Lecture 3 (The Polyakov action, ~~symmetries~~ × Weyl invariance)

- Just like the proper-time action for the point particle, the squareroot in the Nambu-Goto action is awkward and makes it inconvenient for path-integral quantization and the covariant treatment of interactions.

- Now instead of kicking-in the door, we can motivate a better variational principle (and thus, path-integral), by observing that the eoms in the ~~framework~~ a conformal gauge for the induced metric

$$\gamma_{\alpha\beta} = \partial_\alpha X \partial_\beta X = \sqrt{-\gamma}\,\eta_{\alpha\beta}$$

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X_\mu = 0$$

are linear in X (so all non-linearity is in constraints) and the same as those of a simple free field theory

$$S = \frac{1}{2}\int d^2\sigma\; \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu ,$$

while the constraints ~~can~~ b when rewritten via

$$\eta^{\alpha\beta}\partial_\alpha X \partial_\beta X = 2\sqrt{-\gamma} , \quad \text{namely}$$

$$\partial_\alpha X \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\,\eta^{\delta\varepsilon}\partial_\delta X \partial_\varepsilon X = 0$$

(to go back:

$$\det \partial_\alpha \partial_\gamma X = -\frac{1}{4} \eta^{\delta\varepsilon} (\partial_\delta X \partial_\varepsilon X)^2 \quad)$$

can be recognized as the vanishing of the canonical energy-momentum tensor of this 2-d field theory.

$$T^\alpha{}_\beta = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} \partial_\beta X^\mu - \delta^\alpha_\beta \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu .$$

- Moreover, we might know that (at least for a scalar field theory) this $T^\alpha{}_\beta$ ~~isn't~~ is also obtained on the RHS of the Einstein eq. after coupling this field theory to gravity; namely if $h \doteq h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta$ is arbitrary pseudo-Riemannian metric on the worldsheet and

$$S_P(X, h_{\alpha\beta}) = -\frac{T}{2} \int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

then (Polyakov action)

$$T_{\alpha\beta} := -\frac{2}{T\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\delta\varepsilon} \partial_\delta X \partial_\varepsilon X$$

while those for X become

$$\frac{\delta S_P}{\delta X^\mu} = T\, \partial_\alpha \sqrt{-h}\, h^{\alpha\beta} \partial_\beta X_\mu = 0$$

both equations reduce to the previous ones (derived from S_{N-G}) in the conformal gauge for $h_{\alpha\beta}$

$$h_{\alpha\beta} = \sqrt{-h}\, \eta_{\alpha\beta}$$

To repeat: Classical string dynamics is governed by the D linear ~~equations~~ 2^{nd} order equations

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta X_\mu = 0$$

and the two non-linear first-order equations (constraints)

$$(\dot{X} \pm X')^2 = 0$$

all one has to remember for now!

These eqs. obtain

- from N-G action in conformal gauge for induced metric
- from Polyakov action in conformal gauge for auxiliary metric.

Note: The story is similar to "pseudo-energy" action for particle, but there are some important differences.

- eoms + gauge conditions do not completely fix $h_{\alpha\beta}$.
- in particular, $\sqrt{-h}$ is not necessarily equal to $\sqrt{-\gamma}$, which is uniquely & well-defined in conformal gauge

for γ to be equal to $-\dot{X}^2 = +X'^2$

- The underlying reason in turn is that the Polyakov action (and eoms) are in fact invariant under <u>local rescalings of the metric</u> (<u>Weyl invariance</u>). Thus, we can in fact, if we desire, fix $\sqrt{-h}$ to anything we want, it has no physical significance.

[This invariance is the reason why $T_{\alpha\beta} = 0$ has only rank 2:

$$h^{\alpha\beta} T_{\alpha\beta} = \frac{\delta S_P}{\delta f} S_P(X, f h^{\alpha\beta}) = 0$$

automatically]

- It was Polyakov's insight to <u>elevate</u> Weyl invariance to a fundamental physical principle in ~~string theory and for examples~~ generalizing string actions. For example, the "2-d cosmological constant term"

$$\int \Lambda \sqrt{-h}\, d^2\sigma$$

is not Weyl invariant, and <u>verboten</u> by this principle. "massless strings".

- This is a late vindication for H. Weyl who got so much bashing (by W. Pauli) for introducing local scale invariance into 4-d G.R. (mind you, gauge invariance fared well in field theory anyway but this is not exactly what Weyl initially intended).

