constraints arise as gauge conditions. Their conservation is more a matter of their consistency.
- The gist of Polyakov's insights is threefold.
  - conformal invariance means invariance of "action for "gauged fixed action for X_μ" under Conformal Killing transformations
  - instead of merely "gauging conformal symmetry", we could couple X to 2-d auxiliary metric ("X-matter") whence all of Diff ⋉ Weyl is gauge symmetry.
  - We could elevate "gauging conformal symmetry of an arbitrary (not necessarily linear) conformally invariant matter theory to an "axiomatic definition of perturbative (free) string theory". (Interactions are not far away.)

Next steps: Ready the string as a canonical dynamical system for quantization, by bringing to the front the "algebra of observables", which is the stuff we have quantization prescription for, and in particular: Poincaré & conformal charges.

(8) The point being, we already know the most general solution of eom

$$\partial_+\partial_- X^\mu = 0$$

can be written as

$$(*) \quad X^\mu(\tau,\sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

$$\sigma^\pm = \tau \pm \sigma$$

so describing convenient set of observables is really all there is to do.

$$\partial_\tau = \partial_+ + \partial_-$$

worldsheet light-cone coordinates.

$$\partial_\sigma = \partial_+ - \partial_-$$

Note: We can almost invert (*) (up to constants)

$$X^{\mu\,\prime}_R(\sigma^-) = \frac{1}{2}(\dot{X} - X')$$

$$X'_L = \frac{1}{2}(\dot{X} + X')$$

1) Make (infinite-dimensional) dynamical system by writing

$$\Sigma = I \times K$$

where $I = \mathbb{R}_{\partial_\tau}$ "worldsheet time"

$K =$ compact 1-d mf.

↳ finite energy

with appropriate boundary conditions:

· $K = S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni \sigma$ (closed string)

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$$

$$\rightsquigarrow \quad X_L'(\sigma^+ + 2\pi) = X_L'(\sigma^+)$$

$$X_R'(\sigma^- + 2\pi) = X_R'(\sigma^-)$$

(even though, N.B. $\sigma^\pm$ are not by themselves periodic variables)

· $K = [0, \pi]$ (open string)

$$X'(\tau, 0) = X'(\tau, \pi) = 0$$

$$\rightsquigarrow \quad X_L'(\sigma^+) = X_R'(\sigma^- = \sigma^+)$$

$$X_L'(\sigma^+ + 2\pi) = X_L'(\sigma^+).$$

2. In principle, observables of system with 2^{nd} time derivatives are position $X(\sigma)$ and $\dot{X}(\sigma)$ velocity which in canonical formalism, we eliminate in favor of canonical momentum

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}^\mu \qquad T = \frac{1}{2\pi\alpha'}$$

$$\mathcal{L} = -\frac{T}{2}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

$$= \frac{T}{2}\left(\dot{X}^2 - X'^2\right)$$

Then, time evolution of $X(\sigma)$, $\Pi(\sigma)$ is governed by Hamilton equation

$$\frac{d}{dt}(\ ) = \{H, \cdot\}$$

with Poisson bracket

$$\{\Pi_\mu(\sigma), X^\nu(\sigma')\} = \delta_\mu^\nu\, \delta(\sigma-\sigma').$$

and Hamiltonian (worldsheet!)

$$H = \frac{1}{2}\int_0^{2\pi} d\sigma' \left(\frac{\Pi^2}{T} + TX'^2\right)$$

— [motivates looking for sols in $L^2(K)$]

$$\therefore \quad \{H, X^\mu(\sigma)\} = \int d\sigma' \frac{\Pi(\sigma')^\mu}{T}\delta(\sigma-\sigma') = \dot{X}^\mu$$

$$\{\Pi_\mu(\sigma), X'^\nu(\sigma')\} = -\delta_\mu^\nu \delta'(\sigma-\sigma')$$

$$\therefore \quad \{H, \Pi_\mu(\sigma)\} = \int d\sigma' \, TX'_\mu(\sigma')\left(-\delta'(\sigma'-\sigma)\right) = TX''_\mu = \dot{\Pi}_\mu$$

