# Homework problem set 9

Submission deadline on 20 December 2021 at noon

**Problem 1** (Open-string non-linear $\sigma$-model)**.** To set up string perturbation theory we replaced $S_P$ with a non-linear $\sigma$-model with pseudo-Riemannian target $M$. The action for the non-linear sigma model is

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\sqrt{h}\left[\left(h^{ab}g_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi\right] \tag{1}$$

where $\Phi$ is the dilaton, $B_{\mu\nu}$ is the antisymmetric Kalb-Ramond field and $g_{\mu\nu}$ is the metric of the curved spacetime.

(i) Consider the following gauge transformation of the $B$-field:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{2}$$

Consider a flat worldsheet, i.e. $\int d^2\xi\sqrt{h} = \int d\tau d\sigma$ and show that $S_\sigma$ is invariant under this transformation. Show that also the field strength

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\nu} + \partial_\rho B_{\mu\nu}$$

is invariant under this gauge transformation (2).

(ii) No consider open strings along a $D$-brane, i.e. we have Dirichlet boundary conditions in the normal directions labelled $X^n$ and Neumann boundary conditions in the parallel directions labelled by $X^i$. Show that under the gauge transformation (2) there arises a boundary term

$$\delta S_\sigma = -\frac{i}{2\pi\alpha'}\int d\tau(\Lambda_i\partial_\tau X^i + \Lambda_n\partial_\tau X^n)\,|_{\sigma=0}^{\sigma=\pi}$$

and argue that this vanishes only for the coordinates normal to the brane.

(iii) To restore gauge invariance under (2) we generalise the action in the following way. Introduce a $U(1)$ gauge potential $A_i(X)$ along the brane., which arises as the massless excitation of the open string. The coupling of the string world-sheet to the gauge potential $A_i$ is given by

$$S_{coup} = -\frac{i}{2\pi\alpha'}\int d\tau A_i(X)\partial_\tau X^i\,|_{\sigma=0}^{\sigma=\pi}$$

Determine $\delta A_i$ such that $S_\sigma + S_{coup}$ is gauge invariant under the combined gauge transformation $\delta B_{\mu\nu}$ and $\delta A_i$.

(iv) The field strength of the 1-form field $A_\mu$ ($A_n = 0$ in the normal directions) is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Show that the combination

$$\mathscr{F}_{\mu\nu} = B_{\mu\nu} + F_{\mu\nu}$$

is invariant under the combined gauge transformation, while $F_{\mu\nu}$ is not invariant under this transformation.

**Problem 2** (Weyl anomaly). (i) It was mentioned in the lecture that the Weyl anomaly is

$$\langle T^a_a \rangle = -\frac{c}{12} R^{(2)}$$

where $T_{ab}$ is the worldsheet energy momentum tensor and $R^{(2)}$ is the curvature of the worldsheet. Calculate the Weyl anomaly for the sphere with metric

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 d\phi^2.$$

(ii) Calculte the Euler characteristic $\chi$ of the sphere, using the Gauss-Bonnet theorem.

(iii) *Optional:* Use the Gauss-Bonnet theorem to determine the Euler characteristic of a Riemann surface of genus $g = 2$ and deduce that the Weyl anomaly is given by

$$\langle T^a_a \rangle = -\frac{c}{12} \frac{2\pi\chi}{vol}$$

*Hint:* Cover the sphere by the Poincaré disk. This is the unit disk equipped with the metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1-(x^2+y^2))^2}$$

Construct a fundamental region and integrate the curvature of the Poincaré plane over that region[1]. Assume that the curve is equipped with the induced metric. (It might be quite hard to do the integration, but you can find an upper and lower bound for $\chi$).

[1] It might helpful to look at https://www.mathi.uni-heidelberg.de/ banagl/pdfdocs/aufgaben4.pdf.