- All of this works (in exactly this way) only for 2-dimensional ~~[illegible]~~ field theories. (1-d is fine for other reasons, higher-d not)

-

Lecture 4 Canonical symmetries

- As mentioned before, one of the reasons for insisting on a Lagrangian formulation is the convenient (systematic) treatment of symmetries in the canonical formalism of Noether's theorem(s).
- As a potentially useful service, I want to give a fairly complete, if somewhat impressionistic account of the main statements, and apply to our examples as we go along. (The reason this is impressionistic is that as you progress, the (geometric) meaning of various quantities & operations is less and less clear, and a precise treatment requires considerable effort involving jet bundles & variational bicomplex that I am not prepared to make. (Yet to find student writing bachelor thesis on this!))

Instead: Make comments you might not have heard before

① Global symmetries in classical mechanics

~~ mechanical system with dynamical variables

$$q^I(t) \in M \quad (\text{possibly } \infty\text{-dim. mf.})$$

and Lagrange function $L(q,\dot q)$ that is invariant under an (infinitesimal) symmetry V^I - given by a vector field on M, which means

$$\frac{\partial L}{\partial q^I} V^I + \frac{\partial L}{\partial \dot q^I}\frac{d}{dt}V^I(q) = 0$$

Then the variation of the action

$$S[q(\cdot)] = \int_I dt \; L(q, \dot q)$$

under variation of path q^I in the direction $\lambda(t)\, V^I$ where λ is arbitrary function of λ is

$$L_{\lambda V^I} S = \int dt \left(\underbrace{\left(\frac{\partial L}{\partial q^I} V^I + \frac{\partial L}{\partial \dot q^I} \frac{d}{dt} V^I \right)}_{=0} \lambda(t) + \dot\lambda \frac{\partial L}{\partial \dot q^I} V^I \right)$$

integrating by parts, leads to the conclusion that if $q^I(t)$ satisfies eq. of motion (ie. $dS = 0$ in arbitrary directions) then

$$\frac{d}{dt} \underbrace{\left(\frac{\partial L}{\partial \dot q^I} V^I \right)}_{Q_V \,:\, \text{conserved Noether charge.}} = 0$$

Nothing fancy and certainly something you've seen.
Two remedial comments:

(i) (purpose: reminder of canonical formalism) In the canonical formalism we eliminate $\dot q^I$ in favor of canonical momentum

$$p_I = \frac{\partial L}{\partial \dot q^I}$$

at the expense of introducing the Poisson bracket

$$\{ p_J, q^I \} = \delta^I_J$$

~~Then the corollary says that~~

$$Q_V = p_I v^I$$

(on top of being conserved, but don't have H yet.

$$\dot{q} = \{H, q\} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q} = \{H, p\})$$

generates symmetry by Poisson bracket:

$$\{ Q_V, q^I \} = v^I$$

$$\{ Q_V, p_J \} = - p_I \frac{\partial v^I}{\partial q^J}$$

(ii) (~~purpose:~~ ~~make~~ prepare interpretation of τ as "more than just a parameter"), distinction between worldsheet and spacetime quantities).

The relativistic particle ~~...~~ has M = Minkowski space and is invariant under translations

$$(v_\mu)^\nu = \delta_\mu^\nu$$

and Lorentz trafos

$$(v_{\mu\nu})^\rho = \delta_\mu^\rho X_\nu - \delta_\nu^\rho X_\mu$$

global symmetries from worldline pt of view associated charges are (all worldsheet scalars)

- spacetime D-momentum $P_\mu = m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$ $\mu = 0, 1, .. d$
- angular momentum $J_{ij} = P_i X_j - P_j X_i$ $ij = 1, .. d$
- ? $J_{0i} = P_0 X_i - X_0 P_i$

$$= P_0 X_i \ (X_0 = 0).$$

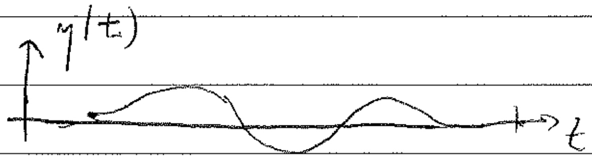
"initial position of particle".