3) Observe that spacetime D-momentum (closed string),

$$p_\mu = \int_0^{2\pi} \Pi_\mu(\sigma)\, d\sigma$$

and "average" or "centre-of-mass" position

$$x^\mu = \frac{1}{2\pi} \int_0^{2\pi} X^\mu(\sigma)\, d\sigma$$

satisfy
$$\{ p_\mu, x^\nu \} = \delta^\nu_\mu \qquad (!)$$

to motivate (if need be) decomposition into Fourier modes, conventionally defined for ~~left~~ right/left movers

$$X_R'(\sigma)^\mu = \frac{1}{2}\left(\frac{\Pi}{T} - X' \right)^\mu$$

$$X_L'(\sigma)^\mu = \frac{1}{2}\left(\frac{\Pi}{T} + X' \right)^\mu$$

via:

$$\alpha_n^\mu = \sqrt{\frac{2}{\alpha'}}\, \frac{1}{2\pi} \int_0^{2\pi} e^{-in\sigma}\, X_R'^\mu(\sigma)\, d\sigma$$

$$\tilde{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}}\, \frac{1}{2\pi} \int_0^{2\pi} e^{in\sigma}\, X_L'^\mu(\sigma)\, d\sigma$$

– complex valued functions on phase space, satisfying $\overline{\alpha_n} = \alpha_{-n}$, $\overline{\tilde{\alpha}_n} = \tilde{\alpha}_{-n}$, $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu = \tilde{\alpha}_0^\mu$ (NB. X is periodic, $X'_{R,L}$ we don't know) whose physical interpretation can be understood by working out Poisson brackets

$$\{X_R'(\sigma), X_R'(\sigma')\} = \frac{1}{4}\left\{\frac{\Pi(\sigma)}{T} - X'(\sigma), \frac{\Pi(\sigma')}{T} - X'(\sigma')\right\}$$

$$= \frac{1}{4}\frac{1}{T}\left(\delta'(\sigma-\sigma') + \delta'(\sigma-\sigma')\right) = \pi\alpha'\delta'(\sigma-\sigma')$$

recall: $\delta'(-\sigma) = -\delta'(\sigma)$.

$$\{X_L'(\sigma), X_L'(\sigma')\} = -\pi\alpha'\delta'(\sigma-\sigma') \quad ; \quad \{X_L', X_R'\} = 0$$

namely $\{\alpha_n, \tilde{\alpha}_m\} = 0$

$$\{\alpha_n^\mu, \alpha_m^\nu\} = \frac{2}{\alpha'}\frac{1}{4\pi^2}\pi\alpha'\int_0^{2\pi} e^{-in\sigma}e^{-im\sigma'}\delta'(\sigma-\sigma')d\sigma d\sigma'$$

$$= \frac{1}{2\pi}\int_0^{2\pi} in\, e^{-i(n+m)\sigma'}d\sigma'$$

$$= in\,\delta_{n+m}\,\eta^{\mu\nu}$$

$$\{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu\} = in\,\delta_{n+m}\,\eta^{\mu\nu}$$

and, after inverting

$$X_R'(\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} e^{in\sigma}\alpha_n$$

$$X_L'(\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} e^{-in\sigma}\tilde{\alpha}_n$$

$$\Pi = T(X_L' + X_R')$$

$$X' = X_L' - X_R'$$

the Hamiltonian $\qquad T = \frac{1}{2\pi\alpha'}$

$$H = \frac{1}{2}\int_0^{2\pi}\left(\frac{\Pi^2}{T} + TX'^2\right)d\sigma \qquad \alpha_0 = \sqrt{\frac{\alpha'}{2}}\,p$$

$$= T\int_0^{2\pi}\left(X_L'^2 + X_R'^2\right)d\sigma$$

$$= T\cdot\frac{\alpha'}{2}\cdot 2\pi\cdot\sum_{n\in\mathbb{Z}}\left(\alpha_n\alpha_{-n} + \tilde{\alpha}_n\tilde{\alpha}_{-n}\right)$$

$$= \frac{1}{2}\sum\left(\alpha_n\alpha_{-n} + \tilde{\alpha}_n\tilde{\alpha}_{-n}\right)$$

$$= \frac{1}{2}\alpha' p^2 + \sum_{n>0}\left(|\alpha_n|^2 + |\tilde{\alpha}_n|^2\right)$$

