Homework problem set 5

Submission deadline on 22 November 2021 at noon

Problem 1 (Not yet Minimal Models)**.** In this exercise we want to study heighest weight representations of the Virasoro algebra. Recall that the Virasoro algebra is generated by $\{L_n\}_{n\in\mathbb{Z}}$ satisfying

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

where c is central. A heighest weight representation V, contains a vector v_0 such that $L_n v_0 = 0$ for all $n > 0$, $L_0 v_0 = h v_0$ and the vectors of the form $v_{\underline{k}} = L_{k_1} L_{k_2} \dots L_{k_n} v_0$ with $k_i < 0$ for all $i = 1, \dots, n$ generate V. We call V the Vermat module with heighest weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$ with $cv = cv$ (by abuse of notation). We call $N = \sum_i k_i$ the level of $v_{\underline{k}}$. We can define an inner product on V by $L_m^\dagger = L_{-m}$ and $v_0 \cdot v_0 = 1$.

(i) Show that for v of level N and w of level N' with $N \neq N'$, the inner product $v \cdot w$ vanishes.

(ii) Consider the Gram matrix M with components

$$M_{\underline{k},\underline{k}'} = v_{\underline{k}} \cdot v_{\underline{k}'}$$

from (i) we know that M is block diagonal with blocks $M^{(N)}$ corresponding to states of level N. Calculate $M^{(N)}$ for $N = 0, \dots, 3$.

(iii) Show that unitarity of V ($v^2 \geq 0$ for all $v \in V$) requires that $\det M^{(N)} \geq 0$. Derive consequences for the allowed combinations of (h, c) for $N \leq 2$.

(iv) Show that if $\det M^{(N)} = 0$, V is reducible.
Hint: Show that V contains a sub-Vermat module build on some $\chi \in V$ of level N.

Remark: The Kac determinant formula calculates the determinant of $M^{(N)}$ for given heighest weight h and central charge c. It states that

$$\det M^{(N)} = a_l \prod_{r,s} [h - h_{r,s}(c)]^{p(N-rs)}. \tag{1}$$

where $p(n)$ denotes the number of partition of $n \in \mathbb{N}_0$. We take the product over positive integers $1 \leq r, s$ such that $rs \leq N$. Rewriting c as $c = 1 - \frac{6}{m(m+1)}$, the $h_{r,s}(c)$ can be expressed as

$$h_{r,s}(c) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}.$$

Problem 2 (Tachyons matter)**.** Consider a $p+1$-dimensional theory of a scalar field $T(x)$ (in flat space) with action

$$S = -\int d^{p+1}x\, e^{-T} \sqrt{1 + \eta^{\mu\nu}\partial_\mu T \partial_\nu T}$$

(i) Write down the Euler-Lagrange equations of motion for T and show explicitly the equivalence with the conservation of the energy-momentum tensor

$$\mathcal{T}^\mu_{\ \nu} = -\frac{e^{-T}}{\sqrt{1 + \partial_\rho T \partial^\rho T}} \partial^\mu T \partial_\nu T + e^{-T}\sqrt{1 + \partial_\rho T \partial^\rho T}\, \delta^\mu_{\ \nu}.$$

(ii) Consider homogeneous solutions, *i.e.*, $\partial_i T = 0$ for all spatial indices. Show that the energy density $\mathscr{T}^0_{\ 0}$ simplifies to

$$\mathscr{T}^0_{\ 0} = \frac{e^{-T}}{\sqrt{1-(\partial_0 T)^2}}$$

(iii) Assume that $0 < \partial_0 T < 1$. Use conservation of $\mathscr{T}^\mu_{\ \nu}$ to show that $T \to \infty$ as time $x^0 = t \to \infty$. Also show that the spatial components (pressure) $\mathscr{T}^i_{\ j} \to 0$ in this limit.

Remark: The above action for T has been used as a model for large-time behaviour of tachyon decay on a Dp-brane.