---

② <u>Time translation & reparametrization invariance</u>

<u>so</u> as before + an ("infinitesimal") diffeomorphism of I.

$$t = \tilde{t} + \eta(\tilde{t})$$



+ ($\eta \to 0$ at bdry of I).

then by definition (or rather F.T.C.):

$$\int_I L(q, \dot{q})\, dt = \int_I L(\tilde{q}, \dot{\tilde{q}})\, d\tilde{t}\, \underbrace{(1 + \dot{\eta})}_{dt}$$

where $\tilde{q}(\tilde{t}) = q(\tilde{t} + \eta(\tilde{t}))$

$$\dot{\tilde{q}} = \left(\frac{d}{dt} q\right)(\tilde{t} + \eta(\tilde{t}))$$

$$\tilde{\dot{q}} = \left(\frac{d}{dt} q\right)\left(\tilde{t} + \eta(\tilde{t})\right) = \frac{d\tilde{t}}{dt} \frac{d\tilde{q}}{d\tilde{t}}$$

$$= (1 - \dot{\eta})\, \dot{\tilde{q}}$$

$$\rightsquigarrow \int_I L(q, \dot{q})\, dt = \int_{\tilde{I}} L(\tilde{q}, \dot{\tilde{q}})\, d\tilde{t} + \dot{\eta}\left(L - \frac{\partial L}{\partial \dot{q}} \dot{q}\right) d\tilde{t}$$

2.1 L time translation invariant: $\int_{\tilde{I}} L(\tilde{q}, \dot{\tilde{q}})\, d\tilde{t} = \int_I L(q, \dot{q})\, dt$

("trivial without explicit t-dependence")

view η (not $\dot{\eta}$!) as variation of q
$\rightarrow$ have to integrate by parts

$$\Rightarrow \frac{d}{dt} \underbrace{\left(L - \frac{\partial H}{\partial \dot{q}} \dot{q}\right)}_{= -H} = 0$$

on equations of motion

2.2 If S is reparametrization invariant, then

$$H = \frac{\partial H}{\partial \dot{q}} \dot{q} - L = 0$$

identically.

Example

$$P_\mu = m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}} \qquad L = -m\sqrt{-\dot{X}^2}$$

$$P_\mu \dot{X}^\mu = m \frac{\dot{X}^2}{\sqrt{-\dot{X}^2}} + m\sqrt{-\dot{X}^2} = 0.$$

3 Global symmetries in field theories

oo n-dimensional fields $X^I(\sigma)$ with action

$$S = \int_\Sigma d^n\sigma \, \mathcal{L}(X, \partial_\alpha X)$$

with invariance

$$\frac{\partial \mathcal{L}}{\partial X^I} V^I(X) + \frac{\partial \mathcal{L}}{\partial \partial_\alpha X} \partial_\alpha V^I = 0$$

⇝ by studying variation w.r.t. $\lambda(\sigma) V^I$,

$$L_{\lambda V} S = \int \left((\quad) \lambda + \partial_\alpha \lambda \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^I} V^I \right) d^n\sigma$$

we find:

$$\partial_\alpha J^\alpha = 0 \quad \text{on eoms where}$$

$$J_V^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^I} V^I(X) \qquad j = J^b$$

~~Conserved current~~ $\cancel{d * j_V = 0}$

This is nominally a vector field, but with some geometric liberty, we can ~~write it as~~ identify it with an $(n-1)$-form

$\cancel{d * j = 0}$ ~~$(n-1)$-form~~

$$* j$$

"conserved current".

and state the conservation $d \star j = 0$ as
de Rham differential

" $\int_K \star j$ depends only on homology class of (compact) $(n-1)$-dimensional subm. $K \subset \Sigma$."

Application: In canonical formalism for field theory we choose (potentially locally) a split

$$\Sigma = \bar{I} \times K \qquad \sigma^0 = t$$

"time, but there's not necessarily metric on Σ"

introduce canonical momentum (to eliminate $\partial_0 X^I$)

$$\Pi_I = \frac{\partial \mathcal{L}}{\partial \partial_0 X^I}$$

and state

$$Q_V = \int_K J_V^0 d^{n-1}\sigma = \int_K \Pi_I(\sigma) V^I d^{n-1}\sigma$$

is time independent (and agrees with charge from 1 by viewing $I = (\sigma^1, \ldots \sigma^{n-1})$

$$q^I(t) = X(t, \sigma = I)$$

and generates symmetry by Poisson brackets.

Example: Poincaré charges of string in conformal gauge

$$K = S^1 \quad \text{closed string}$$
$$= [0, \pi] \quad \text{open string}$$

$$\Pi_\mu = -T \dot{X}_\mu = -T \frac{d}{d\tau} X_\mu$$

$$P_\mu = -T \int_K d\sigma \, \dot{X}_\mu$$

$$J_{\mu\nu} = \int_K d\sigma \left(\Pi_\mu X_\nu - \Pi_\nu X_\mu \right)$$

etc.