<u>Note:</u> · Despite appearances,

$$|\alpha_n|^2 = \alpha_n^\mu\bar{\alpha}_{n\mu} = -|\alpha_n^0|^2 + |\alpha_n^1|^2 + \dots + |\alpha_n^d|^2$$

is not positive definite, and so neither is H. This elephant in the room will be dealt with in due course.

· $[X] = L = [\sqrt{\alpha'}]$, $\quad [p] = \frac{1}{L}$, $[\alpha_n] = [H] = 1$.

· we restrict phase space to solutions with finite energy, i.e. $(\alpha_n), (\tilde{\alpha}_n) \in \ell^2(\mathbb{N})$ etc.

(4) After recovering the time evolution from $\dot{X}'_R = -X''_R$, $\dot{X}'_L = X''_L$, Poisson bracket with H, or any other way. $\dot{\alpha}_n = -in\alpha_n$ $\dot{\tilde{\alpha}}_n = -in\tilde{\alpha}_n$, and reinserting integration constant, x^μ, the mode expansion reads in full glory

$$X(\tau,\sigma) = X_L(\sigma^+) + X_R(\sigma^-)$$

$$X_L^\mu = \frac{1}{2}x^\mu + \left\{ \begin{array}{c} \frac{\alpha'}{2}p^\mu\sigma^+ \\ \parallel \\ \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu\sigma^+ \end{array} \right\} + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma^+}$$

$$X_R^\mu = \frac{1}{2}x^\mu + \left\{ \begin{array}{c} \frac{\alpha'}{2}p^\mu\sigma^- \\ \parallel \\ \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu\sigma^- \end{array} \right\} + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^-}$$

—

For the open string, we can think $X'_R = X'_L$, which means $\tilde{\alpha}_n = \alpha_n$, $\bar{\alpha}_n = \alpha_{-n}$. same algebra.

$$X(\tau,\sigma) = x^\mu + 2\alpha' p\tau + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-in\tau} \cdot 2\cos(n\sigma)$$

$$\uparrow$$

$$\alpha_0 = \sqrt{2\alpha'}\, p \quad !$$

and

$$H = T\int_0^\pi \left(X_L'^2 + X_R'^2\right) d\sigma = T\int_0^{2\pi} X_L'^2\, d\sigma$$

$$= \frac{1}{2}\sum \alpha_{-n}\alpha_n = \alpha' p^2 + \sum_{n>0} |\alpha_n|^2$$

(5) With the mode expansion at hand, it is easy to write out the conserved charges corresponding to Lorentz symmetry

$$J^{\mu\nu} = \int_0^{2\pi} \left(\Pi^\mu X^\nu - \Pi^\nu X^\mu\right) d\sigma$$

$$= \frac{1}{2\pi\alpha'} \int_0^{2\pi} \left[\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\tilde{\alpha}^\mu_n e^{-in\sigma} + \alpha^\mu_n e^{in\sigma}\right)\right] \cdot \left[x^\nu + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n\neq 0} \left(\frac{\tilde{\alpha}^\nu_n}{n} e^{-in\sigma} + \frac{\alpha^\nu_n}{n} e^{in\sigma}\right)\right] d\sigma$$

