Homework problem set 8

Submission deadline on 13 December 2021 at noon

Problem 1 (Virasoro-Shapiro)**.** In this exercise we want to compute the amplitude for 2-to-2 scattering for closed string tachyons at tree-level. Recall that the vertex operator is given by

$$V_p(x) =: e^{ip_\mu X^\mu(x)} :$$

(i) Show that

$$\begin{aligned} A &= \int d^2 z_2 \langle V_{p1}(z_1) V_{p2}(z_2) V_{p3}(z_3) V_{p4}(z_4)\rangle D \\ &= (2\pi)^D \delta^D(p_1 + p_2 + p_3 + p_4) \int d^{z_2} |z|^{\alpha' p_1 \cdot p_2} |1-z|^{\alpha' p_3 \cdot p_2} \end{aligned}$$

where D is some fudge factor, such that A is invariant under global conformal transformations. Argue that this indeed describes 2-to-2 scattering for closed string tachyons.
Hint: Determine D by considerations similar to the open string case.

(ii) Show that

$$\int d^2 z |z|^{2a-2} |1-z|^{2b-2} = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)} \tag{1}$$

where $a+b+c=1$. Introduce mandelstam variables and use the above result to arrive at

$$A(s,t,u) = (2\pi)^{26} \delta^{(26)}(p_1+p_2+p_3+p_4) 2\pi \frac{\Gamma(-1-\frac{\alpha'}{4}s)\Gamma(-1-\frac{\alpha'}{4}t)\Gamma(-1-\frac{\alpha'}{4}u)}{\Gamma(2+\frac{\alpha'}{4}s)\Gamma(2+\frac{\alpha'}{4}t)\Gamma(2+\frac{\alpha'}{4}u)} \tag{2}$$

Note that the amplitude is symmetric in s, t and u.
Hint: Use that $|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^\infty dt\, t^{-a} e^{-|z|^2 t}$.

Problem 2 (A toy model for supersymmetry)**.** Let V be a finite real dimensional vector space. The Grassmann (or exterior) algebra is defined as the space

$$\mathbb{A} = \bigoplus_{k=0}^{n} \bigwedge^k V = \mathbb{C} \oplus V \oplus V^2 \oplus \cdots \oplus \bigwedge^n V$$

A general element in $f \in \mathbb{A}$ has the form

$$f = a + a_i \theta^i + \cdots + \frac{1}{n!} a_{i_1 \ldots n} \theta^{i_1} \ldots \theta^{i_n}$$

Where the θ_i build a basis of V and $\theta_i \theta_j = -\theta_j \theta_i$. One should think of this as the Taylor expansion of a function on the vector space with n fermionic directions. We define the integral of $f \in \mathbb{A}$ via $\int d\theta^i \theta^i = 1$ and $\int d\theta^i 1 = 0$. So for instance

$$\int d^n \theta f = a_{12\ldots n} \tag{3}$$

with $d\theta = d\theta^n d\theta^{n-1} \ldots d\theta^1$.

(i) Assume $n = 2m$. Show for an antisymmetric $n \times n$ matrix B, that

$$\int d^n\theta e^{\frac{1}{2}\theta^i B_{ij}\theta^j} = \mathrm{Pf}(B).$$

where $\mathrm{Pf}(B) = (2^m m!)^{-1}\epsilon_{i_1\dots i_n} B_{i_1 i_2}\dots B_{i_{n-1}i_n}$. Show that $\mathrm{Pf}(B) = \sqrt{\det(B)}$
Hint: The exponential function is defined via its Taylor expansion.

(ii) Show for an $n \times n$-matrix C, that

$$\int d\psi d\chi e^{\chi^i C_{ij}\psi^j} = \det(C)$$

(iii) We consider the following integral

$$I = \frac{1}{(2\pi)^n}\int dx d\psi e^{-x^i A_{ij} x^j - \psi^i B_{ij}\psi^j}$$

where A is a symmetric and B an antisymmetric $n \times n$-matrix. Define the supersymmetry transformation $\delta x^i = \epsilon\psi^i$ and $\delta\psi = \epsilon\delta^{ij}B_{jk}x^k$, where ϵ is some Grassmann parameter. Show that this is a symmetry of I if and only if $A = B^2$.

(iv) Assume I is supersymmetric and B invertible. Calculate I explicitly.

Problem 3 (De Rham cohomology)**.** Let M be a smooth manifold and let Ω^k be the space of differential k-forms on M and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the differential. The k-th de Rham cohomology group is

$$H^k(M) := \ker(d : \Omega^k \to \Omega^{k+1})/\mathrm{im}(d : \Omega^{k-1} \to \Omega^k). \tag{4}$$

In this exercise we will describe $H^1_{dR}(\mathbb{R}^2\backslash(0,0))$. Consider the differential 1-form

$$w_0 = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

(i) Consider a loop γ around $(0,0)$ and show that for a closed differential form $w \in \Omega^1(M)$, it holds that $w = df$ if and only if $\int_\gamma w = 0$. Deduce that $H^1_{dR}(\mathbb{R}^2\backslash(0,0)) \cong \mathbb{R}$ with basis $\{[w_0]\}$.

(ii) Describe the de Rham cohomology of $\mathbb{R}^2\backslash\{(0,0),(1,0)\}$.