$\rightarrow$ better justification of last time's calculation $\rightsquigarrow$ $M^2 \sim \alpha' J$

---

④ Diffeomorphism / reparametrization invariance.

$$\sigma^\alpha = \tilde{\sigma}^\alpha + \zeta^\alpha(\sigma)$$

infinitesimal diffeomorphism, n-dim. vector field

Using as in ② F.T.C. (transformation of integral under diffeo.)

$$d^n\sigma = \left(1 + \partial_\alpha \zeta^\alpha\right) d^n\tilde{\sigma}$$

$$\partial_\alpha \tilde{X} = \frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^\alpha} \partial_{\tilde{\beta}} \tilde{X} = -\partial_\alpha \zeta^\beta \, \partial_{\tilde{\beta}} X$$

we obtain

$$\delta S = \int_\Sigma d^n\sigma \; \partial_\alpha \zeta^\beta \underbrace{\left(\delta^\alpha_\beta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^I} \partial_\beta X^I \right)}_{-T^\alpha{}_\beta}$$

$-T^\alpha{}_\beta$ = "canonical energy-momentum tensor".

4.1 If $\mathcal{L}$ is translation invariant,

$$\partial_\alpha T^\alpha{}_\beta = 0 \qquad \text{on equations of motion.}$$

$\Rightarrow$ (and, in canonical formalism,

$$P_\beta = \int_K T^0{}_\beta$$

"worldsheet n-momentum is conserved charge")

4.2 If $\mathcal{L}$ is diffeomorphism (reparametrisation) invariant then

$$T^\alpha{}_\beta = 0 \qquad \text{identically}$$

example: N-G string.

But this is quite exceptional - most theories are invariant under diffeo only after coupling to gravity.

4.3. Theories with gravity

under infinitesimal diffeomorphism, metric transforms as (Lie-derivative).

$$(L_\xi h)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha .$$

$$\rightsquigarrow \quad \delta S_{\text{matter}} = \int \partial_\alpha \xi^\beta \underbrace{T^\alpha_{\ \beta}}_{= \text{"}\frac{\partial \mathcal{L}}{\partial X}\text{"}} + \underbrace{\frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}}}_{\Theta_{\alpha\beta}} \underbrace{\left(\nabla^\alpha \xi^\beta + \nabla^\beta \xi^\alpha \right)}_{2 \frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}} \nabla^\alpha \xi^\beta .}$$

"Einstein-Hilbert or Rosenfeld-Belinfante" tensor

then diffeomorphism invariance implies

- $T^\alpha_{\ \beta} \sim \Theta_{\alpha\beta}$
- when equations of motion for X are satisfied

$$\frac{\delta S}{\delta \xi^\beta} = 0$$

$$\partial_\alpha \Theta^{\alpha\beta} = 0 \qquad \text{is conserved.}$$

5 Conformal invariance

Def: Let (Σ, h) be a (pseudo-) Riemannian manifold. A conformal Killing vector is v.f. ξ s.t.

$$\mathcal{L}_\xi h = \lambda \cdot h \qquad \text{for some scalar function } \lambda.$$

- ξ generates conformal Killing transformation
- Conformal Killing equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = \lambda h_{\alpha\beta} \qquad \lambda = \frac{2}{n} \nabla_\alpha \xi^\alpha .$$

5.1 Fact: 2-dimensional manifolds locally have infinitely many conformal Killing vectors.

Pf: ~~in conformal gauge $h_{\alpha\beta}$~~ In flat space

$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, introduce

"worldsheet light-cone coordinates"

$$\sigma^\pm = \tau \pm \sigma$$

$$\eta_{++} = 0 \qquad \eta_{+-} = \frac{1}{2} \qquad \eta_{--} = 0$$

CKE: $\quad 0 = \partial_+ \xi^- + \partial_- \xi^+$

$$\lambda = \partial_+ \xi^+ + \partial_- \xi^-$$

$$\zeta^\pm = \zeta^\pm(\sigma^\pm) \quad \text{arbitrary functions.}$$

5.2 Fact. S_P in conformal gauge for $h_{\alpha\beta}$ is (still) invariant under conformal Killing transformations.
: a "global symmetry" (not localizable).

$$S_P = \frac{1}{2}\int d\sigma^+ d\sigma^- \, \partial_+ X \cdot \partial_- X$$

$\rightsquigarrow$ infinitely many conserved charges

$$\int_K T_{++}\zeta^+ \, d\sigma, \quad \int T_{--}\zeta^- \, d\sigma$$