$$- (\mu \leftrightarrow \nu)$$

$$= p^\mu x^\nu - p^\nu x^\mu + \sum_{n\neq 0} \frac{i}{2n} \left(\tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n + \tilde{\alpha}^\mu_n \alpha^\nu_n + \alpha^\mu_n \tilde{\alpha}^\nu_n + \alpha^\mu_{-n} \alpha^\nu_n \right)$$

$$- (\mu \leftrightarrow \nu)$$

$$= p^\mu x^\nu - p^\nu x^\mu + i \sum_{n>0} \frac{\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n + \alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n}$$

and to verify Poisson brackets:

$$\{J^{\mu\nu}, x^\rho\} = \eta^{\mu\rho} x^\nu - \eta^{\nu\rho} x^\mu$$

$$\{J^{\mu\nu}, p^\rho\} = \eta^{\mu\rho} p^\nu - \eta^{\nu\rho} p^\mu$$

$$\{J^{\mu\nu}, \alpha^\rho_n\} = \eta^{\mu\rho} \alpha^\nu_n - \eta^{\nu\rho} \alpha^\mu_n$$

etc.

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = \ldots$$

[$\tilde{\alpha}$ oscillators omitted for open strings]

(6) Finally, we can evaluate the charges associated with "conformal invariance" of "X-matter".

Quick reminder: generated by conformal Killing fields ξ^α satisfying $\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \sim \eta_{\alpha\beta}$, leading to

$J_\xi^\alpha = T^\alpha{}_\beta \xi^\beta$ being conserved in a Weyl (scale)-invariant theory.

$$\because \quad \partial_\alpha T^\alpha{}_\beta \xi^\beta = \underbrace{(\partial_\alpha T^\alpha{}_\beta)}_{=0} \xi^\beta + T^\alpha{}_\beta \partial_\alpha \xi^\beta \sim T^{\alpha\beta} \eta_{\alpha\beta} = 0$$

Using 2-d worldsheet lightcone coordinates $\eta_{-+} = \frac{1}{2}$ $\eta_{++} = 0$

$$\mathcal{L} = 2T\, \partial_+ X \partial_- X = \frac{T}{2} (\partial_\tau X)^2 - (\partial_\sigma X)^2$$

$$T^+{}_+ = 0 \qquad T^-{}_- = 0$$

$$T^+{}_- = 2T\, \partial_- X \partial_- X \qquad T_{--} = T \partial_- X \partial_- X$$

$$T^-{}_+ = 2T\, \partial_+ X \partial_+ X \qquad T_{++} = T \partial_+ X \partial_+ X$$

$$\partial_- \xi^+ = 0, \quad \partial_+ \xi^- = 0.$$

$\Rightarrow \xi^\pm$ arbitrary functions of $\sigma^\pm$, respectively.

Peculiarity: $\xi^\pm$ depend on τ, but still "global", non-localizable symmetry.

As vector fields

$$J_\xi = \left(2T\, \partial_- X \partial_- X \xi^-, \quad 2T\, \partial_+ X \partial_+ X \xi^+ \right)$$

as one-forms

$$J_\xi = T\, \partial_+ X \partial_+ X \xi^+ d\sigma^+ + T\, \partial_- X \partial_- X \xi^- d\sigma^-$$

Conserved charge:

$$Q = \int_K J^0_\xi \, d\sigma = \frac{1}{2} \int_K \left(J^+_\xi + J^-_\xi \right)$$

$$\xi^- = e^{in\sigma^-} \qquad = T \int_0^{2\pi} \left(X_R'^2 \xi^- + X_L'^2 \xi^+ \right) d\sigma$$

- Choosing $\xi^- = e^{in\sigma^-} = e^{in\tau} e^{-in\sigma}$

we pick from $\left(X_L = \sqrt{\frac{\alpha'}{2}} \sum \tilde{\alpha}_p e^{ip\sigma} \right)^2$ the piece where $p+q=n$.

$$\tilde{Q}_n = \frac{1}{2} \sum \tilde{\alpha}_p \tilde{\alpha}_{n-p} \, e^{in\tau}$$

- Similarly, choosing $\xi^+ = e^{in\sigma^+} = e^{in\tau} e^{in\sigma}$ we find

$$Q_n = \frac{1}{2} \sum \alpha_p \alpha_{n-p} \, e^{in\tau}$$

(open string: only one set of charges).

