Homework problem set 7

Submission deadline on 6 December 2021 at noon

Problem 1 (Affine currents and Sugawara construction)**.** Consider a general CFT with primary fields $j^a(z)$ satisfying the OPE's

$$j^a(z)j^b(w) \sim \frac{k^{ab}}{(z-w)^2} + \frac{ic^{ab}_c}{(z-w)} j^c(w)$$

(i) Show that the modes of $j^a(z) = \sum_{m\in\mathbb{Z}} j^a_m z^{-(m+1)}$ satisfy

$$[j^a_m, j^b_n] = mk^{ab}\delta_{m+n} + ic^{ab}_c j^c_{m+n}$$

and deduce that $c^{ab}_c = f^{ab}_c$ must be the structure constants of some Lie algebra $\mathfrak{g}$.
Hint: Use Cauchy's integral formula.

(ii) Assume $\mathfrak{g}$ is simple and finite dimensional. Use the Jacobi identity for j^a_1, j^b_0, j^c_{-1} to show that $k^{ab} = k\delta^{ab}$ for some complex constant k.
Hint: For simple Lie algebra's there exist a up to normalization unique bilinear form B on $\mathfrak{g}$ called the Killing form, such that for every $x, y, z \in \mathfrak{g}$, $B([x,y],z) + B(y,[x,z]) = 0$. You may assume that in the given Basis of $\mathfrak{g}$ the components of B are δ^{ab}.

(iii) Consider the operator

$$T_S(z) = \gamma \sum_{a=1}^{\dim \mathfrak{g}} :j^a j^a:(z)$$

where γ is some constant. We require that the j^a have conformal dimension $h = 1$. Show that this fixes

$$\gamma^{-1} = 2(k + C_{\mathfrak{g}})$$

where $\sum_{b,c} f^{bac} f^{bcd} = -2C_{\mathfrak{g}}\delta^{ad}$ defines the dual Coxeter number of the Lie algebra $\mathfrak{g}$. For $\mathfrak{g} = su(n)$ we have $C_{\mathfrak{g}} = n$.

(iv) Show that $T_S(z)$ satisfies the OPE of the Energy-momentum tensor, i.e.

$$T_S(z)T_S(w) \sim \frac{c^{\mathfrak{g},k}}{2(z-w)^4} + \frac{2}{(z-w)^2} T_S(w) + \frac{1}{z-w} \partial T_S(w)$$

where the central charge is

$$c^{\mathfrak{g},k} = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}}.$$

Hint: You can either work with the OPE of the j^a or you can use that the OPE is equivalent to the commutators of the modes and show that the modes of $T_S(z)$ satisfy the Virasoro algebra with central charge $c^{\mathfrak{g},k}$. Either way, you need to be careful about the normal ordering in the definition of $T_S(z)$.

Problem 2 (More on ghosts)**.** The action of the bc-ghost-system in complex coordinates takes the form

$$S_{bc} = \frac{1}{2\pi} \int d^2x \left[b_{\mu\nu} \partial^\mu c^\nu \right].$$

Here the ghost field c^μ is a worldsheet vector and anti-ghost $b_{\mu\nu}$ is a symmetric traceless tensor. In complex coordinates we write $c = c^z$ and $\overline{c} = c^{\overline{z}}$ and similarly for the non-zero components $b = b_{zz}$ and $\overline{b} = b_{\overline{z}\overline{z}}$. The (holomorphic component of the) energy momentum tensor for this system is given by

$$T(z) =: (2\partial c b + c\partial b):$$

(i) Show that the OPE for b and c in complex coordinates is

$$b(z)c(w) \sim \frac{1}{z-w} \tag{1}$$

Use this result and show by using Wick's theorem that c is a primary field of conformal dimension $h_c = -1$ and b is a primary field of conformal dimension $h = 2$. Calculate $T(z)T(w)$ to compute the central charge.
Hint: In order to find the OPE (1) write $S = \int d^2x d^2y b_{\mu\nu}(x) P^{\mu\nu}_{\alpha}(x,y) c^\alpha(y)$ and find the propagator for P. Use that in complex coordinates $\partial_{\overline{z}} \frac{1}{z} = 2\pi\delta^{(2)}(z,\overline{z})$.

(ii) More generally one can change the energy-momentum tensor by a total derivative, i.e.

$$T(z) =: (\partial b)c : -\lambda\partial(:bc:)$$

for $\lambda \in \mathbb{R}$. Compute the conformal dimensions for the b and c and the central charge.
Hint: Proceed analogous to part (i) and note that the OPE's for b and c and don't change. The conformal dimension and central charge will depend on λ. The result for the central charge should look familiar from earlier exercises.