Homework problem set 6

Submission deadline on 29 November 2021 at noon

Problem 1 (Super-ghosts...)**.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Basis (e_α) and dual basis (c^α) (the ghosts) satisfying $\{c^\alpha, c^\beta\} = 0$. The anti-ghosts are defined as $b_\alpha = \iota_{e_\alpha}$. They satisfy $\{\beta_\alpha, \beta_\beta\} = 0$ and $\{c^\alpha, b_\beta\} = \delta^\alpha_\beta$. In the lecture we saw that we can define a BRST-differential

$$Q = c^\alpha \rho_\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma b_\alpha$$

where $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma$ and $\rho_\alpha = \rho(e_\alpha) \in Hom(M, M)$ for some representation (M, ρ) of $\mathfrak{g}$.

(i) Consider now a grade Lie-algebra $\mathfrak{h}$ with bosonic generators $\{e_\alpha\}$ and fermionic generators $\{f_\beta\}$, i.e. define the Fermion number F as $F(e_\alpha) = 0$ and $F(f_\beta) = 1$. Then for two generators h_α, h_β we have that
$$[h_\alpha, h_\beta] = (-1)^{F(h_\alpha)F(h_\beta)+1} [h_\beta, h_\alpha]$$
(in other words $f^\gamma_{\alpha\beta} = (-1)^{F(h_\alpha)F(h_\beta)+1} f^\gamma_{\beta\alpha}$). Define suitable ghosts and anti-ghosts for the system and construct a BRST-charge Q satisfying $Q^2 = 0$.

Problem 2 (...and no-ghosts...)**.** In the proof of the no-ghost theorem we wrote $Q = Q_1 + Q_0 + Q_{-1}$ with $[N^{l.c}, Q_j] = jQ_j$ where
$$Q_1 = -\alpha_0^+ \sum_{n \neq 0} \alpha_n^-$$
where $\alpha_0^+ = \sqrt{2\alpha'} p^+$. In the lecture the cohomology of Q_1 was calculated rather abstractly. Alternatively one can compute the cohomology directly by considering the action of Q_1 in the occupation basis.

(i) Restrict the sum above to $n = \pm 1$. This gives the truncated operator $Q' = \alpha_0^+ (\alpha_{-1}^- c_1 + \alpha_1^- c_{-1})$. Calculate the cohomology of Q' by considering its action on
$$(b_{-1})^{N^b} (c_{-1})^{N^c} (\alpha_{-1}^+)^{N^+} (\alpha_{-1}^-)^{N^-} v_0.$$

(ii) Generalize this to the full Q_1.

Problem 3 (Schwarz derivative)**.** The Schwarzian derivative of a holomorphic function f of one complex variable z is defined by
$$(Sf)(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

(i) Let f and g be holomorphic functions show that the Schwarzian derivative of $f \circ g$ is given by the chain rule
$$(S(f \circ g))(z) = (Sf)(g(z)) \cdot g'(z)^2 + Sg. \tag{1}$$
Deduce that for a global coordinate transformation $w = w(v)$ we have $S(w)(z) = -(w')^2 S(z)(w)$.

(ii) Show that $Sf = 0$ if and only if
$$f = \frac{ax + b}{cx + d} \tag{2}$$
for complex numbers $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 0$.