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\, p$$

NB . The charges Q_n, $\tilde{Q}_n$ are conserved on account of

$$\dot{\alpha}_n = -in\alpha_n$$

but only if we include $e^{in\tau}$. Traditionally, one also addresses respectfully the "Virasoro generators"

$$L_n = \frac{1}{2}\sum \alpha_p \alpha_{n-p} \qquad \tilde{L}_n = \frac{1}{2}\sum \tilde{\alpha}_p \tilde{\alpha}_{n-p}$$

as conserved charges, event though

$$\dot{L}_n = -in L_n \qquad \dot{\tilde{L}}_n = -in\tilde{L}_n$$

~~They are important, of course~~ In other words, their conservation follows from that of α_n. Their prominence is due to the fact that their vanishing determines "physical string motion" as explained at the beginning.

To prepare our next big task, which is the implementation of all these constraints, and at the quantum level, I want to share what is perhaps the single most important formula of perturbative string theory, resulting from

$$L_0 = \tilde{L}_0 = 0 \qquad\qquad H = L_0 + \tilde{L}_0 \quad \text{closed}$$

Closed string $\quad L_0 = \frac{1}{4}\alpha' p^2 + \sum_{n>0} |\alpha_n|^2$

$$\tilde{L}_0 = \frac{1}{4}\alpha' p^2 + \sum_{n>0} |\tilde{\alpha}_n|^2$$

$$L_0 = 0 \qquad p^2 = -M^2$$

$$\Rightarrow \quad M^2 = \frac{4}{\alpha'}\sum_{n>0}|\alpha_n|^2 = \frac{4}{\alpha'}\sum_{n>0}|\tilde{\alpha}_n|^2$$

Open string $L_0 = \alpha' p^2 + \sum_{n>0} |\alpha_n|^2$

$$M^2 = \frac{1}{\alpha'} \sum |\tilde{\alpha}_n|^2$$

where again, this is finite (by our assumptions) but in fact not manifestly positive definite (despite appearances).

Also, we ought to work out the algebra satisfied by L_n under Poisson brackets (one set of oscillators)

Using $\{\alpha_n, \alpha_m\} = in\,\delta_{n+m}$
and derivative property of Poisson bracket, we find first of all.

$$\{L_n, \alpha_m\} = \{\tfrac{1}{2}\sum \alpha_p \alpha_{n-p}, \alpha_m\}$$

$$= \frac{1}{2}\sum_p \left(\delta_{p+m}\, ip\, \alpha_{n-p} + \alpha_p\, i(n-p)\, \delta_{n-p+m}\right)$$

$$= \frac{1}{2}\left(i(-m)\alpha_{n+m} + i(-m)\,\alpha_{n+m}\right)$$

$$= -im\alpha_{n+m}.$$

(in all these calculations $\eta^{\mu\nu}$ is omitted or implicit)

$$\{L_n, L_m\} = \frac{1}{2}\sum_p \{L_n, \alpha_p \alpha_{m-p}\}$$

$$= \frac{1}{2}\sum_p \left(-ip\,\alpha_{n+p}\alpha_{m-p} - i(m-p)\,\alpha_p\,\alpha_{n+m-p}\right)$$

$$\circledast \quad = \frac{1}{2}\sum_p \left(-i(m-p)\,\alpha_p\alpha_{n+m-p} + \sum_p (-ip\,\alpha_{n+p}\alpha_{m-p}\right)$$

$$\circledast \qquad\qquad p = p-n$$

$$= \frac{1}{2}\sum_p -i(m-p+p-n)\;\alpha_p\alpha_{n+m-p}$$

$$= -i(m-n)\;L_{n+m}$$

$$= i(n-m)\,L_{n+m} \qquad \text{(Witt algebra)}$$

- It's worthwhile pointing out that the splitting and combining of the infinite sums in $\circledast$ is <u>allowed</u> on account of their absolute convergence.

(This analogue assumption in the quantum case is invalid.) to which we now turn.

Lecture 6 Quantization & Virasoro anomaly

Before embarking on the implementation of the constraints at the classical and quantum level, I want (as another remedial) zoom in on one of the feature calculations, the fate of the Witt algebra associated with an infinite set of harmonic oscillators under quantization.

To be sure: We're interested in the Poisson algebra

$$\{p, x\} = 1$$

$$\{\alpha_n, \alpha_m\} = in\,\delta_{n+m,0} \qquad \bar{\alpha}_n = \alpha_{-n}$$

+ possibly $\qquad \{\tilde{\alpha}_n, \tilde{\alpha}_m\} = in\,\delta_{n+m,0}$

$$p = \sqrt{\frac{2}{\alpha'}}\,\alpha_0 \text{ closed} \qquad p = \sqrt{\frac{1}{2\alpha'}}\,\alpha_0 \text{ open}$$

parametrizing motion and oscillations of string in one (spatial!) direction before imposing the constraints.

Canonical quantization: , replace Poisson bracket with $i[\,,\,]$ and represent resulting Lie algebra on a Hilbert space. Spectrum of (Hermitian) self-adjoint operators gives possible results for observables under measurement.

- in general, any observable that is "non-linear" in canonical variables suffers from "ordering ambiguities"

- For example: harmonic oscillator

$$\{a, \bar{a}\} = i\omega \qquad a \sim \sqrt{\frac{\omega}{2}}\,(p - ix)$$

$$[a, a^\dagger] = \omega > 0 \qquad a^\dagger = \sqrt{\frac{\omega}{2}}\,(p + ix)$$

is represented on $\mathcal{H} = \mathrm{Span}\left(|\lambda\rangle \sim (a^\dagger)^\lambda |0\rangle,\ \lambda = 0, 1, \dots\right)$

where $a|0\rangle = 0 \qquad \langle 0|0\rangle = 1$

but $H = \omega a^\dagger a$ is only well-defined "up to an additive constant", traditionally though in honour of Max Planck taken to be $\frac{\omega}{2}$.

- free particle

$$\{p, x\} = 1$$

$$[p, x] = -i$$

represented on $\mathcal{H} = L^2(\mathbb{R}) \qquad p = -i\partial_x \qquad x = x.$

pretend plane wave eigenbasis $|p\rangle \qquad p \in \mathbb{R}$

$$\langle p|p'\rangle = \delta(p - p') \qquad \text{etc.}$$

$$\langle x|p\rangle = e^{ipx}.$$

Now, the algebra $\{\alpha_n, \alpha_m\} = i n \delta_{n+m}$

$$\rightarrow \quad [\alpha_n, \alpha_m] = n\,\delta_{n+m}$$

is simply infinite collection of harmonic oscillators of frequency n, provided we identify

$$\alpha_n = \begin{cases} \text{creation operators} & n < 0 \\ \text{annihilation operators} & n > 0 \end{cases}$$

(conjugate variable of α_0 is related to x).
$[p, x] = -i$

The standard representation of this algebra takes

$$\mathcal{H} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{oscillators}}$$

where $\mathcal{H}_{\text{com}} = L^2(\mathbb{R})$ while

$\mathcal{H}_{\text{oscillators}} = \mathcal{H}_{\text{Fock}}$, which in practice we take to mean

$$\mathcal{H} = \text{Span}_{\mathbb{C}} \left\{ \prod_{k>0} (\alpha_{-k})^{\lambda_k} |p\rangle, \quad p \in \mathbb{R} \quad (\lambda_k) \in \{0, 1, 2, \ldots\} \text{ only } \underline{\text{finitely many}} \text{ non-zero.} \right\}$$

$p|p\rangle = p|p\rangle. \quad \alpha_k |p\rangle = 0 \quad k = 1, 2, \ldots$

Relevant question: What happens to $L_n = \frac{1}{2}\sum \alpha_p \alpha_{n-p}$ under quantisation?

Central observation: something non-trivial.

· At face value, L_n are ∞ sums. Not clear well-defined.

Claim: On any state in Hilbert space basis of $\mathcal{H}$,

L_n for $n \neq 0$ is actually a finite sum.

Pf: $p \to -\infty$: α_{n-p} ~~is annih~~ annihilate state

$p \to +\infty$ α_p " "

and α_p, α_{n-p} commute

On the other hand: $\sum_{p \in \mathbb{Z}} \alpha_p \alpha_{-p}$ is not well-defined

b/c e.g. $\cancel{\alpha_p \alpha_{-p}}$ to $a a^\dagger |0\rangle = |0\rangle$

Rescue: observe ordering ambiguity and define

$$\boxed{L_0 = \sum_{p>0} \alpha_{-p} \alpha_p + \frac{1}{2}\alpha_0^2}$$

"Locally finite ~~ave~~ sum" and well-defined.

$\boxed{\text{"normal ordering"}}$

Now: $$[L_n, \alpha_m] = \frac{1}{2} \sum [\alpha_p \alpha_{n-p}, \alpha_m]$$

$$= -m\, \alpha_{n+m}$$

$$\left([L_0, \alpha_m] = -m\, \alpha_m \quad \text{also} \right)$$

while:

$$[L_n, L_m] = \frac{1}{2} \sum_p [L_n, \alpha_p \alpha_{m-p}]$$

$$= \frac{1}{2} \sum_p \left(-p\, \alpha_{n+p} \alpha_{m-p} + (p-m)\, \alpha_p \alpha_{n+m-p} \right)$$

At this stage, by definition well-defined.

For $n+m \neq 0$, the pairs in each summand commute so sum is 2 ~~sums~~ locally finite sums.

shifting $p \to p-n$ in first sum gives

$n \neq m$: $$[L_n, L_m] = \frac{1}{2} (n-m) L_{n+m}.$$

For $m=-n$, we have to be more careful. Say $n>0$.

$$[L_n, L_{-n}] = \frac{1}{2}\sum_p \left(-p\,\alpha_{n+p}\,\alpha_{-n-p} + (p+n)\,\alpha_p\,\alpha_{-p}\right)$$

For $p\to-\infty$ this is locally finite, for $p\to\infty$ also, but in a non-trivial way:

$$-p\,\alpha_{n+p}\,\alpha_{-n-p} + (p+n)\,\alpha_p\alpha_{-p}$$

$$= \underbrace{(-p)(n+p) + (p+n)p}_{0} - p\,\alpha_{-n-p}\,\alpha_{n+p} + (p+n)\,\alpha_{-p}\,\alpha_p$$

→ split: into three locally finite sums

$$\sum_p = \sum_{p<-n} + \sum_{p=-n}^{-1} + \sum_{p\geq 0}$$

first: ~~locally~~ normal ordered

$$\sum_{p<-n} = \sum_{p=n+1}^{\infty} p\,\alpha_{n-p}\,\alpha_{p-n} + (n-p)\,\alpha_{-p}\,\alpha_p$$

$$= \sum_{p=1}^{\infty}(n+p)\,\alpha_{-p}\,\alpha_p + \sum_{p=n+1}(n-p)\,\alpha_{-p}\,\alpha_p$$

third: first normal order, then split into two locally finite sums:

$$\sum_{p=0}^{\infty} = \sum_{p=0}^{\infty} -p\, \alpha_{-n-p}\alpha_{n+p} + (n+p)\, \alpha_{-p}\alpha_{p}$$

$$= \sum_{p=n}^{\infty} +(n-p)\,\alpha_{-p}\alpha_{p} + \sum_{p=0}^{\infty}(n+p)\,\alpha_{-p}\alpha_{p}$$

second term is already finite, we can manipulate at will

$$\sum_{p=-n}^{-1} = \sum_{p=1}^{n} \underbrace{p\,\alpha_{n-p}\alpha_{p-n}}_{p(n-p)\, +\, p\,\alpha_{p-n}\alpha_{n-p}} + (n-p)\,\underbrace{\alpha_{-p}\alpha_{p}}_{\text{normal ordered}}$$

$$= \sum_{p=1}^{n} p(n-p) + \sum_{p=0}^{n-1}(n-p)\,\alpha_{-p}\alpha_{p} + \sum_{p=1}^{n}(n-p)\,\alpha_{-p}\alpha_{p}$$

All together:

$$[L_n, L_{-n}] = \frac{1}{2}\Bigg[\sum_{p=0}^{\infty}\big((n+p)+(n-p)\big)\,\alpha_{-p}\alpha_{p}$$

$$+ \sum_{p=1}^{\infty}\big((n+p)+(n-p)\big)\,\alpha_{-p}\alpha_{p} + \frac{1}{2}\sum_{p=1}^{n} p(n-p)\Bigg]$$

$$= 2n\,L_0 + \frac{1}{2}\left(\frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2n\,L_0 + \frac{1}{12}\Big(\underbrace{3n^3+3n^2-2n^3-3n^2-n}_{(n^3-n)}\Big)$$

· The infinite-dimensional Lie-algebra

$$\mathcal{V} = \mathrm{Span}_{\mathbb{C}} \{ L_n \quad n \in \mathbb{Z}, \quad c \}$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{1}{12} (n^3 - n) \cdot c \ \delta_{n+m}$$

$$[L_n, c] = 0 \qquad \forall n.$$

is called Virasoro-algebra (with "central charge c")
(formally just a basis element of $\mathcal{V}$)

We have found: The Fock space of ∞ # of oscillators on fixed momentum eigenvector $|p\rangle$ carries natural representation of $\mathcal{V}$ with central charge $c|p\rangle = 1$.

Facts: · $\mathcal{V}$ is a "central extension" of the Witt-algebra, itself Lie algebra of diffeomorphisms of circle.

- up to modifications equivalent to shifts $L_n \to L_n + c$ (and in particular $L_0 \to L_0 + a$) this is the only such extension.

· The central extension is really n^3 term; requiring (for example) that L_0, L_1, L_{-1} form subalgebra (isomorphic to $SL(2,\mathbb{C})$) fixes "ground state energy" / normal ordering constant.

- $\mathcal{W}$ has an interesting representation theory that is of central importance for string theory.

Preview: • At this point, it seems obvious how to proceed:

"Take $\mathcal{H}_{\text{string}} = \bigotimes_{\mu=0}^{D} \mathcal{H}$, under the constraint that

$$L_n = \tilde{L}_n = 0 \quad \forall n \text{"}.$$

Issue: time-like oscillators have

$$[\alpha_n^0, \alpha_m^0] = -n\delta_{n+m}$$

"wrong sign". Using α_n^0 for $n>0$ as annihilation operators, this leads to states with "negative norm" (so-called ghosts).

$$\| \alpha_{-1}^0 |p\rangle \|^2 = \langle p| \alpha_1^0 \alpha_{-1}^0 |p\rangle$$

$$= -\langle p|p\rangle.$$

Using α_n^0 for $n<0$ as annihilation operators leads to unbounded below Hamiltonian (this is naively what happens in classical theory).

$\rightsquigarrow$ ghosts in spacetime.

In either case, somehow, imposing $L_n = 0$ ought to resolve the issue.

Intuition: there are as many constraints as bad oscillators.

Three standard avenues

1. Solve the constraints classically, only quantize physical degrees of freedom.
(Light-cone gauge quantization)

2. Implement constraints after quantization, but only "weakly" $L_n |\psi\rangle = 0 \quad n \geq 0$.

3. BRST.

Either way, the main lesson: This works best/only for $D = 26$.